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# A TREATISE ON INFINITESIMAL CALCULUS;

CONTAINING

DIFFERENTIAL AND INTEGRAL CALCULUS, CALCULUS OF VARIATIONS, APPLICATIONS TO ALGEBRA AND GEOMETRY, AND ANALYTICAL MECHANICS.

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## VOL. III.

## STATICS, AND DYNAMICS OF MATERIAL PARTICLES.

"Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires."—CH. DUPIN.

OXFORD:
AT THE UNIVERSITY PRESS.
M.DCCC.LVI.

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# PREFACE.

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STATICS, Attractions, and the Dynamics of a material particle are contained in the present volume.

The investigations are for the most part confined to subjects which are within the range of the general principles of Mechanics, and are not extended to particular sciences wherein these principles are specifically applied. Thus, although the principles are discussed on which the equilibrium and stability of bridges, arches, and roofs depend; yet the practical rules of the engineer's and the builder's arts are not considered. Also as physical astronomy, the theories of light, heat, and electricity require the explanation and discussion of certain experimental laws which rule their subject-matter, so the inquiry into these is beyond the scope of this work at the present stage of it.

Chapter I is introductory to the whole of this part of the Treatise on Infinitesimal Calculus. It seemed desirable to exhibit at greater length than is usual the relation between "applied Mathematics," as some parts of the subject are called, and the sciences of number and geometrical space; and so I have entered on a discussion of one or two salient points of the subject. I have also ventured to submit to the common judgment of Mathematicians the statement, that Mechanics, enlarged in its idea and principles, as I have attempted to enlarge it, is nothing else than the science of motion, and ought, as such, to be called by that name. Thus there are three principal mathematical sciences, those viz. of number, space, and motion: the last of which it has been my purpose to develope in the following pages.

A course of inquiry somewhat irregular has been found most convenient for a didactic treatise; and Chapters II—VI contain Statics, wherein the laws of pressure as they produce equilibrium, or neutralize each other's effects, so far as motion is concerned, are considered. In Chapter VII the principles, incidents, laws, and conditions of the science of motion are formally drawn out. The Chapter is thus introductory to Dynamics. The mode of investigation and the forms of statement of some of the pregnant principles are different from those which are commonly given. Only two of the three ordinary laws of motion (*axiomata motûs*, as they are called by Newton) are admitted. The truth of these is made to depend on and to flow from an intelligible conception of the idea of motion and its incidents; and on an inductive verification only so far as the science is applied. This distinction is important, and appears to solve some questions which are in dispute between the two schools of writers on Mechanics. The method which I have taken is indeed counter to that of most English authorities on the subject: it is rather in accordance with that of foreign, and chiefly French, writers. If any one after reflection should hesitate or refuse to admit my principles, and the mode of arriving at and of stating them, I must ask him to consider the subject from the point of view which the Infinitesimal Calculus and a reasonable conception of Infinitesimals present to him; and which, with great respect for the great names and the sober judgment of those who take the opposite course, I venture to think to be the most natural and the most rational.

In the following treatise the first principles of the science are drawn from an intelligible conception of motion itself. For the mathematical expression of these, the language and the symbols of Infinitesimals are peculiarly appropriate: effects are produced by causes which act according to continuous laws: thus the effects become continuously developed, and a peculiar system of symbols is required to express them. New ideas necessitate a new language, and new language requires new characters; and such are those of Infinitesimal Calculus.

A license has been taken, for which I must crave some indulgence; certain words are used which are either new or are used in a new relation. In the absence of generally recognised rules for the formation of scientific language, I have used compounded words; and have thereby obtained expressive, though somewhat long, words. This course I found myself obliged to take. For ideas which are in themselves clear and distinct have been so much obscured by ambiguity and indistinctness of language, that there is no source of error more fertile. Let me cite an instance. In former books no word occurs more frequently than the word "force." Indeed Mechanics has been called the science of *forces*. But what does "force" mean? Will any one give an accurate definition of it? a definition, that is, which will be correct, when the word is applied to "the cause of motion," to "accelerating forces," to "effective forces," to "forces lost and forces gained," to "living force," to "labouring force?" In some of these various meanings it indicates effect, in others it indicates cause. Surely herein is confusion; and herein too, as it seems to me, is the reason why the principles of mechanical science, or the science of motion, are so imperfectly understood. Similar is the ambiguity of the word "motion:" it is frequently used synonymously with the word *velocity*: thus "*momentum*" has been called "quantity of motion:" it is *quantity of velocity;* and it is at all events perplexing to most minds to have a thing called by a name which means what it is not. Thus I have endeavoured in those parts of the treatise where first principles are expounded, and where clearness of language no less than clearness of conception is required, to call things by names which are expressive *vi significationis;* although in the more popular parts I have used words in their ordinary and less exact meaning. The subject is not in itself difficult, but it has been made difficult by the maze of indistinct nomenclature by which its fundamental notions have been obscured.

As in the previous Volumes, I am under obligation to many friends, and to many writers on these subjects. It is almost superfluous to mention Euler, Lagrange, Laplace, Poisson, Poinsot, Jacobi, M. Bertrand, Sir W. R. Hamilton of Dublin, because no one has a right to form a judgment, and much less to compose a didactic treatise, on the subject of Mechanics, without a previous and preparatory study of the works of these eminent men. From the works of Dr. Whewell, the Master of Trinity College, Cambridge, I have derived much aid: I know not how much: because in the Appendices to the second volume of his Philosophy of the Inductive Sciences so much suggestive matter on Mechanical Philosophy is contained, that opinions which appear to be one's own may perhaps owe their origin to those essays. The Journals of Crelle and Liouville have given much assistance. To the editors of those Journals and to their contributors, whose names are too many to be mentioned here, I tender my acknowledgments.

The references are made to the numbers of the Articles and of the equation as in the preceding volumes. The colloquial style has been retained.

PEMBROKE COLLEGE, OXFORD,
FEB. 19, 1856.

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# ANALYTICAL MECHANICS.

## CHAPTER I.

### INTRODUCTORY; METHOD OF THE TREATISE.

ARTICLE 1.] Of all parts of Infinitesimal Calculus, Analytical Mechanics, or (as I shall hereafter have reason to call it) the Science of Motion is in its results and its applications the most important; the principles and processes of all mathematical physics are derived from it; and as, for reasons which shall be assigned hereafter, it is in itself the most perfect of physical sciences, so do the others approach more or less to completeness according as the laws and methods of mechanics are more or less satisfied by them; and the object to be attained in all is, to make them parts of this principal and normal science. Now in the process of our application of the science of number to that of motion, new subject-matter, or new kinds of quantity measurable by number, will be introduced; and also as the results of our investigations will be applicable to the phenomena of the external world, and to the unravelling of complex effects, it is necessary to premise some few observations on the method of our inquiry; and especially to shew how, and how far, the pure sciences of number, space, and motion may aid us in the discovery of the proximate causes of such effects; proximate, I say, in order that the objects of our search may be definite and intelligible, and that we may not be lost in the subtleties of metaphysics.

2.] There are generally two processes, by one or other of which our knowledge of natural phenomena is obtained, and with both of which it is in many cases absolutely necessary, and in all cases desirable, that an inquirer into nature's laws should be acquainted; and although in their use one of these processes frequently runs into the other, and they are alternately applied for the purposes of discovery and verification, yet they are in themselves distinct, and for philosophical reasons it is requisite to keep them so. In one of these processes we take the facts of nature as they are presented to us in their simple and concrete forms; and animated by a conviction deep-seated in our nature that they are not isolated, but instances of a grand and comprehensive law, which has been impressed on them, and by virtue of which they are, we seek for that law: with this object in view we study them, analyse them; and in the analysis we subject them to trials of various kinds, if they admit of experiment, or observe them in such varying relations as they exhibit to us, if they do not; we separate what is extraneous and thus accidental from that by virtue of which they seem to us to be, and without which they would not be: and by this process detect the general law which lies latent in the fact; or, in perhaps more precise terms, the cause of which the fact is the effect. Thus we ascend from the fact to the cause; and when many facts have been subjected to a similar process, and the same law has been detected in all, we collect them under a general formula which expresses this law, and thereby a cause of which all the examined facts are the effects; and the human mind, endowed with a love of continuity, extends this to other facts similar in kind, and beside those which have been examined. In this process therefore we interrogate nature as she offers herself to us in her simple forms and particular developments; and so long as any branch of knowledge consists only of such isolated facts it is little else than mere empiricism; but when a bond of union has been imported from some other source, and these facts have been collected into general propositions; when on these phenomena has been induced a distinct idea, and the information obtained from them in their isolated forms has been studied, arranged, and reasoned upon, then, and not until then, has it a right to bear the name of Science; it is then no longer accumulated experience, but it is experience systematized, digested, assimilated, organized into a whole; it has harmony, regularity, and law; and the physical sciences thus formed will be found to satisfy another most exact test; they predict similar effects from similar causes.

3.] All the physical sciences are progressive, and, as such, pass through the stage which I have described; the subjects of them in their infancy have been in this disconnected state. Experience in the way of experiment and observation has been the chief instrument by which their boundaries have been advanced, distinct ideas for the colligation of facts been obtained, and inquirers been led to the discovery and enuntiation of their particular laws: the discovery of the laws of motion by Galileo, of the laws of planetary orbits by Kepler, of the law of refraction by Snell, are, amongst many, early and salient instances. Invariably, so long as any science is in this imperfect condition, its phenomena must be examined for the purpose of discovering such normal laws, and it is in the prosecution of this work that the most eminent philosophers of the present age have earned their glory: in short, the analysis of such facts is the characteristic of the science of the XIXth century: and no mean work is it: it demands the highest intellectual and moral qualities that can adorn human nature; an eager and honest desire after truth; patience and endurance of labour; a courage that never fails under non-success; the keenest intellectual acuteness in detecting resemblances; a mind gifted with a plastic power of framing an idea distinct and pregnant, which shall collect all into one general formula; an inventiveness and a never-failing command of resources: and in our days these qualities have not been wanting, and have not been unrewarded. It is however unnecessary for me to do more than to indicate the methods of experimental philosophy, in order that I may contrast with them, and thus bring into greater prominence, the process of investigating truth which will be developed in the following volume; and the reader desirous of further information on the methods of inductive philosophy must have recourse to works wherein such subjects are specially treated of. Let me refer him to Sir John Herschel's Treatise of Natural Philosophy, a work which contains in a short compass a masterly exposition of the methods, and to Dr. Whewell's Philosophy of the Inductive Sciences, wherein he will find the subject treated by an eloquent author, whose knowledge of physical science seems to be limited only by the limits of science itself *.

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* See also an article on "Whewell on Inductive Sciences" in the Quarterly Review, Vol. LXVIII: evidently written by a master-hand.

4.] It will be seen then that the first step in experimental philosophy is to colligate facts by means of a distinct and appropriate idea; afterwards a consilience of inductions takes place; and hereby we arrive at the last step in the construction of a science, which is the enuntiation of a *theory;* the determination, that is, of a law which rules all the subject facts, and the discovery of a general cause, of which the facts of the science are the single and (as they seem at first) isolated or independent effects; and when such perfection is attained the aggregate of the knowledge receives the name of a science, having all the notes of arrangement, order, system, completeness, which are necessary for such perfection.

And now comes in the second process to which allusion has already been made. If the theory is true, not only is it a true explanation of all the facts which it comprises in its formula, but it has also a prophetic power: when the cause is active, results similar to the former ones must be produced; the theory requires verification; and the verification consists in the prediction of the future: and it is only when such future facts have been shewn to accord with a theory, that it satisfies those stringent rules of induction which have been constructed in a jealous care of truth. The theory may also be pregnant with results different from those out of which it has grown; these must also be traced and examined: the theory must be tested in all ways and in all directions; and when such tests have been satisfied, it has a claim on our allegiance. For this purpose a process, the reverse of the former, is necessary: facts were in that analysed, so that their latent cause might be detected; in this causes are to be developed into their effects; the former is the *historical* process through which the science has grown from an imperfect state to perhaps full maturity; the latter takes the science in its perfect state, and explores the riches which it contains; the former is the process by which the science has been constructed, and is somewhat analogous to the manner in which we individually *learn* it; the latter is the form wherein the man of science *knows* it. Now this distinction is important: for as it is under the latter and more perfect aspect that I shall have to consider the science of motion, so the method is dogmatic; and the fundamental and axiomatic laws will be enuntiated, and no formal proof of them will be given; it may sometimes be desirable to indicate the steps by which historically they have been arrived at, but such an explanation will be only incidental and that the learner may have adequate knowledge of them; and I shall not lose sight of the chief object, which is to trace into their farthest results those general laws which an inductive philosophy has supplied.

5.] And mathematics is the most powerful instrument, which we possess, for this purpose: in many of the most recondite sciences, not only a fair but a profound knowledge of mathematics is indispensable for a successful investigation. In the most delicate researches into the theories of light, heat, and sound it is the only instrument; they have properties which no other language can express; and their argumentative processes are beyond the reach of other symbols; and I am sure that to any one who has taken the pains to compare the general explanation of planetary disturbances given in Sir John Herschel's Outlines of Astronomy with that of the same phenomena as discussed with the aid of mathematical appliances, there cannot be a doubt that, however successful Sir John Herschel may have been, even beyond his expectation, yet for an accurate comprehension of the circumstances the other method is absolutely necessary. The following extract from Sir John Herschel's work * is unimpeachable testimony: "Admission to its sanctuary" (that is, of astronomy) "and to the privileges and feelings of a votary is only to be gained by one means—*sound and sufficient knowledge of mathematics, the great instrument of all exact inquiry, without which no man can ever make such advances in this or any other of the higher departments of science as can entitle him to form an independent opinion on any subject of discussion within their range.*" I may also utter the same language as to the necessity of mathematics in the other higher branches of the science of motion.

6.] Here it may be asked, What are mathematics? Define them. Do they require and apply reasoning processes different from those of the ordinary discourse of men? have they a different logic? and a different language? What distinction exists between pure and mixed mathematics, since they are commonly divided into these two classes? and what does the term include? Many of these questions may be matter of words only; it is not necessary for me to define mathematics, or to inquire how far "science of quantity," or "science of measuring quantity" may be a sufficient definition, and whether there is not a large class of propositions of geometrical *position* which such definitions will not include; it is enough for me to be able to give you such an account of the means which mathematics afford for pursuing our present inquiry that I may excite in you good hope of final success. I would however observe, that the reasoning process is not different from that of any other branch of knowledge; their logic is the same as that of political economy or moral philosophy; it is addressed to the same faculties of man, and does not require any peculiar formation or deformity of human nature, as some seem to think; but there is required, and in a great degree, that attention of mind which is in some part necessary for the acquisition of all knowledge, and in this branch is indispensably necessary. This must be given in its fullest intensity; this is the excellency which Sir Isaac Newton claimed for himself, and thus placed his superiority on moral rather than on intellectual grounds: the other elements especially characteristic of a mathematical mind are quickness in perceiving logical sequence, love of order, methodical arrangement and harmony, distinctness of conception. The language of mathematics is to a certain extent peculiarly its own; its symbols are certainly its own; but these may generally, if it is desirable, be translated into ordinary language; and its language is peculiar, because the subjects of which it treats are peculiar. Now mathematics include three normal sciences; (1) science of number, (2) science of space, (3) science of motion; and under one or other of these all sciences which are treated mathematically may be ranged; or the several parts of any one may come under different normal sciences: thus, formal or geometrical optics is an application of geometry; physical optics of the science of motion; plane astronomy is geometrical, physical astronomy is mechanical. The division of mathematics into pure and mixed is arbitrary and useless, because it leads to no practical result; and therefore I do not care to retain it. I may however observe that the first two sciences, those viz. of number and space, are commonly included under the term pure mathematics, and that the last one and its subordinates are called mixed; the reason being that the subject-matter of the last has been thought to be terrestrial, or, at all events, cosmical matter; and that therefore the science involves considerations of the properties of this matter, and which must be discovered by examination and analysis, and that these processes are extraneous to pure motion; whereas the other sciences consider subjects only which are proper to them, and therefore they are called *pure*.

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* See Outlines of Astronomy, 4th edition, page 5. Longman and Co., London, 1851.

7.] The science of number, or, as the French call it, *le calcul*, has for its subject-matter number in its pure and abstract form; number, that is, as an abstract quantuplicity; not this or that *thing* taken so many times, but the *times* which it is taken; it does not treat therefore of concrete things; and it is important to observe this property of the science, because the truths of number are for this reason so generally, almost universally, applicable; time, space, pressure, weight, velocity, quantity of light, of heat, of electrical action, may be all measured by it; and so long as the conditions imposed by the numerical science are observed, the truths of number have their counterpart in the applied science. The science also includes number in its twofold division of discontinuous and continuous number; the former of which is the subject of arithmetic and algebra, and the latter of infinitesimal calculus; these being distinguished by a difference of species of subject-matter, and not of process. It is most important to observe that the numerical symbols represent abstract quantuplicities, and that the results are true, because they are correct developments of the idea of number, and are independent of the concrete matter to which they are applied. Yet they may be applied, and by the following process: the numerical proposition is operated on by the concrete unit of the matter of the particular science; whether it be linear length, or area, or cubical content, or weight, or velocity; that is, each term of the numerical equation has the concrete unit affixed to it, and thereby itself becomes concrete, and expresses the concrete thing taken a certain number of times; thus suppose that we have a numerical equation

$$4 + 3 = 7,$$

and suppose that the operating concrete unit is an inch: then we have

$$4 \text{ times} \times \text{one inch} + 3 \text{ times} \times \text{one inch} = 7 \text{ times} \times \text{one inch};$$

an inch being matter of such a kind as to be consistent with the fundamental operations of arithmetic; that is, if one inch is added to one inch, no part of either one is absorbed into the other, but the matter is continuously additive. Similarly might the operating unit be a pound, or an unit of velocity, and in both cases the result would be true because the arithmetical equality is correct. This remark is important, because the homogeneity of our equations is hereby preserved; and if the symbol of the operating factor be retained, we have a test, and an important test, whether the results are correct or not.

8.] The second mathematical science is that of space, or, as it is usually called, geometry; the subject-matter is tridimensional space; whatever is the origin of our conception of it, whether it is experience, or whether space is a phenomenal condition of our knowing things at all, or whether it is an intuitive notion, yet at all events the subject-matter of geometry is space, abstracted from all consideration of the space which we occupy, and in which we are: and the science consists in the development of this idea of space. The axioms contain enuntiations of constituent parts and properties of it; the definitions are explanations of terms arising out of, and necessary to, the division of space which flows from the fundamental idea; thus, for instance, space is such that the whole is greater than its part; that if equal spaces are added to equal spaces, the wholes are equal; spaces are equal which occupy equal parts of space, the comparison being made on the principle of superposition. The truths of geometry may be directly deduced from the axioms and definitions by means of postulates and more complex constructions, and the science of space thus treated of is called pure geometry; as such it neither requires nor involves the properties of number; its additions and subtractions and equalities are made on the principle of superposition; thus, if an angle is added to an angle, no reference is made to any unit angle, but one concrete angle is superposed on the other; and the symbols in pure geometry are symbols of the concrete quantities and are not the subjects of arithmetical laws and operations. The old geometricians employed this process only. But Descartes, perceiving that geometrical space accords with the fundamental requirements of number, treated of its properties by means of arithmetic and algebra: in this view we may operate on any numerical equation with a concrete geometrical unit, and it becomes a geometrical proposition; and whatever numerical truths are contained in, and deducible from, the numerical equation, analogous geometrical propositions are also deducible; and therefore if the equation is transformed or operated on according to arithmetical laws, so will the transformation necessitate the correctness of the corresponding geometrical changes; the geometrical process is parallel with, and proved by, the numerical process. Thus suppose that for certain *numerical* values the following equation is true,

$$y^2 = 2ax - x^2:$$

then by operating on each term with the *linear unit*, and interpreting $x$ and $y$ according to the conventional signification of rectangular axes, we have the geometrical property of the curve of which it is the equation, viz. $(y^2)$ times the linear unit $= (2ax)$ times the linear unit $- (x^2)$ times the linear unit; $y$, $x$, and $a$ being numbers. Or otherwise suppose that we operate on the same equation with the (linear unit)$^2$, then the equation becoming arithmetically

$$y \times y = (2a - x)\, x,$$

we have

the square of the ordinate = the rectangle contained by the segments of the base.

By this process algebraical geometry has been constructed: the equations in their original forms are numerical; but as geometrical space satisfies the conditions as to quantity which the science of number requires, we operate on these numerical equations with a geometrical unit, and hereby transform them into geometrical propositions; and we can further employ all the processes of algebra for deducing and proving geometrical truths which are contained in other given geometrical propositions.

In both these sciences it will be observed that the process of inference is the same: the deduction from the fundamental ideas of number and space of the truths with which they are pregnant.

9.] The third and last of the mathematical sciences is that of motion; into the foundation, laws, and processes of which I shall enter at length in the following pages; but as my method is that of a positive deductive science, intended for didactic use, and therefore to a certain extent dogmatical, it is not necessary formally to discuss the history of the laws of motion, or the growth of the fundamental idea, and the successive steps through which it reached that perfect state so that parts of it can be expressed in definite axioms, and thus be made the major premisses of the first syllogisms from which all the other truths of the science are to be inferred. I shall not relate the logomachy of mechanics in the days of Aristotle, and the disputations of the Schoolmen who taught that rest was natural and motion was unnatural, and that some bodies fall faster than others because they are heavier; nor shall I indicate the several steps by which Galileo first obtained a clear insight into the laws of motion, and how Stevinus first proved the laws of oblique pressure by means of a continuous chain resting on two inclined planes: neither shall I detail experiments by which evidence is given to the truth of the axioms. My work, on the contrary, is to take the *idea* of motion as recognized, and its laws as acknowledged, and to deduce from them their results. To this end mathematics, and especially the science of continuous number, will be found most useful instruments of inquiry: a word or two will shew this. Matter of motion, space, time, velocity, and combinations of these, such as momentum, pressure, weight, will come under consideration. All these quantities are continuously additive and subtractive, and satisfy the requirements of the science of number: and they admit of infinite divisibility; nay, more than this, some of these are within the grasp of our minds only when they are resolved into infinitesimal elements: as, for instance, it is necessary to know the law of change of velocity of a particle moving with a varying velocity, before we can determine the actual change of velocity due to a finite time; that is, the infinitesimal increment must be known, and this is determined by the law, before we can find the finite change: in these respects then the subject-matter of our science will be found to harmonize with the laws of the science of number: and these latter may be applied.

10.] Suppose that the axiomatic laws of mechanics are deduced from the fundamental idea of motion, and that we know them: let them be translated into mathematical language and symbols, so that the propositions take the form of equations; let the concrete mechanical unit be removed, and let the equation stand as a numerical equation: to it in this state all the rules of the science of number may be applied, and whatever are the results which can be inferred by means of them, they may be translated by an operating factor into their mechanical equivalents, and these again into ordinary language. If therefore the resources which the science of number supplies become more numerous, the more fruitful is the deductive process; and hence it is that the progress of the sciences is simultaneous; whatever retards the one is also an obstacle to the progress of the other.

The course of our inquiry will therefore be the following: The idea of motion must be first described; it is the fundamental one of mechanics; and pregnant properties of it must be enuntiated; these are commonly called laws of motion, and will be found to be only two; we shall translate them into mathematical language and symbols; and by the processes of infinitesimal calculus deduce from them their results, which we shall in many cases trace in the applications of mechanics, and especially in the phenomena of gravitation, whether in the case of bodies being near to the earth and falling towards it, or in the case of the approximate motion of the planetary bodies, herein laying the dynamical foundations of physical astronomy. By this method the foundations of mechanics will be laid in breadth sufficient to include all kinds of matter; whether cosmical or of that of light, if there is an ethereal medium; and all kinds of motion, whether direct or orbital or oscillatory; the basis therefore will be wide enough to comprehend the mathematical theories of hydrodynamics, light, heat, electricity; these several sciences, as they advance towards perfection, satisfy more and more the notes of the science of motion, but the perfect state will be reached only when they wholly do so.

11.] Such is the philosophical form of the perfect and exact science of motion; and such is the philosophical course of learning it; but there are reasons why another course is more suitable to a didactic treatise. It is better to begin with what is apparently more simple and more concrete, than with an abstract verity; we are not accustomed to analyse cases of motion, but we are familiar with an effect of the same cause as that which produces motion, but which in mechanics is actually more complex; we have all of us a notion more or less exact of pressure or of weight; the tension of a string caused by a weight suspended at the end of it, or a pressure caused by a weight resting on the hand, gives us a notion more distinct than that of a body falling under the action of the earth's attraction. Now let me analyse such a pressure from a dynamical point of view: take the case of a weight resting on a table; the same force which produces the pressure on the table would cause the body to fall towards the earth, if the table were removed; but the falling effort is the same, although the table is there: the earth attracts the body, impresses velocity on it, and causes it to penetrate the table; but the material of the table is elastic, and therefore so often as the body penetrates the table and causes the particles of the table which are in contact with or are near the body to approach each other, an elastic force of recoil is called into action and causes the body to retire: thus an oscillatory motion of the body is established, which is however so slight that the motion of the body is to the senses imperceptible. It may perhaps be thought that this is an indirect and awkward form of considering such a simple case as that of a body resting on a table: perhaps it is; but it is the mode of applying the principles of the science of pure motion to the case of a body resting on a table. Although in the order of the pure science other and more simple cases of motion would be discussed before this, yet as this case of pressure is so simple, as it seems, and so common, it is for didactic purposes desirable, even if it does cost a loss of order scientifically correct, to consider first those forms of problems with which a learner is most familiar; we shall hereby take advantage of his previous knowledge, and lead him from that which is to him more simple to that which is more complex. I propose therefore to defer the pure science of motion to the second part of the treatise; and to consider at present *pressures* only, and these apart from the properties, real or virtual, of motion. The science of pressures is called statics; and in establishing the principles from which I shall begin, I shall appeal to *experience*, to what we see and observe: and whatever assumptions or hypotheses I may make, I shall refer you for proof to your own observation of such pressures and to the common sense of mankind. Let me make one other observation on the difference which exists in the views of the same effect as presented to us in a statical and a dynamical light. Suppose that a pound weight rests on your hand which is at rest; you experience a pressure which your hand bears; and if another pound be added you experience a pressure twice as great: but are you conscious of or do you think about the cause of that pressure? are you aware that it is due to the earth's attraction, and to a motion which the body would have if your hand were removed? I think that you consider it as a pressure only, and not in reference to velocity: this is, I say, the common judgment about such pressures: it does not refer them to motion; and it is to such common judgment that I shall appeal in laying the foundation of statics: it may be that I shall now and then use language appropriate to the conception of a real or virtual motion, and that I thereby elucidate difficulties; but it must be remembered that such conceptions are extraneous to statics thus considered, and are such as the subject does not of itself require.

# ANALYTICAL MECHANICS.

## PART I.

## STATICS.

## CHAPTER II.

### STATICAL PRESSURES ACTING AT THE SAME POINT.

12.] A formal definition of *matter* is not required; it is sufficient for us to conceive of it as existing in space, having volume and form, capable of receiving and of (as we shall hereafter see) transmitting pressures *. Matter is rigid or stiff when its component particles are kept in a state of relative rest by the action of attraction or cohesion, or such like molecular forces (as they are called); of which we require at present only to know, that the external pressures acting on matter are, as to intensity in comparison of these, infinitesimal.

A body is a finite portion of such matter; matter is conceived to be infinitely divisible; and an infinitesimal portion of it is called a material particle, and a material point. The *mass* of a body is the quantity of matter contained in it.

*Force* is a cause which changes or tends to change matter's state as to motion or rest. Force acts on matter at a certain point, in a certain direction, along a certain line, and with a certain intensity; and when all these circumstances are given, the force is said to be given. Mechanics is the science which considers the relations and effects of such forces.

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* M. Poisson says "La matière est tout ce qui peut affecter nos sens d'une manière quelconque." Dr. Whewell, "Body or matter is any thing extended and possessing the power of resisting the action of force." Mechanics, 5th edition, Cambridge, 1836.

*Statics* is that part of mechanics which considers the relations of forces as they produce pressure or a tendency to motion. Statical forces are therefore *pressures*, and act at a fixed point, in a certain direction, along a certain line, and with a certain intensity. Weight is the most common form of a statical pressure.

The *line of action* of a pressure is that straight line passing through the point of application of the pressure along which the pressure tends to make its point of application move; and the direction of the line *towards* which the pressure tends to make the point move is called the *direction* of the pressure.

Two pressures are equal, which acting at the same point, along the same line of action, and in opposite directions, neutralize each other.

Statical pressures are continuously additive, and, as such, satisfy the requirements of the science of number: thus, if one pound is added to one pound, the sum is two pounds; no part of either of the weights is absorbed into the other; the weight of a basket of stones is the same, whatever is the arrangement of the stones. Statical pressures also admit of continuous increase and decrease, and of infinite divisibility: they thus satisfy the requirements of the science of continuous number.

If two statical pressures, thus proved to be equal, act on a point along the same line and in the same direction, the acting pressure is twice each of the original pressures: if three pressures act similarly, the acting pressure is thrice each of the original pressures: and so on. Thus it is that pressures admit of measurement: an *unit* of pressure is chosen, and other pressures are compared with it; and are expressed as being so many times the unit-pressure. Thus pressures are expressed by numbers, being referred to a concrete unit-pressure. The unit-pressure is arbitrary, and may be a finite or an infinitesimal pressure. If pressures are expressed by numbers which are commonly called incommensurable, they possess the properties of commensurables, if they are referred to an infinitesimal unit-pressure. If the unit-pressure is changed, the numbers expressing the pressures which are referred to it are also changed in an inverse ratio. Thus a weight of six pounds is expressed by 6, if a pound is the unit-pressure; by 12, if one-half of a pound is the unit-pressure; by 3, if two pounds is the unit-pressure. It is manifest that general laws connecting the intensity, point of application, direction, and line of action of pressures must be independent of the conventional unit.

13.] Statical pressures will hereafter be expressed by symbols, such as P, Q, R, .... These, it is to be observed, are numbers, expressing the number of times which the concrete unit is taken; hence also when we meet with such symbols as $P^2$, $Q^2$, ... these are also numbers. It is plain that if P represents a concrete pressure, $P^2$ is unintelligible.

Again, pressures may be conveniently represented by geometrical straight lines. As a pressure has a definite point of application, a definite line of action, a definite direction, and is of a definite magnitude, so does a line starting from the point of application of the pressure, and coincident with the line of action, in its direction, and in length containing the same number of linear units that the pressure contains units of pressure, adequately and completely represent the pressure in all its circumstances. And this mode has the advantage not only of simplifying the enuntiation of many theorems, but also of enabling us to infer mechanical propositions from their geometrical analogues; and vice versâ. Of this process we shall hereafter have many instances.

When a material particle is acted on by many pressures simultaneously, there is generally a definite direction along which it experiences a definite pressure, or, in other words, along which it has a tendency to move. Now the one pressure which would produce on this point a pressure equal, along the same line of action, and in the same direction, is called the *resultant* of the acting or impressed pressures: and its line of action is called the line of action of the resultant: and the several impressed pressures are called *components* in reference to it. The resultant is evidently *unique*, definite as to its line of action, its direction, and its magnitude.

If the impressed pressures acting on a point are so related as to produce a resultant whose magnitude is zero, then the pressures are said to be in *equilibrium*.

Hence we infer that when many pressures act on one point, if a new pressure is introduced equal in magnitude to their resultant, and acting along the same line and in an opposite direction, it neutralizes the effects of all the others, and the system is in equilibrium.

### SECTION 1.—*The composition and resolution of many pressures acting on a material particle, the lines of action of which are in one plane.*

14.] Let O, fig. 1, be a material particle; and first let us suppose all the pressures to act along the same line OA, and in the same direction; and to give distinctness to our conceptions let us suppose all the forces to be *pulling* forces, and to act in the direction of the arrow-heads from O towards A; and let them be represented by the symbols $P_1, P_2, \dots P_n$; then, since statical pressures acting at a point along the same line, and in the same direction are continuously additive, the resultant is equal to the sum of all. Let R represent the resultant, then

$$R = P_1 + P_2 + \dots + P_n, \tag{1}$$

$$= \Sigma . P, \tag{2}$$

where P is the type-symbol of a pressure, and $\Sigma$ is the summation-symbol.

Again, suppose O to be acted on by two pressures, along the same line, and in opposite directions: let them be P and Q, of which P is the greater: let P be resolved into two parts, Q and $P-Q$; then at the point O three pressures are acting, viz. $P-Q$, Q, and Q, of which the last two act in opposite directions; therefore they neutralize each other; and, if R is the resultant, we have

$$R = P - Q. \tag{3}$$

And as a similar result is true for any number of pressures acting in either direction, and along the same line of action, the equation (2) may be extended so as to include the *algebraical* sum of the pressures acting on a point and along the same line.

Hence also we infer that a point is in equilibrium under the action of many pressures along the same line, if the sum of those acting in one direction is equal to the sum of those acting in the opposite direction; and the condition of equilibrium is

$$\Sigma . P = 0. \tag{4}$$

Also let us take another simple case: that of three equal forces* P, Q, R, see fig. 2, acting on O, all of which are in the same plane, and the lines of action of which are inclined to each other at 120°. Let the forces be represented, both in direction and in intensity, by the equal definite lines OP, OQ, OR: then the point O is in equilibrium: for by the principle of sufficient reason it cannot move out of the plane of the forces; neither can there be any resultant pressure in the plane by reason of the same principle; the point therefore is in equilibrium; and either of the pressures may be considered to be equal in magnitude to the resultant of the other two, and to act in the same line, but in an opposite direction. Hence we have the following geometrical construction of the resultant. Let P and Q be the components; then R neutralizes the effects of P and Q on O; produce RO to R′ so that OR′ is equal to OR; then the force of which OR′ is the geometrical representative neutralizes R; but the resultant of P and Q also neutralizes R: therefore the force R′ is the resultant of P and Q; and by the geometry OR′ is the diagonal of the parallelogram of which OP and OQ are the adjacent containing sides.

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* The generic word "force" is here and elsewhere used instead of the specific and more correct word "pressure," in order that the hitherto received forms of statical theorems may be preserved.

15.] The parallelogram of forces however, which is so called from the geometrical form of the result, is more general than the result of the last Article, and includes it. The problem is:

Given two pressures acting on a material point, to find the intensity, the line of action, and the direction of the resultant.

First let the meaning of the problem be clearly understood; it is required to determine the line of action, the direction, and the magnitude of a pressure which acting at a given point shall produce the same effect in all respects as two pressures acting simultaneously.

It is evident by the principle of sufficient reason that the line of action of the resultant is in the same plane with the lines of action of the components.

*And first let us take the case of two *equal* forces P and P acting at O, and with their lines of action inclined at an angle $2\theta$. It is manifest that the line of action of the resultant bisects the angle contained between the lines of action of the components; because every reason which can be alleged why it should be on one side of this line is equally valid to prove that it should be on the other: and an integral part of our conception of a resultant is that it should be unique both as to magnitude and as to line of action; hence by the principle of sufficient reason we conclude that the line of action of the resultant bisects the angle between the lines of action of the components.

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* The following proof of the parallelogram of forces is due to M. Poisson, and commonly bears his name. A discussion, more or less complete, on 45 other proofs will be found in "Præcipuorum inde a Neutono conatuum, compositionem virium demonstrandi, recensio. Auctore Carolo Jacobi. Gottingæ, MDCCCXVIII."

To determine the intensity of the resultant. Let OP, $\text{OP}_{,}$ represent (see fig. 3) the two equal forces acting at O; let the angle $\text{POP}_{,} = 2\theta$; let OR be the line of action of the resultant R, so that $\text{POR} = \text{P}_{,}\text{OR} = \theta$. Now the intensity of R can depend on only P and $\theta$; so that if $f$ symbolizes a function which is to be determined,

$$\text{R} = f(\text{P}, \theta); \tag{5}$$

in this equation R and P are numbers depending on the arbitrarily chosen unit of pressure, and varying of course as the unit varies; but the law of relation between R, P, and $\theta$ cannot depend on the choice of this unit; therefore (5) must be such that the unit may be divided out, whatever be its magnitude; and this can only be the case when the equation is of the form

$$\text{R} = \text{P}f(\theta). \tag{6}$$

It remains for us to determine the form of $f$.

Suppose P to be the resultant of two equal forces Q and $\text{Q}_{,}$ acting at equal angles on the sides of P's line of action; and let $\text{QOP} = \text{Q}_{,}\text{OP} = \phi$; therefore by (6)

$$\text{P} = \text{Q}f(\phi); \tag{7}$$

similarly let $\text{P}_{,}$ be the resultant of two forces Q and $\text{Q}_{,}$, equal to each other and to the former Qs, acting at equal angles $\phi$ on the sides of $\text{P}_{,}$'s line of action; so that

$$\text{P}_{,} = \text{Q}f(\phi); \tag{8}$$

therefore from (6)

$$\text{R} = \text{Q}f(\theta)f(\phi). \tag{9}$$

Now R is the resultant of P and $\text{P}_{,}$; and therefore, as P and $\text{P}_{,}$ are the resultants of Q, Q, $\text{Q}_{,}$, and $\text{Q}_{,}$, R is the resultant of these also; let them be taken in pairs, so that R is the resultant of Q, Q and of $\text{Q}_{,}$, $\text{Q}_{,}$; but by (6)

$$\left.\begin{aligned} \text{the resultant of Q, Q} &= \text{Q}f(\theta+\phi) \\ \text{- - - - - } \text{Q}_{,}, \text{Q}_{,} &= \text{Q}f(\theta-\phi) \end{aligned}\right\}; \tag{10}$$

& acts along R
& " " R

therefore substituting in (9) & remembering [illegible]

$$\text{Q}f(\theta+\phi)+\text{Q}f(\theta-\phi) = \text{Q}f(\theta)f(\phi),$$

$$f(\theta+\phi)+f(\theta-\phi) = f(\theta)f(\phi); \tag{11}$$

that is, the form of $f$ is such as to satisfy the functional equation (11).

Expanding the left-hand member of (11) by Taylor's series, we have

$$2\left\{f(\theta)+f''(\theta)\frac{\phi^2}{1.2}+f''''(\theta)\frac{\phi^4}{1.2.3.4}+\dots\right\}=f(\theta)f(\phi)$$

$$\therefore\quad f(\phi)=2\left\{1+\frac{f''(\theta)}{f(\theta)}\frac{\phi^2}{1.2}+\frac{f''''(\theta)}{f(\theta)}\frac{\phi^4}{1.2.3.4}+\dots\right\}; \qquad (12)$$

but as no relation exists between $\theta$ and $\phi$, $\theta$ is constant in reference to $\phi$: therefore in (11), which is the expansion of $f(\phi)$, we may put, if $a$ is constant,

$$\frac{f''(\theta)}{f(\theta)}=-a^2; \qquad \therefore\quad \frac{f''''(\theta)}{f(\theta)}=a^4;$$

and so for the other terms;

$$\begin{aligned}\therefore\quad f(\phi)&=2\left\{1-\frac{a^2\phi^2}{1.2}+\frac{a^4\phi^4}{1.2.3.4}-\dots\right\}\\&=2\cos a\phi;\\ \therefore\quad f(\theta)&=2\cos a\theta;\end{aligned}$$

and therefore (11) becomes

$$\cos a(\theta+\phi)+\cos a(\theta-\phi)=2\cos a\theta\cos a\phi; \qquad (13)$$

an equation which is known to be true: and thus (6) becomes

$$\mathrm{R}=2\mathrm{P}\cos a\theta; \qquad (14)$$

$a$ is still undetermined; it must however be some uneven number, because $\mathrm{R}=0$, when $\theta=90°$, that is, when the two equal forces act in the same line and in opposite directions: and the uneven number can be none other than unity, because if it were 3 or 5, or ... or $2n+1$, R would vanish when $\theta=\frac{\pi}{6}, =\frac{\pi}{10}, = \dots = \frac{\pi}{4n+2}$, and this would be absurd: therefore the functional relation between R, P, and $\theta$ is

$$\mathrm{R}=2\mathrm{P}\cos\theta. \qquad (15)$$

16.] Let us geometrically interpret this; let OP and $\mathrm{OP}_{,}$, fig. 4, represent the components in direction and in magnitude; let OR bisect the angle $\mathrm{P}_{,}\mathrm{OP}$; from P draw PD perpendicular to OR, and produce OD to R, so that $\mathrm{DR}=\mathrm{OD}$; then $\mathrm{OR}=2\mathrm{OP}\cos\theta$, and therefore OR by its length and direction represents the resultant of P and $\mathrm{P}_{,}$; join PR, $\mathrm{RP}_{,}$: then $\mathrm{P}_{,}\mathrm{OPR}$ is manifestly a rhombus, of which OP, $\mathrm{OP}_{,}$ are two adjoining sides, and OR is the diagonal. If therefore two adjoining sides of a rhombus represent two impressed pressures acting on O, the diagonal of the rhombus abutting on O represents the resultant both as to line of action and intensity; hence also, since

$$\text{OR}^2 = \text{OP}^2 + \text{OP}_{,}^{\,2} + 2\,\text{OP}.\text{OP}_{,}\cos \text{P}_{,}\text{OP},$$

$$\text{R}^2 = 2\,\text{P}^2 + 2\,\text{P}^2 \cos 2\theta. \tag{16}$$

Hence also conversely we infer that a pressure acting on a particle O may be equivalently replaced by two equal pressures acting at equal angles on each side of its line of action if, R being the pressure to be replaced, P being one of the equal resolved parts of it, and $\theta$ being the angle between the lines of action of R and P,

$$\text{P} = \frac{\text{R}}{2}\sec\theta; \tag{17}$$

P therefore cannot be less than $\frac{\text{R}}{2}$; and increases as $\theta$ increases, and lastly becomes infinite when $\theta = 90°$: hence we infer that the effect of R on O cannot be produced by any force whose line of action is perpendicular to that of R; and therefore that two pressures whose lines of action are perpendicular to each other do not affect each other's effects.

17.] Next let us consider the case of two unequal pressures P and Q acting on a point O, fig. 5, and with lines of action perpendicular to each other. Let the components be geometrically represented by the lines OP and OQ; complete the rectangle OPRQ, and draw the diagonal OR; let the angle $\text{ROP} = a$; then the force P may be resolved into two forces P′ and P′ acting at equal angles $a$ on either side of OP, so that by reason of (17)

$$\text{P}' = \frac{\text{P}}{2}\sec a, \tag{18}$$

and therefore P′ is geometrically and equivalently represented by half of the diagonal OR. Again, let Q be resolved into two equal pressures Q′ and Q′ acting at equal angles $90° - a$ on each side of OQ, so that by reason of (17)

$$\text{Q}' = \frac{\text{Q}}{2}\operatorname{cosec} a, \tag{19}$$

and therefore Q′ is geometrically and equivalently represented by half of the diagonal of the rectangle. Hence we have two pressures, each of which is represented by half of OR, acting along OR and in the same direction, and of which therefore OR is the resultant both as to line of action and as to intensity;

and also two pressures Q′ and P′ acting at O in the same line and in opposite directions: and as these are equal, both being represented by half of OR, they neutralize each other; and therefore the resultant of the two impressed pressures is represented by the diagonal of the rectangle of which the containing sides are the representatives of the components. Hence if R is the resultant

$$R^2 = P^2 + Q^2; \tag{20}$$

and from (18) and (19)

$$R = P \sec a = Q \operatorname{cosec} a. \tag{21}$$

Hence also conversely, fig. 6; suppose a pressure P to act on a point O, and to be represented in intensity and direction by the line OP; it may be resolved into two pressures along two lines originating at O and perpendicular to each other; so that if X and Y are the resolved pressures, and if the angle between the lines of action of P and X is $\theta$, then by (21)

$$\left.\begin{aligned} X &= P\cos\theta \\ Y &= P\sin\theta \end{aligned}\right\} \tag{22}$$

$$P^2 = X^2 + Y^2. \tag{23}$$

Hence the resolved part of a pressure along any line is equal to the product of the pressure and the cosine of the angle between the given line and that of the given pressure.

18.] Let us next consider the case of two unequal forces P and Q acting on a point O, and with their lines of action inclined to each other at an angle $\gamma$; see fig. 7; let OP and OQ be the geometrical representatives of the pressures, and let QOP $= \gamma$; complete the parallelogram QOPR, and draw the diagonal OR. Now resolve P into two pressures P′ and P″ in directions at right-angles to each other, along OR and perpendicularly to OR, and suppose ROP $= \theta$; then by the last Article

$$P' = P\cos\theta, \qquad P'' = P\sin\theta, \tag{24}$$

so that by reason of the construction of the figure, OP′ is the geometrical representative of P′ and OP″ of P″. Again, resolve Q into two pressures Q′ and Q″, in directions along and at right angles to OR; then

$$Q' = Q\cos(\gamma-\theta), \qquad Q'' = Q\sin(\gamma-\theta), \tag{25}$$

and therefore OQ′ is the geometrical representative of Q′, and OQ″ of Q″. Now the two pressures P″ and Q″ are manifestly equal, and act in the same line but in opposite directions; they therefore neutralize each other; and there remain P′ and Q′ acting along OR in the same direction, and therefore the resultant is equal to the sum of them, and is geometrically represented by OP′ + OQ′, that is, by OR, which is the diagonal of the parallelogram of which OP and OQ are the containing sides; and since

$$\begin{aligned} \text{OR}^2 &= \text{OP}^2 + \text{PR}^2 - 2.\text{OP}.\text{PR} \cos \text{OPR} \\ &= \text{OP}^2 + \text{OQ}^2 + 2.\text{OP}.\text{OQ} \cos \text{POQ}; \qquad (26) \end{aligned}$$

Therefore replacing the geometrical lines by their statical proportionals

$$\text{R}^2 = \text{P}^2 + \text{Q}^2 + 2\,\text{PQ} \cos \gamma. \qquad (27)$$

Of course the former two cases are particular instances of this: for if

$$\gamma = 90^\circ, \qquad \therefore \quad \text{R}^2 = \text{P}^2 + \text{Q}^2;$$
$$\text{P} = \text{Q}, \qquad \text{R} = 2\,\text{P} \cos \frac{\gamma}{2}.$$

Hence in all cases we may enuntiate the proposition in the following form:

If two pressures acting on a point are represented by two sides of a parallelogram meeting at the point, the resultant is represented as to line of action, direction, and intensity by the diagonal of the parallelogram which abuts at the point.

This theorem is known by the name of the parallelogram of forces.

19.] Hence, conversely, if any pressure R acts on a point O, it may be resolved into any two pressures P and Q, whose lines of action are inclined at an angle $\gamma$, provided that the condition (27) is satisfied. And from (24) and (25), if $\theta$ is the angle between the directions of R and P,

$$\text{R} = \text{P} \cos \theta + \text{Q} \cos (\gamma - \theta).$$

Hence, fig. 8, if a pressure R acts at O in the line OR′, but in an opposite direction, and equal, to the resultant of P and Q, the point O is in equilibrium: and either force will be equal to the resultant of the other two; and therefore if QOR $= \alpha$, ROP $= \beta$, POQ $= \gamma$,

$$\left.\begin{aligned} \text{P}^2 &= \text{Q}^2 + 2\,\text{QR} \cos \alpha + \text{R}^2 \\ \text{Q}^2 &= \text{R}^2 + 2\,\text{RP} \cos \beta + \text{P}^2 \\ \text{R}^2 &= \text{P}^2 + 2\,\text{PQ} \cos \gamma + \text{Q}^2 \end{aligned}\right\}. \qquad (28)$$

Also since the three equilibrating forces P, Q, R are proportional to the three lines OP, OQ, OR, or to OP, PR′, R′O; and since the three sides of a triangle are proportional to the sines of the opposite angles, therefore

$$\frac{\text{P}}{\sin \text{OR}'\text{P}} = \frac{\text{Q}}{\sin \text{R}'\text{OP}} = \frac{\text{R}}{\sin \text{OPR}'}, \tag{29}$$

$$\text{or,} \quad \frac{\text{P}}{\sin \alpha} = \frac{\text{Q}}{\sin \beta} = \frac{\text{R}}{\sin \gamma}, \tag{30}$$

that is, if three forces acting on a point are in equilibrium, each is proportional to the sine of the angle contained between the lines of action of the other two.

From (29) we infer that three pressures acting on a point are in equilibrium, if they are proportional to the three sides of any triangle whose sides are parallel to the lines of action of the pressures, and if their directions are those of a point traversing the perimeter of the triangle. This theorem is known by the name of the triangle of forces.

Hence if from any point in R's line of action perpendiculars are drawn to the lines of action of P and Q, of which let the lengths be $p$ and $q$, then

$$\text{P}p = \text{Q}q. \tag{31}$$

Therefore if $\text{P}_1$ and $\text{P}_2$ are pressures acting at a given point along lines of action, the equations to which are

$$\left.\begin{array}{l} x \cos \alpha_1 + y \sin \alpha_1 - \delta_1 = 0 \\ x \cos \alpha_2 + y \sin \alpha_2 - \delta_2 = 0 \end{array}\right\}, \tag{32}$$

which we may represent by the abridged notation $a_1 = 0$, and $a_2 = 0$; then the equation to the line of action of the resultant is

$$\text{P}_1 a_1 + \text{P}_2 a_2 = 0. \tag{33}$$

20.] Let us next consider the case of many pressures acting on a given point, the lines of action of all of which are in one plane.

Let O be the point on which all the pressures act: and through it let two lines, as coordinate axes, be drawn perpendicular to each other, and in the plane in which the pressures act.

Let the pressures be $\text{P}_1, \text{P}_2, \ldots \text{P}_n$, of which let P be the type-pressure: and let the angles between the axis of $x$ and their lines of action severally be $\alpha_1, \alpha_2, \ldots \alpha_n$, of which let $\alpha$ be the type-angle; and let the several pressures be resolved along the axes of $x$ and $y$: then by equations (22), Art. 17, the resolved parts along the axis of $x$ severally are $\text{P}_1 \cos \alpha_1$, $\text{P}_2 \cos \alpha_2, \ldots$ $\text{P}_n \cos \alpha_n$; and those along the axis of $y$ are $\text{P}_1 \sin \alpha_1$, $\text{P}_2 \sin \alpha_2, \ldots$ $\text{P}_n \sin \alpha_n$; and therefore if X and Y symbolize the pressures along the axes of $x$ and $y$ respectively,

$$\left.\begin{aligned} \text{X} &= \text{P}_1 \cos a_1 + \text{P}_2 \cos a_2 + \ldots + \text{P}_n \cos a_n \\ &= \Sigma . \text{P} \cos a \end{aligned}\right\}, \tag{34}$$

$$\left.\begin{aligned} \text{Y} &= \text{P}_1 \sin a_1 + \text{P}_2 \sin a_2 + \ldots + \text{P}_n \sin a_n \\ &= \Sigma . \text{P} \sin a \end{aligned}\right\}. \tag{35}$$

Let R be the resultant of all the forces acting on O, and $\theta$ the angle which its line of action makes with the axis of $x$; then as R produces on O the same effect as to intensity, line of action, and direction which all the impressed pressures taken in combination do, so are the resolved parts of R along the axis equal severally to X and Y:

$$\therefore \quad \left.\begin{aligned} \text{R} \cos \theta &= \text{X} = \Sigma . \text{P} \cos a \\ \text{R} \sin \theta &= \text{Y} = \Sigma . \text{P} \sin a \end{aligned}\right\}, \tag{36}$$

$$\therefore \quad \text{R}^2 = \text{X}^2 + \text{Y}^2, \tag{37}$$

$$\tan \theta = \frac{\text{Y}}{\text{X}}, \tag{38}$$

$$\frac{\sin \theta}{\text{Y}} = \frac{\cos \theta}{\text{X}} = \frac{1}{\text{R}}; \tag{39}$$

and hereby may the magnitude, and direction of the line of action, of the resultant of many pressures acting in one plane on a given point be determined.

If the acting pressures are so related that the particle is in equilibrium, then the resultant vanishes, and therefore

$$\text{R}^2 = \text{X}^2 + \text{Y}^2 = 0;$$

$$\therefore \quad \text{X} = 0, \qquad \text{Y} = 0,$$

$$\Sigma . \text{P} \cos a = 0, \qquad \Sigma . \text{P} \sin a = 0. \tag{40}$$

And as the conditions of equilibrium must be independent of the particular system of coordinate axes, we infer that, if many forces acting on a particle in one plane are in equilibrium, the sum of the resolved parts of the forces along every straight line is equal to zero.

21.] Ex. 1. Four equal forces whose directions are inclined to the axis of $x$ at angles of 15°, 75°, 135° and 225° act on a point: determine the magnitude and direction of their resultant.

Let each pressure be equal to P; then

$$\text{X} = \text{P} \cos 15° + \text{P} \cos 75° + \text{P} \cos 135° + \text{P} \cos 225°$$

$$= \text{P} \frac{3^{\frac{1}{2}} - 2}{2^{\frac{1}{2}}},$$

$$\text{Y} = \text{P} \sin 15^\circ + \text{P} \sin 75^\circ + \text{P} \sin 135^\circ + \text{P} \sin 225^\circ$$
$$= \text{P}\left(\frac{3}{2}\right)^{\frac{1}{2}},$$
$$\therefore \quad \text{R} = \text{P}\,(5 - 3^{\frac{1}{2}})^{\frac{1}{2}}, \qquad \tan\theta = \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}} - 2}.$$

Ex. 2. Three forces act perpendicularly to the sides of a triangle at their middle points, and are proportional to the sides; it is required to prove that they are in equilibrium.

Let ABC, fig. 9, be the triangle, and let the forces be P, Q, R, and act in the directions indicated by the arrow-heads; their lines of action meet at the point O; let $\text{QOR} = \alpha$, $\text{ROP} = \beta$, $\text{POQ} = \gamma$; $\alpha, \beta, \gamma$ being manifestly the supplements of A, B, C; then by the data

$$\frac{\text{P}}{a} = \frac{\text{Q}}{b} = \frac{\text{R}}{c} = k \text{ (say)}; \tag{41}$$

and since the sides are proportional to the sines of the opposite angles,

$$\therefore \quad \frac{\text{P}}{\sin \text{A}} = \frac{\text{Q}}{\sin \text{B}} = \frac{\text{R}}{\sin \text{C}},$$

$$\therefore \quad \frac{\text{P}}{\sin \alpha} = \frac{\text{Q}}{\sin \beta} = \frac{\text{R}}{\sin \gamma};$$

and therefore by (30) P, Q, R are in equilibrium.

Or thus resolving along BC,

$$\begin{aligned} \text{Forces along BC} &= \text{Q} \sin \text{C} - \text{R} \sin \text{B} \\ &= k\,\{b \sin \text{C} - c \sin \text{B}\}, \text{ by (41)} \\ &= 0; \end{aligned}$$

and similarly will the sum of the resolved parts of the forces along any other line vanish. And therefore the system is in equilibrium.

22.] In the application of the preceding principles, statical forces often arise from (1) determinate tension of strings, (2) reacting pressures. It is worth while to say a few words on each of these cases.

Suppose in fig. 1 OA to be a string, fastened at O, and pulled at its other extremity with a certain force = P; then it is (experimentally) plain that O is pulled with a force equal to that exerted on the string at A, and that the tension of the string is the same throughout; the line of the string of course expresses the *line* in which the pressure acts on O, but the length of it is not a measure of the intensity of the pull, although a length may be taken along it which shall be proportional to that intensity. One or two examples, in which such pressures are involved, are subjoined.

Ex. 1. A and B, fig. 10, are two fixed points in a horizontal line; at A is fastened a string of length $c$ with a smooth ring at its other extremity C, through which passes another string fastened at one end at B; the other end of which is attached to a given weight W; it is required to determine the position of C.

Let $AB = 2a$, $AC = c$, $CAB = \theta$, $ABC = \phi$. Let the tension of the string $AC = T$; which is undetermined. Now as the ring at C is smooth, the tension of WCB is the same throughout, and is of course equal to the weight W; and therefore C is kept in equilibrium by three forces, W, W, and T; in the application of equations (40) we shall resolve the forces horizontally and vertically; and equate those that act towards the right-hand to those acting towards the left; and those acting upwards to those acting downwards.

Horizontal forces; $\quad W \cos\phi = T \cos\theta.$

Vertical forces; $\quad W \sin\phi + T \sin\theta = W.$

Therefore eliminating T,

$$\cos\theta = \sin(\theta+\phi)$$

$$2\theta + \phi = 90°. \tag{42}$$

Also from the geometry

$$\frac{\sin(\theta+\phi)}{\sin\phi} = \frac{c}{2a}; \tag{43}$$

and from (42) and (43) $\theta$ and $\phi$ may be found: and thence T may be determined; and thus all the circumstances of the problem are known.

Ex. 2. A and B are two points in a horizontal line; a string fastened at A, fig. 11, passes over a small pully at B, and supports at its other end a weight W; a small and smooth heavy ring of weight W′ slides on the string between A and B; determine the position in which the string rests.

Let C be the point at which the heavy ring rests: as the pully is smooth, and has no friction, and as the ring is also smooth, the tension of the string is the same throughout and is equal to the weight of W; hence the point C is kept in equilibrium by three forces, W along CA, W along CB, and W′ which acts vertically downwards: let CAB $= \theta$, CBA $= \phi$; therefore, taking horizontal and vertical forces, we have

$$\text{Horizontal forces;} \qquad \text{W} \cos\theta = \text{W} \cos\phi.$$

$$\text{Vertical forces;} \qquad \text{W} \sin\theta + \text{W} \sin\phi = \text{W}',$$

$$\therefore \quad \theta = \phi = \sin^{-1}\frac{\text{W}'}{2\text{W}}.$$

23.] Again, suppose the material point, on which the statical pressures act, to be on a smooth plane surface, which is capable of bearing the resultant of the component pressures which acts along the normal and in a direction towards the plane: but by reason of its smoothness does not offer any resistance to motion in the direction of its surface; then, since the actual normal pressure of such a plane is equal, and in direction opposite, to that impressed on it by the component pressures, this normal reaction of the plane is one of the forces by which such a material particle is kept in equilibrium, and, as such, will enter into the equations of equilibrium.

Ex. 1. A material particle of weight W is kept at rest on a given inclined plane by a force P acting at a given angle to the plane; determine the pressure on the plane, and the magnitude of P.

Let fig. 12 be a vertical section of the system; AC the inclined plane; CAB $= \alpha$, PQC $= \beta$, R $=$ the reaction of the plane against the particle Q: then, as the lines along which forces may be resolved are arbitrary, let us resolve along, and perpendicularly to, the plane.

$$\text{Forces along the plane,} \qquad \text{P} \cos\beta = \text{W} \sin\alpha,$$

$$\text{Forces perpendicular to the plane,} \qquad \text{R} + \text{P} \sin\beta = \text{W} \cos\alpha;$$

$$\therefore \quad \text{P} = \text{W}\frac{\sin\alpha}{\cos\beta}, \qquad \text{R} = \text{W}\frac{\cos(\alpha+\beta)}{\cos\beta}.$$

The force P therefore acts to the greatest advantage when $\beta = 0$.

Ex. 2. Two forces P and Q acting respectively parallel to the base and length of an inclined plane will each singly sustain on it a particle of weight W; to determine the weight of W.

Let $\alpha$ be the inclination of the plane to the horizon; then in each case resolving along the plane, so that the normal pressures may not enter into the equations,

$$\mathrm{P}\cos\alpha = \mathrm{W}\sin\alpha, \qquad \mathrm{Q} = \mathrm{W}\sin\alpha,$$

$$\therefore \quad \mathrm{W} = \frac{\mathrm{PQ}}{(\mathrm{P}^2-\mathrm{Q}^2)^{\frac{1}{2}}}.$$

24.] The resultant of forces acting on a point in one plane must be, as to line of action and intensity, independent of the particular origin and the particular system of coordinates; and we may in the following manner deduce this property from the preceding results:

$$\left.\begin{aligned} \mathrm{X} &= \Sigma.\mathrm{P}\cos\alpha = \mathrm{P}_1\cos\alpha_1 + \mathrm{P}_2\cos\alpha_2 + \ldots + \mathrm{P}_n\cos\alpha_n \\ \mathrm{Y} &= \Sigma.\mathrm{P}\sin\alpha = \mathrm{P}_1\sin\alpha_1 + \mathrm{P}_2\sin\alpha_2 + \ldots + \mathrm{P}_n\sin\alpha_n \end{aligned}\right\}, \tag{44}$$

$$\begin{aligned} \therefore \quad \mathrm{R}^2 &= \mathrm{X}^2+\mathrm{Y}^2 \\ &= \mathrm{P}_1^2+\mathrm{P}_2^2+\ldots+\mathrm{P}_n^2 \\ &\qquad + 2\,\{\mathrm{P}_1\mathrm{P}_2\cos(\alpha_1-\alpha_2)+\mathrm{P}_1\mathrm{P}_3\cos(\alpha_1-\alpha_3)+\ldots \\ &\qquad\qquad \ldots + \mathrm{P}_{n-1}\mathrm{P}_n\cos(\alpha_{n-1}-\alpha_n)\} \qquad (45) \\ &= \Sigma.\mathrm{P}^2 + 2\Sigma.\mathrm{P}\mathrm{P}'\cos(\alpha-\alpha'), \qquad (46) \end{aligned}$$

where $\mathrm{P}\mathrm{P}'$ are the symbols for any two of the forces, and $\alpha-\alpha'$ is the angle contained between their lines of action; and the sign of summation prefixed to $\mathrm{P}\mathrm{P}'\cos(\alpha-\alpha')$ indicates the sum of the products corresponding to the $n$ forces taken two and two together; and therefore (46) is independent of the system of coordinate axes. Equation (27) is a particular case of (46).

25.] We have also the following relation between the several components and their lines of action, and any point in the line of action of the resultant.

Let the equations to the lines of action of the components be

$$\left.\begin{aligned} x\cos\alpha_1 + y\sin\alpha_1 &= 0 = \alpha_1 \\ x\cos\alpha_2 + y\sin\alpha_2 &= 0 = \alpha_2 \\ \cdots\cdots&\cdots\cdots \\ x\cos\alpha_n + y\sin\alpha_n &= 0 = \alpha_n \end{aligned}\right\}, \tag{47}$$

the point at which they act being the origin, $\alpha$ being the angle between the axis of $x$ and the normal to the line of action; and the $\alpha$ on the right-hand side of the equation being the length of the perpendicular from the point $(x, y)$ to the line.

Now if the components are $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$, and the resultant is $\mathrm{R}$, and $\alpha$ is the angle between the normal to $\mathrm{R}$'s direction and the axis of $x$, then the equation to $\mathrm{R}$'s line of action is

$$x\cos\alpha + y\sin\alpha = 0, \tag{48}$$

$$\therefore \quad x\,\mathrm{R}\cos\alpha + y\,\mathrm{R}\sin\alpha = 0;$$

but $\quad \text{R} \cos a = \Sigma . \text{P} \cos a, \qquad \text{R} \sin a = \Sigma . \text{P} \sin a;$

$$\therefore \quad x \Sigma . \text{P} \cos a + y \Sigma . \text{P} \sin a = 0,$$

$$\therefore \quad \text{P}_1 \{x \cos a_1 + y \sin a_1\} + \text{P}_2 \{x \cos a_2 + y \sin a_2\} + \dots$$
$$\dots + \text{P}_n \{x \cos a_n + y \sin a_n\} = 0, \tag{49}$$

$$\text{P}_1 a_1 + \text{P}_2 a_2 + \dots + \text{P}_n a_n = 0, \tag{50}$$

where $a_1, a_2, \dots a_n$ are the perpendiculars from $(x, y)$, any point in the line of action of R, on the lines of action of the components; therefore,

If from any point in the line of action of the resultant perpendiculars are drawn to the lines of action of the components, the sum of the products of each component and its corresponding perpendicular is equal to zero.

26.] There is also another geometrical interpretation of the conditions of equilibrium in equations (40), which I must not omit to notice.

It is a well-known property of a closed polygon that the sum of the projections of its sides on any given straight line is zero; the projections of the sides being affected with positive or negative signs according as the angles made by them with the given straight line are acute or obtuse, and care being taken to estimate the angles between the given line and the sides of the polygon which are turned all towards the inside or all towards the outside of the figure. Hence, if $l_1, l_2, \dots l_n$ are the lengths of the sides, and $a_1, a_2, \dots a_n$ are the angles between them and the given straight line,

$$\Sigma . l \cos a = 0. \tag{51}$$

Now if $n$ forces act on a point, the condition of equilibrium is

$$\Sigma . \text{P} \cos a = 0. \tag{52}$$

Hence if $n$ forces, having their lines of action parallel to the successive sides of a closed polygon, their directions the same as that of a point traversing the sides of the polygon, and their intensities represented by the lengths of those sides, act on a point, (51) assumes the analogous mechanical form (52), and the forces are in equilibrium: hence conversely, if many pressures act on a point in one plane and are in equilibrium, their directions are parallel to the sides of a closed polygon, the sides being proportional to the intensities of the forces. This proposition is known by the name of the polygon of forces, and the triangle of forces proved in Article 19 is a particular case of it.

## SECTION 2.—*Composition and resolution of pressures acting in any directions on a material particle.*

27.] Here and elsewhere in considering the effects of forces acting in space we shall refer them to a system of rectangular coordinates; because the results are not more general, and are much more complicated, when they are referred to a system of oblique axes. And let me in the first place take the case of three forces X, Y, Z acting at the origin O, see fig. 13, and along the coordinate axes. Let the resultant of X and Y, which are at right-angles to each other in the plane of $xy$, be $R'$; then, by Article 17,

$$R'^2 = X^2 + Y^2.$$

Again, in like manner compounding $R'$ and Z, which are at right-angles to each other and in the same plane, and of which let the resultant be R,

$$\begin{aligned} R^2 &= R'^2 + Z^2 \\ &= X^2 + Y^2 + Z^2, \end{aligned} \tag{53}$$

and R is the resultant of the three forces. To determine the direction of its line of action, let its direction-angles be $a\,b\,c$; then, by equation (22),

$$X = R \cos a, \qquad Y = R \cos b, \qquad Z = R \cos c. \tag{54}$$

Hence, conversely, any force P, acting at O, the direction-angles of whose line of action are $\alpha\,\beta\,\gamma$, may be resolved into three forces X, Y, Z acting along the coordinate axes, such that

$$X = P \cos \alpha, \qquad Y = P \cos \beta, \qquad Z = P \cos \gamma. \tag{55}$$

28.] Next let us consider the case of many forces acting in any directions at the point O.

Let the forces be $P_1, P_2, \dots P_n$; and let the direction-angles of their lines of action be $\alpha_1\,\beta_1\,\gamma_1$, $\alpha_2\,\beta_2\,\gamma_2, \dots \alpha_n\,\beta_n\,\gamma_n$; let these be resolved severally along the coordinate axes, and let X, Y, Z be the sums of the resolved parts along the axes respectively of $x$, $y$, and $z$; then

$$\left.\begin{aligned} X &= P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \dots + P_n \cos \alpha_n \\ &= \Sigma . P \cos \alpha \\ \text{Similarly}\quad Y &= \Sigma . P \cos \beta \\ Z &= \Sigma . P \cos \gamma \end{aligned}\right\}. \tag{56}$$

Let R be the resultant of all the impressed pressures; and let the direction-angles of its line of action be $a, b, c$; then

$$\text{R} \cos a = \text{X}, \quad \text{R} \cos b = \text{Y}, \quad \text{R} \cos c = \text{Z}; \tag{57}$$

and squaring and adding,

$$\text{R}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2, \tag{58}$$

$$\cos a = \frac{\text{X}}{\text{R}}, \quad \cos b = \frac{\text{Y}}{\text{R}}, \quad \cos c = \frac{\text{Z}}{\text{R}}; \tag{59}$$

and therefore the equations to the line of action of the resultant of many forces acting on a particle at the origin are

$$\frac{x}{\Sigma.\text{P} \cos a} = \frac{y}{\Sigma.\text{P} \cos \beta} = \frac{z}{\Sigma.\text{P} \cos \gamma}. \tag{60}$$

Also from (58),

$$\begin{aligned} \text{R} &= \text{X}\frac{\text{X}}{\text{R}} + \text{Y}\frac{\text{Y}}{\text{R}} + \text{Z}\frac{\text{Z}}{\text{R}} \\ &= \text{X} \cos a + \text{Y} \cos b + \text{Z} \cos c, \end{aligned}$$

that is, R is equal to the sum of the forces along the coordinate axes resolved along the line of action of R.

If the coordinates to the point on which all the forces act are $x', y', z'$, so that the equations to the lines of action of the components are

$$\left.\begin{aligned} \frac{x-x'}{\cos a_1} &= \frac{y-y'}{\cos \beta_1} = \frac{z-z'}{\cos \gamma_1} \\ \frac{x-x'}{\cos a_2} &= \frac{y-y'}{\cos \beta_2} = \frac{z-z'}{\cos \gamma_2} \\ &\cdots\cdots\cdots \\ \frac{x-x'}{\cos a_n} &= \frac{y-y'}{\cos \beta_n} = \frac{z-z'}{\cos \gamma_n} \end{aligned}\right\}, \tag{61}$$

then the equations to the line of action of the resultant are

$$\frac{x-x'}{\Sigma.\text{P} \cos a} = \frac{y-y'}{\Sigma.\text{P} \cos \beta} = \frac{z-z'}{\Sigma.\text{P} \cos \gamma}. \tag{62}$$

Now from the point on which all the forces act let straight lines be drawn, which are in length and direction geometrical representatives of the forces: and let the extremities of these lines be $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, ... $(x_n, y_n, z_n)$, and let their lengths be $s_1, s_2, \ldots s_n$; then

$$\left.\begin{aligned} x'-x_1 &= s_1 \cos a_1 \\ y'-y_1 &= s_1 \cos \beta_1 \\ z'-z_1 &= s_1 \cos \gamma_1 \end{aligned}\right\} \quad \left.\begin{aligned} x'-x_2 &= s_2 \cos a_2 \\ y'-y_2 &= s_2 \cos \beta_2 \\ z'-z_2 &= s_2 \cos \gamma_2 \end{aligned}\right\} \cdots\cdots \tag{63}$$

So that

$$\left.\begin{aligned} \Sigma.\text{P} \cos a &= \Sigma(x'-x) = nx' - (x_1 + x_2 + \ldots + x_n) \\ \Sigma.\text{P} \cos \beta &= \Sigma(y'-y) = ny' - (y_1 + y_2 + \ldots + y_n) \\ \Sigma.\text{P} \cos \gamma &= \Sigma(z'-z) = nz' - (z_1 + z_2 + \ldots + z_n) \end{aligned}\right\}, \tag{64}$$

and therefore (62) become

$$\frac{x-x'}{x'-\dfrac{x_1+x_2+\ldots+x_n}{n}}=\frac{y-y'}{y'-\dfrac{y_1+y_2+\ldots+y_n}{n}}=\frac{z-z'}{z'-\dfrac{z_1+z_2+\ldots+z_n}{n}}, \tag{65}$$

which are the equations to the line of action of the resultant. The point whose coordinates are

$$\frac{x_1+x_2+\ldots+x_n}{n}, \quad \frac{y_1+y_2+\ldots+y_n}{n}, \quad \frac{z_1+z_2+\ldots+z_n}{n},$$

is that which is known by the name of the geometrical *centre of mean distances* of the points which are the extremities of the line-representatives of the forces: and therefore from (65) it appears that the line of action of the resultant passes through this point.

Also the magnitude of the resultant of the pressures, which is of course independent of the particular system of coordinate axes, may thus be found; since

$$\left.\begin{aligned} X &= P_1\cos\alpha_1+P_2\cos\alpha_2+\ldots+P_n\cos\alpha_n \\ Y &= P_1\cos\beta_1+P_2\cos\beta_2+\ldots+P_n\cos\beta_n \\ Z &= P_1\cos\gamma_1+P_2\cos\gamma_2+\ldots+P_n\cos\gamma_n \end{aligned}\right\}, \tag{66}$$

$$\begin{aligned} \therefore\ R^2 = {} & P_1^2+P_2^2+\ldots+P_n^2 \\ & +2P_1P_2\{\cos\alpha_1\cos\alpha_2+\cos\beta_1\cos\beta_2+\cos\gamma_1\cos\gamma_2\} \\ & +\ \ldots\ldots\ldots\ldots\ldots\ldots \\ & +2P_{n-1}P_n\{\cos\alpha_{n-1}\cos\alpha_n+\cos\beta_{n-1}\cos\beta_n+\cos\gamma_{n-1}\cos\gamma_n\} \qquad (67) \\ = {} & \Sigma.P^2+2\Sigma.PP'\cos(P,P'), \qquad (68) \end{aligned}$$

where $P, P'$ are the symbols for any two of the forces, and $\cos(P,P')$ is the cosine of the angle between their lines of action. And from the forms, which the resolved parts of R take in equations (64), it follows that the geometrical representative of it is $n$ times the length of the line joining the point of application of the forces and the centre of mean distances of the extremities of the geometrical representatives of them. This theorem is due to M. Chasles, and is the true generalization of the parallelogram of forces.

If the pressures are in equilibrium, $R=0$; in which case, by reason of (58), $X=0$, $Y=0$, $Z=0$; therefore

$$\Sigma.P\cos\alpha=0, \qquad \Sigma.P\cos\beta=0, \qquad \Sigma.P\cos\gamma=0; \tag{69}$$

that is, the sum of the resolved parts of the forces along each of three coordinate axes is equal to zero.

## SECTION 3.—*Conditions of equilibrium of many pressures acting on a material point which is in contact with a smooth surface or a smooth curve.*

29.] Let us first take the case of a smooth surface; and let us suppose a material particle acted on by many forces to be in contact with it at a given point. As the surface is smooth, the only direction along which it can offer any resistance to the particle's motion is that of its normal; and as it is conceived to have no active power of its own, but only a capacity of resisting any force that acts against it along its normal, so must the resultant of the impressed pressures act along the normal and towards the surface: these conditions therefore are sufficient for the equilibrium of the particle.

Let the equation to the surface be

$$\mathrm{F}(x, y, z) = c; \tag{70}$$

and for convenience of notation let

$$\left(\frac{d\mathrm{F}}{dx}\right) = \mathrm{U}, \qquad \left(\frac{d\mathrm{F}}{dy}\right) = \mathrm{V}, \qquad \left(\frac{d\mathrm{F}}{dz}\right) = \mathrm{W}, \tag{71}$$

$$\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2 = \mathrm{Q}^2,$$

so that the direction-cosines of the normal at a point $(x, y, z)$ are

$$\frac{\mathrm{U}}{\mathrm{Q}}, \quad \frac{\mathrm{V}}{\mathrm{Q}}, \quad \frac{\mathrm{W}}{\mathrm{Q}};$$

then as this line is to be coincident in direction with the resultant of the acting pressures, whose direction-cosines are proportional to $\mathrm{X}$, $\mathrm{Y}$, $\mathrm{Z}$, we have the equations

$$\frac{\mathrm{X}}{\mathrm{U}} = \frac{\mathrm{Y}}{\mathrm{V}} = \frac{\mathrm{Z}}{\mathrm{W}}, \tag{72}$$

as the conditions of equilibrium; and if these are not, and cannot be, satisfied, equilibrium on the surface cannot exist. If therefore it is required to determine the point on a given surface, at which a material particle under the action of given forces will rest in equilibrium, we must find the point on the surface at which the preceding equations are satisfied.

The normal pressure of the surface, which arises from the action of the impressed forces, may thus be determined. Let N represent the normal pressure, then the resolved parts of it along the coordinate axes are

$$\text{N}\frac{\text{U}}{\text{Q}}, \quad \text{N}\frac{\text{V}}{\text{Q}}, \quad \text{N}\frac{\text{W}}{\text{Q}},$$

and these together with the acting forces must be in equilibrium: therefore

$$\left.\begin{aligned} \Sigma.\text{P}\cos\alpha &= \text{X} = \text{N}\frac{\text{U}}{\text{Q}} \\ \Sigma.\text{P}\cos\beta &= \text{Y} = \text{N}\frac{\text{V}}{\text{Q}} \\ \Sigma.\text{P}\cos\gamma &= \text{Z} = \text{N}\frac{\text{W}}{\text{Q}} \end{aligned}\right\},$$

whence squaring and adding

$$\text{N}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2.$$

We subjoin some examples of the preceding formulæ.

Ex. 1. A particle is placed on the surface of an ellipsoid and is acted on by attracting forces which vary directly as the distance of the particle from the principal planes of section; it is required to determine the position of equilibrium.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 = \text{F}(x, y, z);$$

$$\therefore \quad \text{U} = \frac{2x}{a^2}, \quad \text{V} = \frac{2y}{b^2}, \quad \text{W} = \frac{2z}{c^2};$$

let $\quad \text{X} = \mu_1 x, \quad \text{Y} = \mu_2 y, \quad \text{Z} = \mu_3 z,$

then equations (72) become

$$\frac{\mu_1}{a^{-2}} = \frac{\mu_2}{b^{-2}} = \frac{\mu_3}{c^{-2}} = \frac{\mu_1 + \mu_2 + \mu_3}{a^{-2} + b^{-2} + c^{-2}},$$

therefore $\mu_1 : \mu_2 : \mu_3 = a^{-2} : b^{-2} : c^{-2}$; and if these conditions are fulfilled, the particle will rest at all points of the surface.

Ex. 2. Again, take the same surface, and let the forces vary inversely as the distances of the point from the principal planes: it is required to determine the position of equilibrium.

$$\text{X} = \frac{\mu_1}{x}, \quad \text{Y} = \frac{\mu_2}{y}, \quad \text{Z} = \frac{\mu_3}{z};$$

therefore (72) become

$$\frac{\frac{x^2}{a^2}}{\mu_1} = \frac{\frac{y^2}{b^2}}{\mu_2} = \frac{\frac{z^2}{c^2}}{\mu_3} = \frac{1}{\mu_1 + \mu_2 + \mu_3} = \frac{1}{\mu}, \text{ (say)};$$

$$\therefore \quad x = a\left(\frac{\mu_1}{\mu}\right)^{\frac{1}{2}}, \quad y = b\left(\frac{\mu_2}{\mu}\right)^{\frac{1}{2}}, \quad z = c\left(\frac{\mu_3}{\mu}\right)^{\frac{1}{2}},$$

$$\begin{aligned} \text{N}^2 &= \frac{\mu_1{}^2}{x^2} + \frac{\mu_2{}^2}{y^2} + \frac{\mu_3{}^2}{z^2} \\ &= \mu\left\{\frac{\mu_1}{a^2} + \frac{\mu_2}{b^2} + \frac{\mu_3}{c^2}\right\}. \end{aligned}$$

Ex. 3. A particle of given weight is placed inside a smooth sphere, and is acted on by a repulsive force varying inversely as the square of the distance from the lowest point of the sphere: it is required to find the position of rest of the particle.

Let the lowest point of the sphere be taken for the origin, and let the axis of $z$ be vertical; then the equation of the sphere, whose radius is $a$, is

$$x^2 + y^2 + z^2 - 2az = 0.$$

Let $\text{W}$ = the weight of the particle, and $r$ = the distance of it from the lowest point; then

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ &= 2az. \end{aligned}$$

Also let the repulsive force $= \dfrac{\mu}{r^2} = \dfrac{\mu}{2az}$;

$$\therefore \quad \text{X} = \frac{\mu}{2az}\,\frac{x}{r}, \quad \text{Y} = \frac{\mu}{2az}\,\frac{y}{r}, \quad \text{Z} = \frac{\mu}{2az}\,\frac{z}{r} - \text{W}.$$

Let $\text{N}$ = the normal pressure of the curve,

$$\therefore \left.\begin{aligned} \frac{\mu}{2az}\,\frac{x}{r} &= \text{N}\,\frac{x}{a} \\ \frac{\mu}{2az}\,\frac{y}{r} &= \text{N}\,\frac{y}{a} \\ \frac{\mu}{2az}\,\frac{z}{r} - \text{W} &= \text{N}\,\frac{z-a}{a} \end{aligned}\right\},$$

from which we have

$$r^3 = \frac{\mu a}{\text{W}}, \qquad z = \frac{\mu^{\frac{2}{3}}}{2a^{\frac{1}{3}}\text{W}^{\frac{2}{3}}};$$

whence the position of the particle is known for a given weight of it, and for a given value of $\mu$.

If another force of the same kind, and in which $\mu$ is replaced by $\mu'$, makes the particle to rest at a distance $r'$ from the lowest point, then

$$r'^3 = \frac{\mu' a}{\text{W}}, \qquad \therefore \quad \frac{\mu}{\mu_1} = \frac{r^3}{r'^3},$$

that is, the absolute values of the repulsive forces at an unit-distance vary as the cubes of the distances from the lowest point of their positions of rest.

30.] Next let us consider the circumstances of pressure of a particle resting, or (to fix our thoughts) of a small ring sliding, on a given curved line which is smooth and offers no resistance to motion along itself.

As the curve is smooth, the resultant of the impressed forces is manifestly perpendicular to the tangent of the curve at the point of equilibrium; therefore if the curve is of double curvature, so that the direction-cosines of its normal are proportional to $dx$, $dy$, $dz$, the required condition is

$$\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz = 0; \tag{73}$$

and if $\mathrm{N}$ is the normal pressure, and $\lambda$, $\mu$, $\nu$ are the direction-angles of its line of action,

$$\left.\begin{array}{l} \mathrm{N}\cos\lambda = \mathrm{X} \\ \mathrm{N}\cos\mu = \mathrm{Y} \\ \mathrm{N}\cos\nu = \mathrm{Z} \end{array}\right\}, \tag{74}$$

$$\therefore\quad \mathrm{N}^2 = \mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2; \tag{75}$$

whence $\mathrm{N}$, $\lambda$, $\mu$, $\nu$ are known. If the equation (73) cannot be satisfied at any point of the curve, equilibrium is impossible; and if the forces are given, the point, at which equilibrium takes place, may be determined by means of (73) and the equations to the curve.

If the curve is a plane curve, (73) becomes

$$\mathrm{X}\,dx+\mathrm{Y}\,dy = 0, \tag{76}$$

which, if $\mathrm{F}(x, y) = c$ is the equation to the curve, may be written in the form

$$\frac{\mathrm{X}}{\left(\frac{d\mathrm{F}}{dx}\right)} = \frac{\mathrm{Y}}{\left(\frac{d\mathrm{F}}{dy}\right)}. \tag{77}$$

Also (75) becomes

$$\mathrm{N}^2 = \mathrm{X}^2+\mathrm{Y}^2. \tag{78}$$

Ex. 1. A ring is capable of sliding on a smooth helix, and is acted on by a constant force perpendicular to the axis; shew that equilibrium is impossible, unless the force parallel to the axis of $z$ is zero.

The equations to the helix are

$$\begin{aligned} x &= a\cos\phi, & \therefore\quad dx &= -y\,d\phi, \\ y &= a\sin\phi, & dy &= x\,d\phi, \\ z &= ka\phi, & dz &= ka\,d\phi, \end{aligned}$$

and if $\mu a$ is the constant force which acts towards and perpendicular to the axis,

$$\mathrm{X} = -\mu x, \qquad \mathrm{Y} = -\mu y;$$

and therefore substituting in (73), we have

$$\mu xy - \mu xy + \mathrm{Z}ka = 0,$$

which can be satisfied only when $\mathrm{Z} = 0$.

Ex. 2. A small ring, capable of sliding on a smooth ellipse, whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

is acted on by forces parallel to the axes of $x$ and $y$ represented by $\mu x^n$ and $\mu y^n$; find the position of equilibrium.

In this case (77) becomes

$$a^2 x^{n-1} = b^2 y^{n-1},$$

$$\therefore\quad x = ab^{\frac{n+1}{n-1}}\left\{a^{2\frac{n+1}{n-1}} + b^{2\frac{n+1}{n-1}}\right\}^{-\frac{1}{2}};$$

and a similar value may be found for $y$.

Ex. 3. Two weights P and Q are fastened to the ends of a string, fig. 14, which passes over a pully O; and Q hangs freely, when P rests on a plane curve AP in a vertical plane; it is required to find the position of rest when the curve is given.

The forces which act on P are, (1) the tension of the string in the line OP, and which is equal to the weight of Q, (2) the weight of P acting vertically downwards, (3) the normal reaction of the curve, viz. R.

Let $\mathrm{F}(x, y) = c$ be the equation to the plane curve, O being the origin, and the axis of $x$ being vertical. Let $\mathrm{OM} = x$, $\mathrm{MP} = y$, $\mathrm{OP} = r$, $\mathrm{POM} = \theta$, $\mathrm{AO} = a$. Then

$$\mathrm{X} = \mathrm{P} - \mathrm{Q}\cos\theta - \mathrm{R}\frac{dy}{ds}, \qquad \mathrm{Y} = -\mathrm{Q}\sin\theta + \mathrm{R}\frac{dx}{ds},$$

therefore from (76)

$$(\mathrm{P} - \mathrm{Q}\cos\theta)\,dx - \mathrm{Q}\sin\theta\,dy = 0,$$

$$\mathrm{P}\,dx - \mathrm{Q}\frac{x\,dx + y\,dy}{r} = 0;$$

but since $\quad x^2 + y^2 = r^2, \qquad \therefore\quad x\,dx + y\,dy = r\,dr;$

$$\therefore \quad \text{P}\,dx - \text{Q}\,dr = 0; \tag{79}$$

and this condition must be satisfied by P, Q, and the equation to the curve. Also

$$\text{R}^2 = \text{P}^2 - 2\,\text{PQ}\cos\theta + \text{Q}^2. \tag{80}$$

(1) Let the curve AP be an hyperbola of which O is the centre; then

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$$\therefore \quad r^2 = x^2 + y^2 = e^2x^2 - b^2, \qquad r\,dr = e^2x\,dx,$$

$$\therefore \quad r\text{P}\,dx - \text{Q}\,e^2x\,dx = 0,$$

$$\therefore \quad x = \frac{b\,\text{P}}{e\,\{\text{P}^2 - e^2\text{Q}^2\}^{\frac{1}{2}}}.$$

(2) Let it be required to find the equation to the curve, on all points of which P will rest. In this case (79) must be satisfied at all points of the curve; therefore

$$\begin{aligned} \text{P}x - \text{Q}r &= \text{a constant} \\ &= (\text{P} - \text{Q})\,a, \ \ (\text{say}), \end{aligned}$$

if the curve passes through A, and $\text{OA} = a$; therefore

$$r = \frac{\left(1 - \frac{\text{P}}{\text{Q}}\right)a}{1 - \frac{\text{P}}{\text{Q}}\cos\theta}; \tag{81}$$

which is the equation to a conic section, of which the focus is the pole; and is an ellipse, parabola, or hyperbola, according as P is less than, equal to, or greater than, Q.

(3) Let the curve be a circular quadrant, convex downwards, with a horizontal radius passing through O, which is also a point on the circle, and let $\text{P} = 2\,\text{Q}$; then the equation to the circle is, if $a$ is the radius,

$$r = 2a\sin\theta;$$

and therefore (79) becomes

$$4\,(\cos\theta)^2 - \cos\theta - 2 = 0,$$

whence $\theta$ may be determined.

(4) Another form of the problem is, The length of the string being given, and Q always resting on a given curve, to find the curve on which P shall rest in all positions.

Let the tension of the string be equal to T, and let $r'$ and $\theta'$

refer to the curve BQ, fig. 15, on which Q rests, and of which let the equation be

$$r' = f(\theta'), \tag{82}$$

where $f$ is the symbol of a known function: then we have from (79)

$$\mathrm{Q}\, dx' - \mathrm{T}\, dr' = 0,$$

$$\text{also} \quad \mathrm{P}\, dx - \mathrm{T}\, dr = 0;$$

and since

$$r + r' = 2c = \text{length of the string}, \tag{83}$$

$$\therefore \quad dr + dr' = 0, \qquad \therefore \quad \mathrm{Q}\, dx' + \mathrm{P}\, dx = 0; \tag{84}$$

and by means of (82), (83), and (84), $r'$ and $\theta'$ are to be eliminated, and the resulting equation in terms of $r$ and $\theta$ will be that required.

Let the curve on the left-hand side be a parabola of which O is the focus; then

$$r' = \frac{2b}{1 - \cos\theta'}, \tag{85}$$

and from (84)

$$\mathrm{Q}x' + \mathrm{P}x = 2k\mathrm{Q},$$

where $k$ is an arbitrary constant; therefore from (85)

$$r' - r'\cos\theta' = 2b, \qquad 2c - r - \frac{2k\mathrm{Q} - \mathrm{P}r\cos\theta}{\mathrm{Q}} = 2b,$$

$$\therefore \quad r = \frac{2(c-b-k)}{1 - \frac{\mathrm{P}}{\mathrm{Q}}\cos\theta}, \tag{86}$$

which is the equation to a conic section of which the focus is O.

In recapitulation of the preceding results it appears that, (1) if the point on which certain pressures act is entirely free, so that three variables are independent, the pressures must satisfy three conditions; (2) if the point is constrained to be on a given surface, there are two equations of equilibrium; and (3) only one condition is requisite, when the point is on a given curve.

# CHAPTER III.

## COMPOSITION AND RESOLUTION OF STATICAL PRESSURES ACTING ON A RIGID BODY.

31.] Before we enter on the formal inquiry into the mode and results of the composition of many statical pressures acting on a rigid body, it is necessary to remove some obscurities as to properties of such bodies, and to state principles which are fundamental and necessary to the discussion; and here I must again remind the reader that I am appealing, not to abstract scientific principles, but to his common sense and experimental knowledge of those bodies which he meets with in his contact with the external world.

A rigid body is such that its component material particles are in a state of relative rest by the action of unknown molecular forces, such as attractions, cohesions, &c.: and the intensity of these forces is so great, that the relative equilibrium of the particles, which is due to them, is not disturbed by the pressures which act on the body.

And suppose a pressure to act at a definite point of a body in a definite direction; it produces a pressure of the particle on which it acts *against* the contiguous particle in the line of its action, and *from* the contiguous particle in the opposite direction: and this pressure on these particles, although infinitesimal in comparison of the molecular forces, is propagated from one particle to another along the whole line of action of the impressed force; and is manifestly the same, at whatever point along this line the pressure is applied. Hence we infer that the effect of a pressure on a rigid body, acting at a definite point and in a definite direction, is unaltered, whatever is the point in its line of action at which it is applied. This principle is commonly called that of Transmissibility of Pressure, and the truth of it depends on the rigidity of the body, which appears to involve such a mode of action as that described above.

Now two equal pressures acting on a material point in the same line and in opposite directions neutralize each other; and this property may be extended by means of the preceding principle, so that, Two equal pressures acting in the same line and in opposite directions on any points of a rigid body neutralize each other. Hence we infer, that when many forces are acting on a rigid body, any two which are equal and have the same line of action and are opposite may be omitted; and similarly the introduction of two equal forces along the same line of action and in opposite directions does not change the circumstances of the system as to resultant pressure.

The effects of the pressures which have been considered in the preceding chapter are a tendency to motion in a given straight line, and, so far as we have considered them, along that straight line only: these are called *pressures of translation*. But suppose a point O, fig. 16, of a rigid body to be fixed, so that there cannot be any motion of translation of the whole body; and suppose a pressure P to act on the body at a definite point M in the direction MP; join OM, and resolve P into two parts, one along, and the other perpendicular to, OM; then the part along OM produces a pressure at O, which being fixed is capable of bearing it without the body having thereby any tendency to motion: but the other component causes a pressure on M in a direction at right-angles to OM; but as O is fixed, M can only describe a circle about O as the centre; the effect therefore of this latter component is a tendency to *circular* motion of M about O; a pressure producing such an effect is called *a pressure of rotation* about a given point; and we shall consider these, their measures, and their laws more at length. Single material particles are subject to pressures of translation, but, having neither magnitude nor parts, not to pressures of rotation.

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## SECTION 1.—*Composition and resolution of forces acting on a rigid body in one plane—Composition and resolution of couples.*

32.] Composition of two pressures acting at definite points on a rigid body in one plane.

Let the two pressures be P and Q, and suppose them to act in the plane of the paper at the points A and B; fig. 17; join AB, and let the angles between AB and the lines of action of P and Q be respectively $\alpha$ and $\beta$; produce the lines of action of P and Q so as to meet in O (O being supposed to be in the rigid body); and, by virtue of the principle of transmissibility, we may suppose P and Q to be applied at O. Let R be the resultant of them so transferred, and let the line of action of R intersect AB in the point C; then we have to determine the magnitude of R, its line of action, and a point in that line; these last two will be conveniently known, if we find AC, and the angle between AB and CO.

$$\text{Let} \quad \text{AC} = x, \quad \text{CB} = y, \quad \text{AB} = a; \qquad \therefore \quad x+y=a,$$

$$\text{OAB} = \alpha, \quad \text{OBA} = \beta, \quad \text{OCB} = \theta;$$

then, by the parallelogram of forces,

$$\text{R}^2 = \text{P}^2 - 2\text{PQ} \cos(\alpha+\beta) + \text{Q}^2; \tag{1}$$

whereby the intensity of the resultant is known. And resolving P, Q, R at O along lines through O, parallel, and perpendicular to, AB, we have

$$\left.\begin{aligned} \text{R} \cos\theta &= \text{P} \cos\alpha - \text{Q} \cos\beta \\ \text{R} \sin\theta &= \text{Q} \sin\beta + \text{P} \sin\alpha \end{aligned}\right\}, \tag{2}$$

$$\therefore \quad \tan\theta = \frac{\text{Q} \sin\beta + \text{P} \sin\alpha}{\text{P} \cos\alpha - \text{Q} \cos\beta}; \tag{3}$$

and by reason of equations (30) Article 19,

$$\frac{\text{P}}{\sin(\theta+\beta)} = \frac{\text{Q}}{\sin(\theta-\alpha)} = \frac{\text{R}}{\sin(\alpha+\beta)}. \tag{4}$$

Let $p$ and $q$ be the lengths of the perpendiculars on the lines of action of P and Q from any point in the line of action of R, say from C; then

$$\left.\begin{aligned} p &= \text{CO} \sin(\theta-\alpha) \\ q &= \text{CO} \sin(\theta+\beta) \end{aligned}\right\}; \tag{5}$$

therefore from the first two terms of (4)

$$\text{P}p = \text{Q}q; \tag{6}$$

and therefore, since $p = x \sin\alpha$, $q = y \sin\beta$,

$$\text{P}x \sin\alpha = \text{Q}y \sin\beta, \tag{7}$$

$$\begin{aligned} \frac{x}{\text{Q} \sin\beta} = \frac{y}{\text{P} \sin\alpha} &= \frac{a}{\text{P} \sin\alpha + \text{Q} \sin\beta} \\ &= \frac{a}{\text{R} \sin\theta}, \end{aligned} \tag{8}$$

whereby $x$ and $y$ are given in terms of known quantities: the intensity, line of action, and point of application on the line AB of the resultant are therefore determined.

33.] The equation (6) requires especial consideration; two pressures, P and Q, act on the body, each of which alone produces a pressure of translation along its line of action: but the resultant of the two taken in combination is a single pressure R, the position of whose line of action is given by the equation (3); a pressure therefore equal to R, along the same line of action, and opposite in direction to R, will with P and Q produce equilibrium. Now this force may be applied at *any* point in the line of action of R; let C be the point of application; and thus the system is in equilibrium, and is as if C were a fixed point. Let us consider this in the light of the remarks of Art. 31; P and Q severally produce a pressure of rotation about C, and manifestly in opposite directions; and they neutralize each other, for the body is at rest: therefore their rotatory effects are equal. But what relation exists between them? because we may thence infer a measure of their rotatory effects with reference to the centre C. P and Q balance, when (6) is satisfied; that is, the rotatory effect due to one force is equal to, and neutralized by, that due to the other, when the products of each force and the perpendicular distance from C on its line of action are equal. This product therefore may be taken as the measure of the rotatory effect of a force. And as it is desirable to have a distinctive name for such an effect, it is called a force's *moment;* and therefore we define as follows:

DEF. *Moment* of a pressure is the rotatory effect of a pressure with reference to a given point; and is measured by the product of the numbers which represent the pressure and the perpendicular distance from the point on the line of action of the pressure.

Two pressures are said to be *equimomental* with respect to a given point when their moments with respect to that point are equal.

As the forces act in one plane we have spoken of the moments with respect to a point: it is more correct to say, with respect to an axis passing through the point and perpendicular to the plane in which the forces act, because it is about this *line* that the forces *per se,* and all other things neglected, tend to make the body turn. However, in a subsequent part of the course, and when the body, on which the forces act, moves, we shall have a modification of this statement.

A force may tend to make a body turn about an axis in either one or the other of two directions; it is necessary therefore to distinguish these, and to affect them with different signs: let therefore the moment of a force be positive if it tends to turn a body from right to left, that is, in the direction in which the hands of a clock revolve, when it is opposite to us; and let the moment of a force be negative, when it turns a body in the opposite direction.

Moments of forces, being quantities measurable by number, are capable of addition and subtraction, and also of being represented by lines.

34.] Let us return to equation (7), and consider C as a point fixed, by means of the force R acting on it and being in equilibrium with P and Q: then resolving P and Q along and perpendicular to AB, we have $\text{P} \sin \alpha$ and $\text{Q} \sin \beta$ perpendicular to AB, and $\text{P} \cos \alpha$ and $\text{Q} \cos \beta$ along AB: these latter forces produce a pressure on C which is equal to their difference; but the former components produce a rotatory pressure about C, and equilibrate when the moments of the two are equal, that is, when

$$x \text{P} \sin \alpha = y \text{Q} \sin \beta ;$$

and this is equation (7).

Again, suppose that the components are $\text{P}_1$ and $\text{P}_2$, and that the equations to their lines of action are given; and let it be required to find that of the line of action of the resultant R.

Let the equations to the lines of action of the components be

$$\left.\begin{aligned} x \cos \alpha_1 + y \sin \alpha_1 - p_1 = 0 = a_1 \\ x \cos \alpha_2 + y \sin \alpha_2 - p_2 = 0 = a_2 \end{aligned}\right\}, \qquad (9)$$

$a_1$ and $a_2$ being abridged forms of notation for the left-hand members of the equations: then, if $x$ and $y$ refer to any point in the line of action of the resultant, by equation (6) we have

$$\text{P}_1 a_1 + \text{P}_2 a_2 = 0,$$

$$\therefore \quad (\text{P}_1 \cos \alpha_1 + \text{P}_2 \cos \alpha_2)\, x + (\text{P}_1 \sin \alpha_1 + \text{P}_2 \sin \alpha_2)\, y - p_1 \text{P}_1 - p_2 \text{P}_2 = 0, \quad (10)$$

which is the equation to the line of action of R.

Hence if $r$ is the perpendicular from the origin on the line of action of R,

$$\begin{aligned} r &= \frac{p_1 \text{P}_1 + p_2 \text{P}_2}{\{(\text{P}_1 \cos \alpha_1 + \text{P}_2 \cos \alpha_2)^2 + (\text{P}_1 \sin \alpha_1 + \text{P}_2 \sin \alpha_2)^2\}^{\frac{1}{2}}} \\ &= \frac{p_1 \text{P}_1 + p_2 \text{P}_2}{\{\text{P}_1^2 + 2 \text{P}_1 \text{P}_2 \cos (\alpha_1 - \alpha_2) + \text{P}_2^2\}^{\frac{1}{2}}} \\ &= \frac{p_1 \text{P}_1 + p_2 \text{P}_2}{\text{R}}, \end{aligned}$$

$$\therefore \quad \text{R}r = p_1\text{P}_1 + p_2\text{P}_2; \tag{11}$$

that is, the moment of the resultant is equal to the sum of the moments of the components.

35.] To consider the subject from another point of view, I will take the case of two *parallel* forces acting in the same direction on a rigid body.

Let P, Q be the two parallel forces acting at A and B: fig. 18: join AB, and let $a$ be the angle between AB and the lines of action of P and Q; at A and B introduce two equal forces S and S which act along AB, and in opposite directions: the circumstances of pressure are not hereby altered. Let P′ be the resultant of P and S at A, and Q′ the resultant of Q and S at B; let the lines of action of P′ and Q′ be produced to meet in O, O being supposed to be rigidly connected with the body: at O resolve P′ and Q′ into the forces of which they were compounded; the components along the line parallel to AB manifestly cancel each other, and there remains P+Q acting in a line parallel to the lines of action of P and Q. Let this resultant be R, so that

$$\text{R} = \text{P} + \text{Q}; \tag{12}$$

that is, the resultant is the sum of the two parallel forces.

Let $\text{AC} = x$, $\text{CB} = y$, $\text{AB} = a$; therefore $x + y = a$; then P′ is the resultant of P and S, and these pressures are parallel to the sides of the triangle ACO,

$$\frac{\text{S}}{x} = \frac{\text{P}}{\text{CO}}; \qquad \text{similarly} \quad \frac{\text{S}}{y} = \frac{\text{Q}}{\text{CO}};$$

$$\therefore \quad \text{P}x = \text{Q}y. \tag{13}$$

Let $p$ and $q$ be the perpendicular distances from C on the lines of action of P and Q: then $p = x \sin a$, $q = y \sin a$, and therefore (13) becomes

$$\text{P}p = \text{Q}q, \tag{14}$$

that is, the moments of P and Q about C are equal.

Again, from (13)

$$\frac{x}{\text{Q}} = \frac{y}{\text{P}} = \frac{x+y}{\text{P}+\text{Q}}$$

$$= \frac{a}{\text{R}}, \tag{15}$$

whence $x$ and $y$ are known; and are reciprocally proportional to the forces at their extremities.

If $\text{P} = \text{Q}$, $\qquad y = x = \frac{a}{2}$, $\qquad \text{R} = 2\text{P}$;

that is, the resultant is equal to twice one of the forces, and is applied at the point of bisection of the line joining the points of application of the forces.

As (14) is independent of the angle between AB and the direction of the forces, C is the same whatever that angle is; C is for this reason called the centre of the two parallel forces.

36.] Suppose one of the parallel forces of the last Article to act in a direction contrary to that of the other: then, fig. 19, introducing as before two equal forces S, S acting along AB and in opposite directions, and compounding P and S into P′, and Q and S into Q′, let us suppose the lines of action of P′ and Q′ to meet at O, O being rigidly connected with the body; and at O let P′ and Q′ be resolved into the forces of which they were compounded; the forces parallel to the line AB cancel each other, and there remain P and Q acting in a line parallel to the original lines of action of P and Q, the resultant of which is equal to their difference: let us suppose Q to be the greater, then

$$\text{R} = \text{Q} - \text{P}. \tag{16}$$

Let $\text{AB} = a$, $\text{AC} = x$, $\text{BC} = y$; therefore $x - y = a$; and let $\alpha$ be the angle between AB and the lines of action of P and Q. Since P′ is the resultant of P and Q,

$$\frac{\text{S}}{x} = \frac{\text{P}}{\text{CO}}, \qquad \text{similarly,} \quad \frac{\text{S}}{y} = \frac{\text{Q}}{\text{CO}},$$

$$\therefore \quad \text{P}x = \text{Q}y. \tag{17}$$

Let $p$ and $q$ be the perpendicular distances from C on the lines of action of P and Q; then $p = x \sin\alpha$, $q = y \sin\alpha$; therefore (17) becomes

$$\text{P}p = \text{Q}q; \tag{18}$$

that is, the moments of P and Q about C, and similarly about every point in the line of action of R, are equal.

Again, from (17)

$$\frac{x}{\text{Q}} = \frac{y}{\text{P}} = \frac{x-y}{\text{Q}-\text{P}}$$

$$= \frac{a}{\text{R}}, \tag{19}$$

whence $x$ and $y$ are known; and are reciprocally proportional to the forces acting at their extremities.

37.] Now let us suppose the pressure Q, which I have imagined to be the larger of the two, to diminish, and to approach in intensity nearer and nearer to P; then R becomes less and less, and $x$ becomes greater; and ultimately, when $\mathrm{Q} = \mathrm{P}$, $\mathrm{R} = 0$, and $x = y = \infty$; that is, there is no single force of translation which will be equivalent to such a pair of forces, and therefore there is no one force of translation which will be in equilibrium with them. And it is, by the principle of sufficient reason, manifest that such a system cannot have a single resultant of translation; because such a resultant is *unique;* and whatever is the process of reasoning, by which its line of action is assigned in respect of one of the forces, by the same will it be assigned in a similar position with respect to the other force.

Such a pair of forces, equal and acting in parallel lines and in opposite directions, is called *a couple*;* its effect is evidently *a pressure of rotation* about a line perpendicular to the plane in which the forces act, and which line is called the axis of the couple. Now in statics, as the motion is only *virtual* and not *actual,* the *direction* of the axis is fixed, but not the *position* of it; it is *some* line perpendicular to the plane in which the forces act. The perpendicular distance between the lines of action of the forces is called the *arm of the couple.*

In estimating the *moment,* that is, the rotatory effect, of the couple, we must examine all possible positions of the axis. Let the couple be that indicated in fig. 20; and (1) let us suppose the axis to pierce the plane of the couple at the point O which lies between the forces; then

$$\begin{aligned} \text{the moment of the couple} &= \mathrm{P} \times \mathrm{OA} + \mathrm{P} \times \mathrm{OB} \\ &= \mathrm{P} \times \mathrm{AB}. \qquad (20) \end{aligned}$$

(2) Suppose the axis to pass through A, one of the extremities of the arm: then the force which acts at A produces no pressure of rotation, and we have

$$\text{the moment of the couple} = \mathrm{P} \times \mathrm{AB}. \qquad (21)$$

(3) Suppose the axis to pierce the plane of the couple at a point O, fig. 21, in the arm produced: then

$$\begin{aligned} \text{the moment of the couple} &= \mathrm{P} \times \mathrm{OB} - \mathrm{P} \times \mathrm{OA} \\ &= \mathrm{P} \times \mathrm{AB}. \qquad (22) \end{aligned}$$

In all cases therefore the moment of the couple is equal to the product of the numbers expressing the force and the length of the arm. Thus if the force contains 6 units of pressure, and the arm 3 units of linear length, the moment of the couple is expressed by 18; that is,

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* See Poinsot, "Mémoire sur la composition des Moments et des Aires dans la Mecanique." The tract is appended to "Élements de Statique" of the same author, $8^{\text{me}}$ edition, Paris. 1842.

$$\text{moment of couple} = \text{pressure} \times \text{length of arm}. \tag{23}$$

A couple may evidently tend to make a body revolve in either one or the other of two opposite directions; that is, in the direction of the hands of a watch, as it is before us, or in the opposite direction; and it is desirable to affect these different directions with different signs; let the former therefore be positive or right-handed couples, and the latter, negative or left-handed couples. In fig. 20 and 21 right-handed couples are represented.

38.] The equation to the line of action of the resultant of two parallel forces $\text{P}_1$ and $\text{P}_2$ may be determined as follows:

Let the equations to the lines of action of the components be

$$\left.\begin{array}{l} x\cos a + y\sin a - \delta_1 = 0 = a_1 \\ x\cos a + y\sin a - \delta_2 = 0 = a_2 \end{array}\right\}; \tag{24}$$

therefore the equation to the line of action of the resultant is

$$(\text{P}_1+\text{P}_2)\,x\cos a + (\text{P}_1+\text{P}_2)\,y\sin a - (\delta_1\text{P}_1+\delta_2\text{P}_2) = 0. \tag{25}$$

Let $\text{P}_1+\text{P}_2 = 0$, that is, let the forces be equal and act in opposite directions,

$$\therefore \quad (\delta_1-\delta_2)\,\text{P}_1 = 0, \tag{26}$$

which is the equation to a straight line at an infinite distance; therefore the resultant of two equal and opposite forces acts at an infinite distance.

39.] Before we proceed to the consideration of other cases of forces acting on a rigid body in one plane, it is necessary to discuss the properties and laws of couples. Couples are called *coaxal* when their axes are parallel, that is, when the planes of their forces are parallel.

THEOREM I. The effect of a couple on a rigid body is not altered, if the length of the arm and the force being the same, the arm is turned about its extremity through any angle in the plane of the couple.

Let AB, fig. 22, be the arm of the original couple, and P, P its forces; through A draw *any* straight line AB′ in the plane of the couple equal to AB, and at A and B′ respectively insert in the plane of the couple two forces equal to P, with their lines of action perpendicular to the arm AB′, and opposite in direction to each other. Let $\text{BAB}' = 2\theta$; then the resultant of P acting at B, and of P acting at B′, whose lines of action meet at Q, is $2\text{P}\sin\theta$, and acts along the line AQ: similarly the resultant of P acting at A perpendicularly to AB, and of P perpendicularly to AB′ is 2P sin $\theta$, and acts along the line AQ in a direction opposite to that of the former resultant: these two resultants therefore neutralize each other, and there remains the couple whose arm is AB′ and the forces P, P: and this is equimomental with the original couple and replaces it.

THEOREM II. The effect of a couple on a rigid body is not altered, if the plane of the forces is transferred to any other parallel plane, the arm being parallel to its original direction, and of an equal length, and the forces being unaltered in intensity.

Let AB, fig. 23, be the arm, and P, P the forces of the given couple: let A′B′ be an arm equal and parallel to AB; at A′ and B′ respectively introduce two forces equal to P, acting perpendicularly to A′B′, and in opposite directions, and in a plane parallel to the plane of the original couple: join AB′, A′B; these lines evidently intersect and bisect each other in O; then P at A and P at B′, acting in parallel lines and in the same direction, are equivalent to a force 2P acting at O: similarly P at B and P at A′, acting in parallel lines and in the same direction, are equivalent to 2P acting at O in a direction parallel to their direction: at O therefore these two resultants, being equal and opposite, neutralize each other; and there remains the couple whose arm is A′B′, and whose forces are P, P, acting in the same direction as the original couple, in a parallel plane, and with an equal arm and equal forces: it is therefore coaxal and equimomental.

The proof which is here given for a parallel plane is of course valid for the less general case of the same plane: and therefore from this and Theorem I we infer, that the effect of a couple on a rigid body is not changed whatever is the position of its plane, if the direction of the axis is unaltered, and the arm and the forces are equal.

40.] THEOREM III. The effect of a couple on a rigid body is not altered, whatever is the position of its plane, arm, and force, provided that its axis and moment are unaltered.

In fig. 24, let AB be the arm, and P, P the forces of the given couple; at A and B introduce any equal forces S and S acting along AB and in opposite directions. Let P′ be the resultant of P and S at A, and P′ also be the resultant of P and S at B: the lines of action of P′ and P′ are of course parallel; produce P′A backwards, and from B draw $BA'$ perpendicular to $AA'$: then we have a couple whose arm is $BA'$, and each of whose forces is $P'$; let $BAA' = \theta$; then $A'B = AB \sin\theta$; $P' = P \operatorname{cosec}\theta$; $S = P'\cos\theta = P\cot\theta$; then

$$\begin{aligned} \text{moment of new couple} &= P' \times A'B \\ &= P \operatorname{cosec}\theta \times AB \sin\theta \\ &= P \times AB \\ &= \text{moment of original couple.} \qquad (27) \end{aligned}$$

It will be observed that S is arbitrary, and that $\theta$ and therefore the length of the new arm depends on it; and so does also the force of the new couple: and the only restriction is (27), which requires the new couple to be equimomental with the original one. And thus it appears that a couple is equivalent to, and may be replaced by, another couple, of which the moment is the same, the forces are in the same plane, and the arms have a common extremity.

Combining this Theorem with the preceding, we conclude that a couple is equivalent to, and may be replaced by, any other equimomental and coaxal couple.

41.] Now in all these transformations, the axis of the couple, that is, the direction of the line about which the couple tends to make the body turn, has not been altered; the arm and the force have been altered in position, in length, in intensity; and the plane in which the forces act has been changed from any one into any other parallel plane; but the normal to the plane, which is the axis, has continued the same; and the moment has continued the same; and these quantities cannot be changed without changing the effect of the couple; the former of these then is a fixed direction, and the latter is a fixed quantity. It is convenient, as of forces of translation, so of these forces of rotation, to have geometrical lengths as adequate representatives; and such we shall obtain, if along the *axis* we take lengths containing the same number of linear units as the moment of the couple contains units of pressure. Thus if the force of a couple is 4 and the length of the arm is 3, the moment is represented by the number 12; and if along the axis 12 linear units are measured, this length is a full and adequate representative of the couple; and moreover as couples may be right-handed or left-handed, that is, have positive or negative signs, so from a fixed point (the origin) on the axis may the line be taken in one or the other direction, and thus indicate the sign of the couple. Now the line of a definite length and subject to these conditions is technically called *the axis of the couple*, the word being used in a sense different to the former one: *there* it indicated line of rotation only; *here* it indicates three things, viz. the line of rotation, a finite length of that line measured from a given point on it, and the direction in which it is measured. This axis therefore fully determines all the circumstances of the couple. Some confusion may arise from the ambiguous use of the word, and therefore I shall always take care to specify axis as to rotation, and axis as to rotation and moment, by calling the former *rotation-axis*, and the latter *moment-axis*, bearing in mind however that the latter is indicative of direction as well as the former; and when couples are said to be coaxal, it is with respect to the former meaning of the word only; and when two couples are statically equivalent they are coaxal and equimomental.

42.] THEOREM IV. The resultant of many coaxal couples is a coaxal couple whose moment is equal to the algebraical sum of the moments of the component couples.

Let the forces of the several couples be $P_1, P_2, \dots P_n$; and the lengths of the arms $p_1, p_2, \dots p_n$; so that their moments are $P_1 p_1, P_2 p_2, \dots P_n p_n$. Let all, by virtue of Theorem II, be transferred to the same plane, and let all the arms have a common extremity; again, by virtue of Theorem III, let all be transformed into equivalent couples with arms of the same length, equal to $r$, and let the forces thereby changed be $P_1', P_2', \dots P_n'$; so that

$$\left.\begin{array}{l} P_1' r = P_1 p_1 \\ P_2' r = P_2 p_2 \\ \;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\ P_n' r = P_n p_n \end{array}\right\}; \tag{28}$$

and lastly, by virtue of Theorem I, let all the arms be turned about their common extremity, and become coincident; then the length of it is $r$, and at each extremity there are equal and opposite forces, of which let the sum be $R$, and

$$R = P_1' + P_2' + \dots + P_n'; \tag{29}$$

so that the moment of the resultant couple is

$$\begin{aligned} Rr &= P_1' r + P_2' r + \dots + P_n' r \\ &= P_1 p_1 + P_2 p_2 + \dots + P_n p_n \\ &= \Sigma . Pp; \end{aligned} \tag{30}$$

that is, the moment of the resultant couple is equal to the sum of the moments of the several component couples.

If some of the couples are negative, the forces belonging to them will in (29) have negative signs, and R will be equal to the difference of the forces which have positive signs and of those which have negative signs: and the same result will appear in (30), so that the right-hand member symbolizes the algebraical sum.

The moment axis of the resultant is equal to the sum of the moment axes of the component couples.

A close analogy therefore exists between parallel forces of translation applied at the same point and coaxal couples: in either case the effect of the resultant is equal to the algebraical sum of the effects of the components. We shall trace this analogy further in the succeeding Article. As to the geometrical representatives of the effects, in the case of couples the moment-axis may be transferred parallel to itself in any manner; in the case of forces of translation, the representative line can, by the principle of transmissibility, be transferred only along its own line of action.

43.] THEOREM V. If two lines meeting at a point represent the moment-axes of two couples, the diagonal of the parallelogram originating at the same point, and of which the two lines are adjacent sides, will represent the moment-axis of a single equivalent couple.

Suppose that two couples act in planes which are inclined to each other at an angle $\gamma$; let the couples be transferred in their own planes so as to have the same arm lying along the line of intersection of the two planes; let the forces of the couples thus transferred be P and Q. And, fig. 25, let AB be the common arm, and let us suppose it to lie in the plane of the paper: then compounding P and Q at A into a single force R, and P and Q at B in the same way, since $\text{PAQ} = \gamma$, we have

$$\text{R}^2 = \text{P}^2 + 2\,\text{PQ}\cos\gamma + \text{Q}^2; \tag{31}$$

and the R at B is equal and parallel to the R at A. At A draw A$a$, A$b$ perpendicular respectively to the planes PBAP, QBAQ, and of lengths equal to the moment-axes of the couples; complete the parallelogram A$acb$, and draw the diagonal A$c$; then A$c$ is the moment-axis of the resultant couple whose arm is AB and whose force is R. For since $\text{A}a = \text{P} \times \text{AB}$, and $\text{A}b = \text{Q} \times \text{AB}$, therefore A$a$ and A$b$ are proportional to P and Q, that is to AP and AQ; and they are also perpendicular to these lines, and are in the same plane with them; therefore the diagonal A$c$ is perpendicular, and proportional in the same ratio, to AR; therefore $\text{A}c = \text{R} \times \text{AB}$, and is the moment-axis of the resultant couple. Therefore, if A$a$ and A$b$ are the moment-axes of two couples, A$c$, the diagonal of the parallelogram of which A$a$ and A$b$ are the two containing sides, is the moment-axis of the resultant couple. Hence if L and M are the moment-axes of two couples, and are inclined to each other at an angle $\gamma$, and if G is the moment-axis of the resultant couple,

$$\text{G}^2 = \text{L}^2 + 2\,\text{L}\,\text{M}\cos\gamma + \text{M}^2. \tag{32}$$

Attention must of course be paid to the direction of the couple; thus, if A$a$ is the moment-axis, to an eye placed at A and looking along A$a$, the couple is right-handed.

Hence also we are authorized to resolve a couple whose moment-axis is given into any two couples, such that their moment-axes are the sides of the parallelogram of which the given moment-axis is the diagonal.

44.] Suppose that the moment-axes of two couples are perpendicular to each other; then $\gamma = 90°$; and therefore by (32)

$$\text{G}^2 = \text{L}^2 + \text{M}^2; \tag{33}$$

and let the rotation-axis of G make an angle $\lambda$ with that of L, then

$$\text{L} = \text{G}\cos\lambda, \qquad \text{M} = \text{G}\sin\lambda, \tag{34}$$

$$\tan\lambda = \frac{\text{M}}{\text{L}}; \tag{35}$$

a couple therefore whose moment-axis is G may be resolved into any two couples such that their moment-axes are the sides of the rectangle whose diagonal is the given moment-axis.

Hence also a couple, whose moment-axis is equal to G, but is in an opposite direction, neutralizes L and M, and the whole system is in equilibrium.

Also from (32) by a process analogous to that of Article 19 we can shew that if, fig. 26, OL, OM, ON represent the moment-axes of three couples L, M, N; and if $\text{MON} = \alpha$, $\text{NOL} = \beta$, $\text{LOM} = \gamma$, and if

$$\frac{\text{L}}{\sin\alpha} = \frac{\text{M}}{\sin\beta} = \frac{\text{N}}{\sin\gamma}, \tag{36}$$

then the three couples are in equilibrium; and conversely, if three couples are in equilibrium, the moment-axis of each is proportional to the sine of the angle contained between the rotation-axes of the other two.

Hence also if many couples acting on a rigid body are in equilibrium, their rotation-axes are parallel to the sides of a closed polygon, the sides themselves being the moment-axes.

Finally therefore we conclude that couples may by means of their moment-axes, which are their geometrical representatives, be resolved and compounded according to the same laws as forces of translation by means of their equivalent lines of action. It is to M. Poinsot that we are indebted for this great simplification of a problem which it is very difficult to follow in its complex form.

We return now to the consideration of the composition of pressures acting on a rigid body in one plane, and of the conditions of equilibrium of such pressures.

45.] Composition of many parallel forces acting on a rigid body or a rigid system of material particles in one plane.

Let the plane in which the forces act be the plane of $xy$; and let the origin O be, fig. 27, any point which is in, or rigidly connected with, the body; and let the forces be $P_1, P_2, \dots P_n$, of which let P be the type: let $p_1, p_2, \dots p_n$ be the perpendiculars from the origin on their lines of action, of which let $p$ be the type-perpendicular: let $x, y$ be the current coordinates to the line of action of the type-force P, and let $a$ be the angle between the line of action of P and the axis of $x$: then the equation to the line of action of P is

$$x \sin a - y \cos a - p = 0. \tag{37}$$

Let two forces each equal to P, with their lines of action parallel to that of P, and acting in opposite directions, be introduced at the origin O; so that instead of the original force P, we have P acting at O in a parallel line and the same direction, and a couple whose moment is P$p$ and whose rotation-axis is perpendicular to the plane of the forces.

Let P at O be resolved into two forces along the coordinate axes, viz. P cos $a$, and P sin $a$; and let all the forces be similarly transformed; then, if X and Y are the resultants of the forces severally along the axes of $x$ and $y$,

$$\begin{aligned} X &= P_1 \cos a + P_2 \cos a + \dots + P_n \cos a \\ &= \cos a \, \Sigma . P, \end{aligned} \tag{38}$$

$$\begin{aligned} Y &= P_1 \sin a + P_2 \sin a + \dots + P_n \sin a \\ &= \sin a \, \Sigma . P. \end{aligned} \tag{39}$$

Also the moment of the couple arising from P is equal to $\text{P}p$, the tendency of which is to turn the body from the axis of $x$ towards that of $y$; and, as a similar couple and moment will arise from every one of the forces, we shall, by reason of Art. 42, if G is the moment of the resultant couple, have

$$\begin{aligned} \text{G} &= \Sigma.\text{P}p \\ &= \Sigma.\text{P}(x \sin a - y \cos a) \\ &= \sin a\, \Sigma.\text{P}x - \cos a\, \Sigma.\text{P}y, \end{aligned} \tag{40}$$

placing $\sin a$ and $\cos a$ outside the signs of summation, because they are the same for all the forces: and observing that $x$ and $y$ refer to some point in the line of action of each pressure, and which will generally be different for each. It is worth noticing that in (40) G consists of two parts, which are affected with different signs; the resultant couple therefore is the difference between the resultants of two systems of couples acting in contrary directions: $\sin a\, \Sigma.\text{P}x$ tend to turn the body from the axis of $x$ towards that of $y$, and $\cos a\, \Sigma.\text{P}y$ act in the opposite direction.

Suppose now that all the forces are capable of being reduced to a single force R; or, in other words, suppose that one force R will have the same effect on the rigid body as all the impressed forces taken in combination. Let $\alpha$ be the angle at which the line of action of R is inclined to the axis of $x$, and let $\bar{x}$, $\bar{y}$ be the current coordinates of the line of action of R, and $\bar{r}$ the perpendicular distance from the origin on it. Then introducing at O two forces, each equal to R, with their lines of action parallel to that of R, and acting in opposite directions, we have the force of translation R acting at the origin, and a couple $\text{R}\bar{r}$; whence, resolving R at the origin along the coordinate axes, and equating the resolved parts to the sum of the resolved parts of the impressed pressures, we have

$$\left.\begin{aligned} \text{R} \cos \alpha &= \Sigma.\text{P} \cos a = \cos a\, \Sigma.\text{P} \\ \text{R} \sin \alpha &= \Sigma.\text{P} \sin a = \sin a\, \Sigma.\text{P} \end{aligned}\right\}; \tag{41}$$

therefore

$$\text{R} = \Sigma.\text{P}, \qquad \alpha = a; \tag{42}$$

that is, the resultant is equal to the algebraical sum of the components, and its line of action is parallel to those of the several components.

Again, the couple $\text{R}\bar{r}$ must be equal to G;

$$\therefore \quad \text{R}(\bar{x} \sin a - \bar{y} \cos a) = \text{G},$$

$$\bar{x} \sin a - \bar{y} \cos a = \frac{\text{G}}{\text{R}}, \tag{43}$$

and as $\bar{x}\ \bar{y}$ are the current coordinates of the line of action of

R, this is the equation of the line; and the length of the perpendicular from the origin on it is

$$\bar{r} = \frac{G}{R} = \frac{\Sigma.Pp}{R} = \frac{\Sigma.Pp}{\Sigma.P}, \tag{44}$$

$$\therefore \quad R\bar{r} = \Sigma.Pp; \tag{45}$$

that is, the moment of the resultant couple is equal to the algebraical sum of the moments of the component couples.

**46.]** The magnitude and position of the line of action of R are of course independent of any particular system of coordinate axes; and therefore, as $a$ is indeterminate, we have the following property of a system of parallel forces.

Suppose the forces to act at definite points of their lines of action, and $(x_1, y_1)$ $(x_2, y_2) \dots (x_n, y_n)$ to be the points of application of $P_1, P_2, \dots P_n$, and $(\bar{x}, \bar{y})$ to be that of R; then (45) becomes

$$\begin{aligned} R(\bar{x} \sin a - \bar{y} \cos a) &= \Sigma.P(x \sin a - y \cos a) \\ &= \sin a \, \Sigma.Px - \cos a \, \Sigma.Py, \end{aligned} \tag{46}$$

$$\therefore \quad \frac{R\bar{x} - \Sigma.Px}{\cos a} = \frac{R\bar{y} - \Sigma.Py}{\sin a}; \tag{47}$$

and as $a$ is indeterminate the numerator of each of these fractions must vanish: therefore

$$\left.\begin{aligned} \bar{x} &= \frac{\Sigma.Px}{R} = \frac{\Sigma.Px}{\Sigma.P} \\ \bar{y} &= \frac{\Sigma.Py}{R} = \frac{\Sigma.Py}{\Sigma.P} \end{aligned}\right\}. \tag{48}$$

The point of which these are the coordinates is called *the centre of parallel forces;* being independent of the direction of the line of action of the forces, and therefore the same whatever that line is, provided that their points of application are the same.

Ex. 1. Suppose six parallel pressures proportional to the numbers 1, 2, ... 6 to act at points whose coordinates are severally $(-2, -1)$, $(-1, 0)$, $(0, 1) \dots (3, 4)$; find the resultant, and its point of application.

$$\begin{aligned} R = \Sigma.P &= 1+2+\dots+6 \\ &= 21 \\ \Sigma.Px &= -2-2+4+10+18 \\ &= 28 \\ \Sigma.Py &= -1+3+8+15+24 \\ &= 49, \end{aligned}$$

$$\therefore \quad \bar{x} = \frac{28}{21}, \qquad \bar{y} = \frac{49}{21}.$$

47.] Given the equations to the lines of action of many parallel forces acting on a rigid body in one plane, it is required to find the equation to the line of action of the resultant.

Let us employ the abridged form of the equation to a straight line; and let the equations to the lines of action of $P_1, P_2, \dots P_n$ be

$$a_1 = 0, \quad a_2 = 0, \quad \dots \quad a_n = 0, \tag{49}$$

where $a$ is the length of the perpendicular from any point $(x, y)$ on the line of action of $P$. Now since $R\bar{r} = G$, it is plain that in reference to any point in the line of action of the resultant, $G = 0$; therefore

$$P_1 a_1 + P_2 a_2 + \dots + P_n a_n = \Sigma . P a = 0, \tag{50}$$

which is the equation to the line of action of $R$; and written at length is

$$x \cos \alpha \, \Sigma . P + y \sin \alpha \, \Sigma . P - \Sigma . P p = 0; \tag{51}$$

and therefore the perpendicular distance from the origin on the line of action of $R$ is

$$\frac{\Sigma . P p}{\Sigma . P}.$$

If $\Sigma . P = 0$, and $\Sigma . P p$ is a finite quantity, then $R = 0$, and the line of action of $R$ is at an infinite distance; in which case, by reason of the explanation given in Article 37, the forces are reducible to a couple, but not to a single force of translation; and the moment of the couple is $\Sigma . P p$.

Suppose however that the impressed forces are so related as to keep the body on which they act at rest: this condition will be fulfilled, if (1) any one point of the body is at rest, and (2) if there is no pressure of rotation about that point. Let this point be the origin; then if it is at rest

$$\Sigma . P = 0; \tag{52}$$

and if there is no pressure of rotation about that point, $G = 0$, and therefore

$$\Sigma . P p = 0; \tag{53}$$

(52) and (53) are therefore the conditions requisite that many parallel forces acting on a rigid body in one plane should be in equilibrium.

48.] Composition of many pressures acting in one plane on a rigid body or a rigid system of material particles.

Let the plane in which the forces act be that of $xy$; and let O, the origin, fig. 27, be a point of the body, or rigidly connected with it: let the forces be $P_1, P_2, \dots P_n$: let $\alpha_1, \alpha_2, \dots \alpha_n$ be the angles between their lines of action and the axis of $x$: let

$p_1, p_2, \dots p_n$ be the lengths of the perpendiculars drawn from the origin on the lines of action: and of these quantities let P, $a$, and $p$ be the types: so that

$$p = x \sin a - y \cos a. \tag{54}$$

At O let there be introduced two forces equal to P, with their lines of action parallel to that of P, and in opposite directions; so that, in the place of the original force P, we have P acting at O in a parallel line and the same direction, and a couple whose moment is P$p$, and whose rotation-axis is perpendicular to the plane of the forces. Let P at O be resolved into parts along the coordinate axes, so that we have P cos $a$ along the axis of $x$, and P sin $a$ along that of $y$; and let all the forces be similarly transformed. Then if X and Y are the sums of the resolved parts of the forces along the axes of $x$ and $y$ respectively,

$$\begin{aligned} \text{X} &= \text{P}_1 \cos a_1 + \text{P}_2 \cos a_2 + \dots + \text{P}_n \cos a_n \\ &= \Sigma.\text{P} \cos a, \end{aligned} \tag{55}$$

$$\begin{aligned} \text{Y} &= \text{P}_1 \sin a_1 + \text{P}_2 \sin a_2 + \dots + \text{P}_n \sin a_n \\ &= \Sigma.\text{P} \sin a. \end{aligned} \tag{56}$$

Also the moment of the couple arising from P is P$p$: the tendency of which is to turn the body from the axis of $x$ towards that of $y$; and as a similar couple will arise from each one of the forces, and as all the couples are coaxal, the moment of their resultant is equal to the sum of the moments of the components. Let G be the moment of the resultant couple; then

$$\begin{aligned} \text{G} &= \text{P}_1 p_1 + \text{P}_2 p_2 + \dots + \text{P}_n p_n \\ &= \Sigma.\text{P}p \\ &= \Sigma.\text{P}\,(x \sin a - y \cos a) \\ &= \Sigma.\text{P}x \sin a - \Sigma.\text{P}y \cos a. \end{aligned} \tag{57}$$

Now suppose R to be a force which will produce on the body the same effect as the system of the impressed forces; let $a$ be the angle between the line of action of R and the axis of $x$, and let $r$ be the perpendicular distance from the origin on the line of action of R: then introducing at the origin two forces equal to R, with their lines of action parallel to that of R, and acting in opposite directions, and combining one of these with the original R, so that a couple may be formed, and resolving the other along the coordinate axes, since R is the resultant of all the forces, we have

$$\left.\begin{aligned} \text{R} \cos a = \text{X} = \Sigma.\text{P} \cos a \\ \text{R} \sin a = \text{Y} = \Sigma.\text{P} \sin a \end{aligned}\right\}; \tag{58}$$

$$\therefore \quad R^2 = X^2 + Y^2, \tag{59}$$

$$\tan a = \frac{Y}{X}; \tag{60}$$

whereby we have the magnitude and the direction of the line of action of R.

And to find its equation; let $x$ and $y$ be the current coordinates, then since $Rr = G$, and

$$r = x \sin a - y \cos a,$$

$$\therefore \quad xR \sin a + yR \cos = G, \tag{61}$$

$$\text{or,} \quad xY - yX = G; \tag{62}$$

either of which equations is that of the line along which R acts.

The equation to the line of action of R may also be expressed in terms of the equations of the lines of action of the components. Thus let the impressed forces act along lines whose equations are $a_1 = 0, a_2 = 0, \ldots a_n = 0$; then the equation to the line of action of R is

$$P_1 a_1 + P_2 a_2 + \ldots + P_n a_n = 0, \tag{63}$$

$$x \Sigma . P \cos a + y \Sigma . P \sin a = \Sigma . Pp. \tag{64}$$

49.] Now let the impressed forces be so related as to magnitude, line of action and direction, that the body is at rest: that is, so that (1) the origin is at rest, and (2) there is no pressure of rotation tending to turn the body about an axis passing through the origin; then for these two conditions it is necessary that $R = 0$, and $G = 0$; and therefore

$$\left.\begin{aligned} \Sigma . P \cos a &= 0 \\ \Sigma . P \sin a &= 0 \\ \Sigma . Pp &= 0 \end{aligned}\right\}; \tag{65}$$

and therefore as the origin is arbitrary, and the directions of the axes are arbitrary, a system of forces acting in one plane on a body is in equilibrium, if the sums of the resolved parts of the forces along any two straight lines in the plane perpendicular to each other vanish, and if the sum of the moments of the forces about any point in the plane also vanishes.

If $R = 0$, and G is finite, the forces are reduced to a single couple, whose moment is G or $\Sigma . Pp$; if $G = 0$, and R is finite, the resultant force of translation passes through the origin; and if R and G are finite, then, as the last Article shews, the forces can be reduced to a single force of translation.

If one point of the body on which the forces act is fixed, and if that point is in the plane of the forces, it will of course bear all the pressures of translation, and the only motion of which the body is capable is that of rotation about the point. If therefore the point be taken as the origin of coordinates, the conditions of equilibrium become reduced to the single one

$$\text{G} = 0.$$

The examples in which the equations of equilibrium (65) are applied are extremely numerous; and a large supply will be found in any of the ordinary collections; it is desirable however to insert a few, that the reader may understand the mode of application.

50.] Ex. 1. A heavy uniform beam AB rests in a vertical plane, fig. 28, with one end A on a smooth horizontal plane and the other end B against a smooth vertical wall: the end A is prevented from sliding by a horizontal string of given length fastened to the end of the beam and to the wall: determine the tension of the string and the pressures against the horizontal plane and the wall.

Let the length of the beam be $2a$, and let W be its weight; which, as the beam is uniform, we may suppose to act at its middle point G; let R be the vertical pressure of the horizontal plane against the beam; and R′ the horizontal pressure of the vertical wall, and T the tension of the horizontal string AC; let $\text{BAC} = \alpha$, which is a known angle, as the lengths of the beam and the string are given. Then equations (65) become,

$$\begin{array}{ll} \text{Horizontal forces;} & \text{T} = \text{R}', \\ \text{Vertical forces;} & \text{W} = \text{R}, \\ \text{Moments about A;} & \text{W}a\cos\alpha = \text{R}'2a\sin\alpha; \end{array}$$

$$\therefore \quad \text{R}' = \text{T} = \frac{\text{W}}{2}\cot\alpha.$$

Ex. 2. A heavy uniform beam rests on two given smooth inclined planes: it is required to find the position of the beam, and the pressures on the planes.

Let AB, fig. 29, be the beam, whose length is $2a$, and whose weight is W acting at the centre of gravity G: let the inclinations of the planes AC and BC to the horizon be respectively $\alpha$ and $\beta$; and let the inclination of the beam be $\theta$; let R and R′ be the pressures of the planes on the beam, and the lines of action of which are perpendicular to the planes by reason of their smoothness. Then we have

Horizontal forces; $\text{R} \sin \alpha = \text{R}' \sin \beta$;
Vertical forces; $\text{W} = \text{R} \cos \alpha + \text{R}' \cos \beta$;
Moments about G; $\text{R} a \cos (\alpha - \theta) = \text{R}' a \cos (\beta + \theta)$;

$$\therefore \quad \tan \theta = \frac{\sin (\alpha - \beta)}{2 \sin \alpha \sin \beta},$$

$$\text{R} = \frac{\text{W} \sin \beta}{\sin (\alpha + \beta)}, \qquad \text{R}' = \frac{\text{W} \sin \alpha}{\sin (\alpha + \beta)}.$$

Ex. 3. An uniform heavy beam AB, fig. 30, rests with one end A against a smooth vertical wall, and the other B is fastened by a string BC of given length to a point C in the wall; the beam and the string are in a vertical plane: it is required to determine the pressure against the wall, the tension of the string, and the position of the beam and the string.

Let $\text{AG} = \text{GB} = a, \quad \text{AC} = x, \quad \text{BC} = b,$

Weight of beam = W, Tension of string = T, Pressure of wall = R,

$$\text{BAE} = \theta, \qquad \text{BCA} = \phi;$$

Horizontal forces; $\text{R} = \text{T} \sin \phi$;
Vertical forces; $\text{W} = \text{T} \cos \phi$;
Moments about A; $\text{W} a \sin \theta = \text{T} x \sin \phi$;

$$\therefore \quad a \sin \theta = x \tan \phi;$$

and, by the geometry of the figure,

$$\frac{b}{\sin \theta} = \frac{2a}{\sin \phi} = \frac{x}{\sin (\theta - \phi)},$$

$$\therefore \quad x = \left\{ \frac{b^2 - 4a^2}{3} \right\}^{\frac{1}{2}},$$

$$\cos \phi = \frac{2}{b} \left\{ \frac{b^2 - 4a^2}{3} \right\}^{\frac{1}{2}},$$

$$\sin \theta = \frac{1}{2a} \left\{ \frac{16a^2 - b^2}{3} \right\}^{\frac{1}{2}};$$

whence R and T are known.

Ex. 4. A system of forces acting on a rigid body in one plane is represented by the sides of a closed polygon taken in order; it is required to determine the resultant.

Let some point within the polygon be taken for the origin, and two lines drawn perpendicularly to each other for coordinate axes. Let the lengths of the sides of the polygon be $s_1$, $s_2, \ldots s_n$; and let their angles of inclination to the axis of $x$ be $\alpha_1$,

$a_2, \dots a_n$, and the perpendiculars from the origin on the lines of action be $p_1, p_2, \dots p_n$: at the origin let pairs of equal and opposite forces be introduced, equal and parallel to those along the sides of the polygon: so that the system is changed into (1) a system of forces acting at the origin, which are in equilibrium by reason of Article 26, and (2) a system of coaxal couples, the moment of the resultant of which is equal to $s_1 p_1 + s_2 p_2 + \dots + s_n p_n$; that is, to a moment of which the geometrical representative is twice the area of the polygon.

A particular case is that of a triangle, whose sides are geometrical representatives of three impressed forces; of which the resultant of translation vanishes, and the moment of the resultant couple is represented by twice the area of the triangle.

Ex. 5. A heavy and smooth circular ring rests on two horizontal bars, which are not in the same horizontal plane: determine the pressure on each bar.

Let fig. 31 represent a vertical section of the system; P and Q being the two bars, R and R′ the pressures of the ring against them, W the weight of the ring acting at its centre O; let the angle $\text{POQ} = a$, which is known; and let the angles of inclination to the vertical of the lines of action of R and of R′ be $\beta$ and $\gamma$; then, as the three forces meet in the centre of the ring, we have

$$\frac{\text{R}}{\sin\gamma} = \frac{\text{R}'}{\sin\beta} = \frac{\text{W}}{\sin a}.$$

Ex. 6. A parabolic curve, fig. 32, is placed in a vertical plane with its axis vertical and vertex downwards, and inside of it and against a peg in the focus a smooth uniform and heavy beam rests: required the position of rest.

Let PQ be the beam of length $2c$ and of weight W; let $\text{SA} = a$, $\text{SP} = r$, $\text{PSA} = \theta$,

$$\therefore \quad r = \frac{2a}{1+\cos\theta};$$

$$\text{also} \qquad \text{SPT} = \text{STP} = 90^\circ - \frac{\theta}{2}; \qquad \text{PG} = \text{GQ} = c,$$

$$\left.\begin{array}{lr} \text{Forces along PQ;} & \text{R}\sin\text{SPT} = \text{W}\cos\theta, \\ \text{Moments about S;} & \text{R}r\cos\text{SPT} = \text{W}(r-c)\sin\theta, \end{array}\right\} \qquad (66)$$

$$\therefore \quad \theta = 2\cos^{-1}\left(\frac{a}{2c}\right)^{\frac{1}{4}}.$$

Suppose that it were required to find the curve AP such that the beam should rest in all positions; then $\tan\text{SPT} = r\dfrac{d\theta}{dr}$; therefore from (66)

$$\frac{d\theta}{dr} = \frac{\cos\theta}{(r-c)\sin\theta}, \qquad \therefore \quad r = c + a\sec\theta, \tag{67}$$

where $a$ is an arbitrary constant; and this is the equation to the conchoid with an arbitrary modulus.

Ex. 7. To discuss the properties and conditions of equilibrium of a balance, fig. 33.

Let AB be the arm of the balance; $AC = CB = a$; and let the balance be suspended by a point O in a line perpendicular to AB at its middle point C, and let $OC = c$; let the balance be symmetrical with respect to the line OC, and let the centre of gravity of the beam, scales, &c. be at G; let $OG = h$, and let the weight of the whole machine, short of the weights in the scales, $= W$; and to consider the general case suppose the weights in the scales P and Q to be unequal, Q being greater than P; and let the arm of the balance be inclined to the horizontal line at an angle $\theta$. Then

$$\text{Vertical pressure on O} = P + Q + W;$$

and taking moments about O,

$$Q(a\cos\theta - c\sin\theta) = P(a\cos\theta + c\sin\theta) + Wh\sin\theta;$$

$$\therefore \quad \tan\theta = \frac{(Q-P)a}{c(P+Q)+Wh}. \tag{68}$$

Now the conditions required in a balance are (1) horizontality of the beam when the arms and weights are equal; (2) sensibility, which is estimated by the angle through which the arm is turned when the weights are unequal; (3) stability, or the tendency to return after the cause of displacement is removed.

Condition (1) is fulfilled if $Q = P$, since, by (68), in that case, $\theta = 0$.

Condition (2) is more or less satisfied according as $\theta$ is larger or smaller for a small difference between P and Q; now in (68), if $Q - P$ is very small, $\tan\theta$, and therefore $\theta$, is large,

(1) if $a$ is large, that is, if the arms of the balance are long;

(2) if $c$ is small, that is, if the point of suspension is not far above the beam;

(3) if $P + Q$ is small, that is, if the weights are small;

(4) if $W$ is small, that is, if the weight of the whole balance is small;

(5) if $h$ is small, that is, if the centre of gravity of the machine is not far below the beam;

and either $c$ or $h$ or both may be negative; and then as a limiting case we may have $\tan\theta = \infty$, and $\theta = 90°$; in which case the beam becomes vertical when it is displaced at all, and may have no tendency to return to its horizontal position; and thus the sensibility of the balance may be very great, but there may be no stability, and one of the necessary conditions is not satisfied: this last condition therefore may be inconsistent with the second, and the two must be adjusted as is practically most convenient.

51.] Although in all cases it is possible, and in most cases scarcely less general, to refer forces and conditions of equilibrium to rectangular coordinates, yet it is desirable to indicate the forms which the reduced resultants take, if the coordinate axes are oblique.

Let the angle of ordination be $\omega$; let the forces be $\text{P}_1, \text{P}_2, \dots \text{P}_n$; $(x_1, y_1), (x_2, y_2), \dots (x_n, y_n)$ their points of application; $p_1, p_2, \dots p_n$ the perpendiculars from the origin on their lines of action; $a_1\beta_1$, $a_2\beta_2, \dots a_n\beta_n$ the angles between the perpendiculars to the lines of action and the axes of $x$ and $y$ respectively; then, employing the symbols without any subscripts as the type-symbols, we have for the line of action of P

$$x\cos a + y\cos\beta - p = 0. \tag{69}$$

Let two equal and opposite forces, each of which is equal to P and has its line of action parallel to that of P, be introduced at the origin; so that, instead of the one force P applied at $(x, y)$, there are (1) a parallel and equal force at the origin, (2) a couple whose arm is $p$ and whose force is P. Let the former be resolved into parts along the coordinate axes, viz. $-\text{P}\sin a$, and $-\text{P}\sin\beta$; and let all the forces be similarly reduced; let X and Y be the sums of the resolved parts along the axes of $x$ and $y$ respectively; then

$$\begin{aligned} -\text{X} &= \text{P}_1\sin a_1 + \text{P}_2\sin a_2 + \dots + \text{P}_n\sin a_n \\ &= \Sigma.\text{P}\sin a, \\ -\text{Y} &= \text{P}_1\sin\beta_1 + \text{P}_2\sin\beta_2 + \dots + \text{P}_n\sin\beta_n \\ &= \Sigma.\text{P}\sin\beta; \end{aligned} \tag{70}$$

and therefore if R is the resultant of X and of Y,

$$\text{R}^2 = \text{X}^2 + 2\text{X}\text{Y}\cos\omega + \text{Y}^2.$$

And let G be the moment of the resultant couple: then

$$\begin{aligned} \text{G} &= \text{P}_1 p_1 + \text{P}_2 p_2 + \dots + \text{P}_n p_n \\ &= \Sigma.\text{P}p \\ &= \Sigma.\text{P}(x\cos a + y\cos\beta). \end{aligned}$$

If the impressed forces are in equilibrium, $R = 0$, and $G = 0$;

$$\therefore \quad \Sigma . P \sin \alpha = 0, \qquad \Sigma . P \sin \beta = 0, \qquad \Sigma . P p = 0.$$

If the equations to the lines of action of the impressed forces are given, that to the line of action of the resultant may thus be found; let the equation to the lines of action of the forces be

$$\left. \begin{array}{r} x \cos \alpha_1 + y \cos \beta_1 - p_1 = 0 \\ x \cos \alpha_2 + y \cos \beta_2 - p_2 = 0 \\ \cdots\cdots\cdots\cdots \\ x \cos \alpha_n + y \cos \beta_n - p_n = 0 \end{array} \right\} ;$$

then in reference to any point in the line of action of the resultant, $\Sigma . P p = 0$; therefore we have

$$\Sigma . P (x \cos \alpha + y \cos \beta - p) = 0,$$

$$x \, \Sigma . P \cos \alpha + y \, \Sigma . P \cos \beta - \Sigma . P p = 0.$$

52.] In Article 46 it is proved that if a system of parallel forces acts on a body at given points, the point of application and the intensity of the resultant are the same, whatever is the direction of the line of action of the forces; and this property of the centre of parallel forces (as the point of application of the resultant is called) induces us to inquire, whether (1) at all, (2) under what circumstances, a system of forces acting in a plane on a rigid body at definite points and being in equilibrium, will also be in equilibrium when the forces are applied at the same points as before, and their lines of action are inclined, all at the same angle to their former lines of action; or, in other words, when the body is turned through any angle about an axis perpendicular to the plane of the forces, and the forces are applied at the same points of the body, and in lines of action parallel to their former ones.

Let the body be referred to coordinate axes, which are fixed with respect to the body and to space; so that the conditions of equilibrium of the body thus referred are

$$\left. \begin{array}{l} \Sigma . P \cos \alpha = 0, \qquad \Sigma . P \sin \alpha = 0, \\ \quad \Sigma . P (x \sin \alpha - y \cos \alpha) = 0. \end{array} \right\} \qquad (71)$$

Now (1) let the body be shifted so that every point of it is moved over a distance equal to $a$ parallel to the axis of $x$, and over a distance equal to $b$ parallel to the axis of $y$; and (2) let the body be turned through an angle $\theta$ about an axis perpendicular to the plane of $xy$; and let it after these displacements be referred to the original axes: then if $x'y'$ are the coordinates of that point of the body, of which the coordinates in the original position were $x$ and $y$, these being type-symbols,

$$\left.\begin{aligned} x' &= a + x\cos\theta - y\sin\theta \\ y' &= b + x\sin\theta + y\cos\theta \end{aligned}\right\}; \tag{72}$$

and as the line of action of each force is in its new position parallel to its former position, and as the forces are applied at the same points as before, and are still in equilibrium, we have

$$\Sigma.\mathrm{P}\,(x'\sin a - y'\cos a) = 0;$$

and therefore substituting from (72) and reducing, we have

$$a\,\Sigma.\mathrm{P}\cos a - b\,\Sigma.\mathrm{P}\sin a$$
$$+\cos\theta\,\Sigma.\mathrm{P}\,(x\sin a - y\cos a) - \sin\theta\,\Sigma.\mathrm{P}\,(x\cos a + y\sin a) = 0;$$

of which expression the first three terms vanish by reason of (71): and therefore the required condition is

$$\Sigma.\mathrm{P}\,(x\cos a + y\sin a) = 0; \tag{73}$$

and this is evidently such a condition as a system of forces may satisfy: and when it is satisfied, the requirements of the theorem are fulfilled.

And the condition (73) deserves closer consideration; hitherto we have resolved the forces along the coordinate axes of $x$ and $y$: now imagine a line (a radius vector) to be drawn from the origin to the point of application of each force; and let each force be resolved along its radius vector and perpendicular to it; and let $r_1, r_2, \dots r_n$ be the several radii vectores. Let U be the pressure along the radius vector, and acting from the origin, and which I shall call the *central* pressure; and let V be the pressure acting at right angles to the radius vector, and tending to increase the angle between that radius vector and the axis of $x$; U and V referring to P, and being type-symbols. Then

$$\begin{aligned} \mathrm{U} &= \mathrm{P}\cos a\frac{x}{r} + \mathrm{P}\sin a\frac{y}{r} \\ &= \frac{\mathrm{P}x\cos a + \mathrm{P}y\sin a}{r}; \end{aligned}$$

$$\mathrm{V} = \frac{\mathrm{P}x\sin a - \mathrm{P}y\cos a}{r};$$

$$\therefore\quad \left.\begin{aligned} \Sigma.\mathrm{P}\,(x\sin a - y\cos a) &= \Sigma.r\mathrm{V} = \mathrm{G} \\ \Sigma.\mathrm{P}\,(x\cos a + y\sin a) &= \Sigma.r\mathrm{U} \end{aligned}\right\}; \tag{74}$$

of which two expressions, the first is evidently the moment of the resultant couple; and the second, depending on forces all of which act from or towards the same point (the origin), I shall call *the central moment;* some few properties of which are investigated in the following Article. And I would in the first place observe, that if the moment of the resultant couple and also the central moment vanish, the system is subject to the condition of the Theorem at the beginning of this Article; and the point at which the expressions (74) vanish is called the *centre of the forces* of the system.

And the centre of forces has also the following property: Suppose a system of forces, to act on a body in one plane, to have a centre, and to be capable of reduction to a single force of translation; then if the body is turned about an axis perpendicular to the plane of the forces, and if the forces act in their former points of application, and along lines of action parallel to the former lines, the line of action of the resultant, whatever is the angle through which the body is turned, will always pass through the centre; and the centre of forces therefore is the point at which it may always be applied, and about which its line of action will turn.

53.] (1) Since $\Sigma . r \text{U} = \Sigma . \text{P} x \cos a + \Sigma . \text{P} y \sin a,$

it appears that the central moment of the whole system is equal to the sum of the central moments of the two systems of the resolved forces along the axes.

(2) It is evident that the value of $\Sigma . r \text{U}$ is not altered, if the origin remains the same, whatever is the position of the coordinate axes.

(3) Let the origin be moved to the point $(a, b)$; so that

$$x = a + x', \qquad y = b + y';$$

then
$$\begin{aligned} \Sigma . r\text{U} &= \Sigma . \text{P}(x' \cos a + y' \sin a) + a\, \Sigma . \text{P} \cos a + b\, \Sigma . \text{P} \sin a \\ &= \Sigma . r'\text{U}' + a\, \Sigma . \text{P} \cos a + b\, \Sigma . \text{P} \sin a; \qquad (75) \end{aligned}$$

the central moment therefore varies as we pass from one origin to another.

Suppose $(a, b)$ to be a point at which the central moment vanishes; that is, at which $\Sigma . r'\text{U}' = 0$; then

$$a\, \Sigma . \text{P} \cos a + b\, \Sigma . \text{P} \sin a = \Sigma . r\text{U}; \qquad (76)$$

and as this is the only relation to which $a$ and $b$ are subject, it is the equation to a straight line; and therefore at any point in this straight line the central moment vanishes. The line is called the line of central moments.

The equation of the line of action of the resultant of the system by reason of (62) is

$$x\text{Y}-y\text{X} = \text{G},$$

and as the equation to the above line is

$$x\text{X}+y\text{Y} = \Sigma.r\text{U},$$

it follows that the line of central moments is perpendicular to the line of action of the resultant. The centre of the forces is evidently the point in which these two lines intersect; and we have therefore, if $x_0\, y_0$ are its coordinates,

$$x_0 = \frac{\text{X}\Sigma.r\text{U}+\text{GY}}{\text{R}^2}, \qquad y_0 = \frac{\text{Y}\Sigma.r\text{U}-\text{GX}}{\text{R}^2}. \qquad (77)$$

If the lines of action of the impressed forces are parallel to each other,

$$x_0 = \frac{\Sigma.\text{P}x}{\Sigma.\text{P}}, \qquad y_0 = \frac{\Sigma.\text{P}y}{\Sigma.\text{P}}.$$

If the resultant of translation = 0, from (75) it follows that the central moment is the same for all points in the plane of the forces.

The centre of two forces acting in a plane on two given points is determined in the following manner by a geometrical construction. Let the forces be P, Q, and let their points of application be A and B; let the lines of action of the forces meet in O; describe a circle passing through O, A, B; and let OC be the line of action of the resultant R, and let it cut the circle in C; then C is the centre of P, Q. Whatever is the position of O in the circumference of the circle between A and B, and suppose it to be at O′, the angles AO′B, BO′C, CO′A are equal severally to AOB, BOC, COA; and as the equilibrating relation between P, Q, R depends on these angles only, it is the same whatever is the position of O′: but in all cases C remains the same; therefore C is the centre of the forces.

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## SECTION 2.—*Composition and resolution of forces acting on a rigid body or system of material particles in any directions.*

54.] We proceed now to the most general case of statical pressures acting in any directions on a rigid body; and, as a preliminary of the investigation, to the composition of couples whose rotation-axes have any positions in space.

Let all the forces of the couples be brought into one or other of three planes which are perpendicular to each other; and let all the coaxal ones be compounded into a single resultant couple: so that there will be three couples whose rotation-axes are perpendicular to each other. Let these three lines meet at the point O, fig. 34, and form a system of rectangular coordinates in space: and let the moment-axes along the axes of $x$, $y$, and $z$ severally be L, M, N; then compounding L and M according to the law of Article 44, and calling their resultant moment-axis G′, we have

$$\text{L}^2+\text{M}^2 = \text{G}'^2;$$

and compounding G′ and N, which are perpendicular to each other, and calling the resultant moment-axis G, we have

$$\begin{aligned} \text{G}^2 &= \text{G}'^2+\text{N}^2 \\ &= \text{L}^2+\text{M}^2+\text{N}^2. \end{aligned} \tag{78}$$

Let $\lambda$, $\mu$, $\nu$ be the direction angles of G: then by the law of Article 44,

$$\text{L} = \text{G} \cos\lambda, \qquad \text{M} = \text{G}\cos\mu, \qquad \text{N} = \text{G}\cos\nu; \tag{79}$$

$$\therefore\quad \cos\lambda = \frac{\text{L}}{\text{G}}, \qquad \cos\mu = \frac{\text{M}}{\text{G}}, \qquad \cos\nu = \frac{\text{N}}{\text{G}}; \tag{80}$$

so that if L, M, N are given, we can find G and the direction of its axis; and if a moment-axis is given, we can resolve it into three component and rectangular moment-axes.

55.] Composition of many forces acting on a rigid body in space.

Let any point, either of the body, or rigidly connected with it, be taken as the origin, and at it let a system of rectangular coordinate axes originate. Let the forces be $\text{P}_1, \text{P}_2, \dots \text{P}_n$; the direction-angles of their lines of action, $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2, \dots \alpha_n\beta_n\gamma_n$; the coordinates to a point in the line of application of each, $x_1y_1z_1$, $x_2y_2z_2, \dots x_ny_nz_n$; the perpendiculars from the origin on their lines of action, $p_1, p_2, \dots p_n$; and of these quantities let the types be P, $\alpha\beta\gamma$, $xyz$, $p$. At the origin O, fig. 35, let there be introduced a pair of equal and opposite forces, each of which is equal to P, and has its line of action parallel to that of P; from O let the perpendicular OD $(=p)$ be drawn to the line of action of P: then instead of the original P, we have P at O equal to the former force and acting in the same direction along a parallel line of action, and a couple each of whose forces is P, whose arm is OD, and whose rotation-axis is perpendicular to the plane PODP: and let a similar process be performed on all the other forces. As to the forces of translation at O, let P be resolved into three components P cos $a$, P cos $\beta$, P cos $\gamma$ along the axes of $x, y, z$ respectively; and let X, Y, Z be the sums of the resolved parts of all the forces along these axes; then

$$\left.\begin{aligned} \mathrm{X} &= \mathrm{P}_1 \cos a_1 + \mathrm{P}_2 \cos a_2 + \dots + \mathrm{P}_n \cos a_n \\ &= \Sigma.\mathrm{P} \cos a \end{aligned}\right\}, \tag{81}$$

$$\left.\begin{aligned} \mathrm{Y} &= \mathrm{P}_1 \cos \beta_1 + \mathrm{P}_2 \cos \beta_2 + \dots + \mathrm{P}_n \cos \beta_n \\ &= \Sigma.\mathrm{P} \cos \beta \end{aligned}\right\}, \tag{82}$$

$$\left.\begin{aligned} \mathrm{Z} &= \mathrm{P}_1 \cos \gamma_1 + \mathrm{P}_2 \cos \gamma_2 + \dots + \mathrm{P}_n \cos \gamma_n \\ &= \Sigma.\mathrm{P} \cos \gamma \end{aligned}\right\}; \tag{83}$$

and therefore, if R is the resultant of the three forces,

$$\mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2; \tag{84}$$

and if $a, b, c$ are the direction-angles of the line of action of R,

$$\cos a = \frac{\mathrm{X}}{\mathrm{R}}, \quad \cos b = \frac{\mathrm{Y}}{\mathrm{R}}, \quad \cos c = \frac{\mathrm{Z}}{\mathrm{R}}; \tag{85}$$

so that the intensity, the line of action, and the direction, of R are known.

As to the couple which arises from P, its moment is P$p$: and, as $p$ is the perpendicular distance from the origin on a line passing through a point $(x, y, z)$, and having direction-angles $a, \beta, \gamma$,

$$p^2 = (y \cos \gamma - z \cos \beta)^2 + (z \cos a - x \cos \gamma)^2 + (x \cos \beta - y \cos a)^2, \tag{86}$$

and as the rotation-axis of the couple is perpendicular to the plane passing through the origin and containing this line, its direction-cosines are

$$\frac{y \cos \gamma - z \cos \beta}{p}, \quad \frac{z \cos a - x \cos \gamma}{p}, \quad \frac{x \cos \beta - y \cos a}{p}; \tag{87}$$

in accordance therefore with the law of the last Article let us resolve the moment-axis of the couple along the three coordinate axes; then the resolved parts are P $(y \cos \gamma - z \cos \beta)$, P $(z \cos a - x \cos \gamma)$, P $(x \cos \beta - y \cos a)$, which are the moment-axes of the three component couples, and whose rotation-axes are along the three coordinate axes. Let the couples corresponding to all the impressed forces be similarly resolved, and let L, M, N be the sums of the moment-axes of those couples whose rotation-axes are severally along the three coordinate axes: so that by reason of (30) Article 42,

$$\mathrm{L} = \mathrm{P}_1 (y_1 \cos \gamma_1 - z_1 \cos \beta_1) + \dots + \mathrm{P}_n (y_n \cos \gamma_n - z_n \cos \beta_n) \tag{88}$$

$$\text{similarly} \quad \left.\begin{aligned} \mathrm{L} &= \Sigma.\mathrm{P} (y \cos \gamma - z \cos \beta) \\ \mathrm{M} &= \Sigma.\mathrm{P} (z \cos a - x \cos \gamma) \\ \mathrm{N} &= \Sigma.\mathrm{P} (x \cos \beta - y \cos a) \end{aligned}\right\}; \tag{89}$$

and if G is the resultant moment-axis of these three couples, by reason of the last Article,

$$G^2 = L^2 + M^2 + N^2; \tag{90}$$

and if the direction-angles of the resultant rotation-axis are $\lambda, \mu, \nu$,

$$\cos\lambda = \frac{L}{G}, \quad \cos\mu = \frac{M}{G}, \quad \cos\nu = \frac{N}{G}; \tag{91}$$

so that both the moment-axis and the rotation-axis of the resultant couple are determined. It appears then that the forces are reduced to a force of translation, viz. R acting at the origin, and to a couple G, whose moment-axis is determined by (90) and (91).

56.] The formulæ (89) require closer consideration; the right-hand member of each of the equations consists of two parts, one of which is affected with a positive, and the other with a negative sign. Thus L is composed of two sets of coaxal couples, viz. $\Sigma . P y \cos\gamma$ and $-\Sigma . P z \cos\beta$; the former of which is the sum of a system of couples, the force in each of which is the $z$-component of the impressed force, and the arm is the $y$-ordinate of its point of application; and in the latter system, the force of each couple is the $y$-component of the impressed force, and the arm is the $z$-ordinate of its point of application. Imagine therefore the force P to be, at its point of application, resolved into three components along lines parallel to the coordinate axes; and let these be $P\cos\alpha$, $P\cos\beta$, $P\cos\gamma$; and let couples be considered positive, which having for their rotation-axes severally the coordinate axes of $x$, $y$, and $z$, tend to turn the body from the $y$-axis to the $z$-axis, from the $z$-axis to the $x$-axis, from the $x$-axis to the $y$-axis; and let those couples be negative which act in a contrary direction: which arrangement, it will be observed, is cyclical. Now consider $P\cos\gamma$; and, fig. 36, introduce at M and at O two equal and opposite forces, equal to it and acting parallel to its line of action; so that we have a parallel and equal force acting at O, and two couples, of one of which the arm is OM, and of the other the arm is MN; of which the former has the axis of $y$ for its rotation-axis and is negative, and the latter has the axis of $x$ for its rotation-axis and is positive; hence $P\cos\gamma$ acting at P is replaced by

A parallel and equal force $= P\cos\gamma$ acting at O,

And a couple whose moment is $P\cos\gamma\, y$, and whose rotation-axis is the axis of $x$, And a couple whose moment is $-\text{P}\cos\gamma\, x$, and whose rotation-axis is the axis of $y$.

By a similar process will $\text{P}\cos\alpha$ and $\text{P}\cos\beta$ be replaced: and the same process having been performed on all the impressed forces, we have ultimately

$\Sigma.\text{P}\cos\alpha$ acting at O along the axis of $x$,
$\Sigma.\text{P}\cos\beta$ - - - - - - - - - - - $y$,
$\Sigma.\text{P}\cos\gamma$ - - - - - - - - - - - $z$;

and the couples whose moments are
$\Sigma.\text{P}(y\cos\gamma - z\cos\beta)$, the rotation-axis of which is the axis of $x$,
$\Sigma.\text{P}(z\cos\alpha - x\cos\gamma)$, - - - - - - - - - - - - - - - $y$,
$\Sigma.\text{P}(x\cos\beta - y\cos\alpha)$, - - - - - - - - - - - - - - - $z$,

which results are the same as those investigated in the preceding Article.

The principle on which signs are affixed to couples is of course arbitrary; we have chosen the order which is most natural; in the preceding Article the conventionality of the sign and direction is involved in the sign of $p$ in (87), which may be either positive or negative.

57.] Now as a rigid body is at rest, when any one of its particles is at rest, and when there is no tendency to rotation about any axis passing through that point, so it follows that, if the point, which is to be at rest, is taken as the origin, $\text{R} = 0$ and $\text{G} = 0$; and therefore by (84) and (90)

$$\text{X} = 0, \qquad \text{Y} = 0, \qquad \text{Z} = 0, \tag{92}$$

$$\text{L} = 0, \qquad \text{M} = 0, \qquad \text{N} = 0; \tag{93}$$

or,
$$\Sigma.\text{P}\cos\alpha = 0, \qquad \Sigma.\text{P}\cos\beta = 0, \qquad \Sigma.\text{P}\cos\gamma = 0, \tag{94}$$

$$\left.\begin{array}{l} \Sigma.\text{P}(y\cos\gamma - z\cos\beta) = 0 \\ \Sigma.\text{P}(z\cos\alpha - x\cos\gamma) = 0 \\ \Sigma.\text{P}(x\cos\beta - y\cos\alpha) = 0 \end{array}\right\}; \tag{95}$$

that is, the sums of the resolved parts of the forces along any three rectangular axes vanish; and the sums of the moments of the couples whose rotation-axes coincide with the axes of any system of rectangular coordinates also vanish.

Also when equilibrium exists, $\text{X} = 0$, $\text{Y} = 0$, $\text{Z} = 0$; and multiplying these severally by undetermined quantities $x, y, z$ aud adding, we have

$$\text{X}x + \text{Y}y + \text{Z}z = 0;$$

and replacing X, Y, Z by their equivalents

$$\begin{aligned}
&\mathrm{P}_1(x\cos\alpha_1+y\cos\beta_1+z\cos\gamma_1)\\
&+\mathrm{P}_2(x\cos\alpha_2+y\cos\beta_2+z\cos\gamma_2)\\
&+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\\
&+\mathrm{P}_n(x\cos\alpha_n+y\cos\beta_n+z\cos\gamma_n)=0.
\end{aligned}$$

Now if $x, y, z$ are the coordinates of any point in space, $x\cos\alpha_1+y\cos\beta_1+z\cos\gamma_1$ is the projection on the line of action of $\mathrm{P}_1$ of the distance of $(x, y, z)$ from the origin; and therefore, as the origin also is an arbitrary point, we conclude that if a system of forces is in equilibrium, the sum of the products of each force and of the projection of a line joining two given points (fixed arbitrarily) on its line of action is equal to zero.

Also $\mathrm{L}=0$, $\mathrm{M}=0$, $\mathrm{N}=0$; and multiplying these severally by $x, y, z$ we have

$$\mathrm{L}x+\mathrm{M}y+\mathrm{N}z=0;$$

and expanding these we have

$$\begin{aligned}
&\mathrm{P}_1\{(y_1\cos\gamma_1-z_1\cos\beta_1)x+(z_1\cos\alpha_1-x_1\cos\gamma_1)y\\
&\qquad\qquad\qquad+(x_1\cos\beta_1-y_1\cos\alpha_1)z\}\\
&+\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\\
&+\mathrm{P}_n\{(y_n\cos\gamma_n-z_n\cos\beta_n)x+(z_n\cos\alpha_n-x_n\cos\gamma_n)y\\
&\qquad\qquad\qquad+(x_n\cos\beta_n-y_n\cos\alpha_n)z\}=0.
\end{aligned}$$

But if $p_1, p_2, \ldots p_n$ are the lengths of the perpendiculars drawn from the origin on the lines of action of the forces, then

$$\begin{aligned}
p_1^2=(y_1\cos\gamma_1-z_1\cos\beta_1)^2&+(z_1\cos\alpha_1-x_1\cos\gamma_1)^2\\
&+(x_1\cos\beta_1-y_1\cos\alpha_1)^2,
\end{aligned}$$

and similar values for $p_2\ldots p_n$; so that, in the preceding expression, the coefficients of $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$ severally divided by $p_1, p_2, \ldots p_n$ are the perpendiculars from the point $(x, y, z)$ on the plane passing through the origin, and containing the lines of action of $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$, and of which the equation corresponding to $\mathrm{P}_1$ is

$$\begin{aligned}
(y_1\cos\gamma_1-z_1\cos\beta_1)x&+(z_1\cos\alpha_1-x_1\cos\gamma_1)y\\
&+(x_1\cos\beta_1-y_1\cos\alpha_1)z=0;
\end{aligned}$$

let these perpendiculars be $\delta_1, \delta_2, \ldots \delta_n$; the equation becomes

$$\mathrm{P}_1 p_1\delta_1+\mathrm{P}_2 p_2\delta_2+\ldots+\mathrm{P}_n p_n\delta_n=0.$$

Suppose that along the lines of action of the forces lengths are taken proportional to the intensity of the forces, and thus proportional to $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$: then $\mathrm{P}p$ is twice the area of the triangle whose vertex is at the origin, and of which the base is the straight line represented by $\mathrm{P}$: and as $\delta$ is the perpendicular distance from $(x, y, z)$ on this plane the above equation may be geometrically interpreted as follows:

If a system of forces is in equilibrium, and any two points taken anywhere in space are joined by a straight line, the sum of the tetrahedra which have for one edge lines along the lines of action of the forces and proportional to the forces' intensity, and for the opposite edge the given straight line, is equal to zero.

58.] In Article 55 the forces acting on a rigid body are reduced to a force of translation, viz. R acting at an arbitrary point, the origin, and along a given line whose direction-cosines are given by equations (85), and to a couple whose moment is G, and whose rotation-axis is determined by (91). Let us investigate the conditions under which the forces may be reduced to a force of translation only.

Let R be the single force of translation, $(x, y, z)$ its point of application, $a, b, c$ the direction-angles of its line of action, $r$ the perpendicular distance from the origin on that line; so that

$$r^2 = (y\cos c - z\cos b)^2 + (z\cos a - x\cos c)^2 + (x\cos b - y\cos a)^2. \quad (96)$$

Let there be introduced at the origin two equal and opposite forces, each of which is equal to R, and whose line of action is parallel to that of R: so that we have now R acting at the origin, and a couple whose moment is $\mathrm{R}r$; and resolving each of these along the three coordinate axes, and equating the resolved parts to the corresponding parts of the aggregate of the impressed forces, we have

$$\left.\begin{aligned} \mathrm{R}\cos a &= \Sigma.\mathrm{P}\cos\alpha = \mathrm{X} \\ \mathrm{R}\cos b &= \Sigma.\mathrm{P}\cos\beta = \mathrm{Y} \\ \mathrm{R}\cos c &= \Sigma.\mathrm{P}\cos\gamma = \mathrm{Z} \end{aligned}\right\}; \quad (97)$$

$$\therefore \quad \mathrm{R}^2 = \mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{Z}^2; \quad (98)$$

$$\left.\begin{aligned} \mathrm{R}(y\cos c - z\cos b) &= \mathrm{L} = \mathrm{Z}y - \mathrm{Y}z \\ \mathrm{R}(z\cos a - x\cos c) &= \mathrm{M} = \mathrm{X}z - \mathrm{Z}x \\ \mathrm{R}(x\cos b - y\cos a) &= \mathrm{N} = \mathrm{Y}x - \mathrm{X}y \end{aligned}\right\}; \quad (99)$$

and if these last three equations coexist, we may multiply them severally by X, Y, Z, and we have

$$\mathrm{LX} + \mathrm{MY} + \mathrm{NZ} = 0; \quad (100)$$

and this relation must be satisfied if the forces are reducible to a single resultant of translation.

Let us consider the meaning of the condition; L, M, N are proportional to the direction cosines of the rotation-axis of the resultant couple G; and X, Y, Z are proportional to the direction-cosines of the line of action of the resultant of translation, viz. R; and therefore, by reason of (100), these two lines are perpendicular to each other; or, in other words, the line of action of the resultant force of translation is in the plane of the forces of the resultant couple. Now as the arm of this couple may be turned round in its own plane, let it be so arranged that the line of action of each of its forces may be parallel to that of R; so that if R′ is a force such, and $a$ an arm such, that $R'a = G$, we shall have three parallel forces R′, −R′, and R acting in one and the same plane, and these manifestly are capable of being reduced to a single resultant, whose magnitude is R.

And as $x, y, z$ in (99) are the coordinates to *any* point in the line of action of R, (99) are the equations to that line; and the equations may be put into the following form:

$$\begin{aligned} M - N &= X(y+z) - x(Y+Z) \\ &= X(x+y+z) - x(X+Y+Z); \end{aligned}$$

$$\therefore \quad \frac{x + \dfrac{M-N}{X+Y+Z}}{X} = \frac{x+y+z}{X+Y+Z};$$

and therefore from the symmetry of the right-hand member

$$\frac{x + \dfrac{M-N}{X+Y+Z}}{X} = \frac{y + \dfrac{N-L}{X+Y+Z}}{Y} = \frac{z + \dfrac{L-M}{X+Y+Z}}{Z}. \tag{101}$$

Or the equations (99) may be put into the following form: multiplying the second by Z and the third by Y, and subtracting, we have

$$\begin{aligned} MZ - NY &= X(Zz+Yy) - (Z^2+Y^2)x \\ &= X(Xx+Yy+Zz) - x(X^2+Y^2+Z^2) \\ &= X(Xx+Yy+Zz) - xR^2; \end{aligned}$$

$$\therefore \quad \frac{x + \dfrac{MZ-NY}{R^2}}{X} = \frac{Xx+Yy+Zz}{R^2};$$

$$\therefore \quad \frac{x + \dfrac{MZ-NY}{R^2}}{X} = \frac{y + \dfrac{NX-LZ}{R^2}}{Y} = \frac{z + \dfrac{LY-MX}{R^2}}{Z}; \tag{102}$$

and either system, (101) or (102), is that of the equations to the line of action of the single resultant, which is plainly parallel to the line of action of $\mathrm{R}$ acting at the origin. If $\mathrm{L}=\mathrm{M}=\mathrm{N}=0$, that is, if $\mathrm{G}=0$, the line of action of the resultant passes through the origin.

59.] If the lines of action of the impressed forces are parallel, the equations (81)...(83) and (88), (89) undergo the following modifications. Let us suppose the forces to be applied at given points $(x_1,y_1,z_1)$ $(x_2,y_2,z_2)\dots(x_n,y_n,z_n)$, and $\alpha_1=\alpha_2=\dots=\alpha_n=\alpha$; $\beta_1=\beta_2=\dots=\beta_n=\beta$; $\gamma_1=\gamma_2=\dots=\gamma_n=\gamma$; then

$$\left.\begin{aligned} \mathrm{X} &= \Sigma.\mathrm{P}\cos\alpha = \cos\alpha\,\Sigma.\mathrm{P} \\ \mathrm{Y} &= \Sigma.\mathrm{P}\cos\beta = \cos\beta\,\Sigma.\mathrm{P} \\ \mathrm{Z} &= \Sigma.\mathrm{P}\cos\gamma = \cos\gamma\,\Sigma.\mathrm{P} \end{aligned}\right\}; \tag{103}$$

$$\begin{aligned} \therefore\quad \mathrm{R}^2 &= \mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2 \\ &= (\Sigma.\mathrm{P})^2; \end{aligned}$$

$$\therefore\quad \mathrm{R} = \Sigma.\mathrm{P}; \tag{104}$$

and therefore from (85)

$$\cos a = \frac{\mathrm{X}}{\mathrm{R}} = \cos\alpha, \qquad \cos b = \cos\beta, \qquad \cos c = \cos\gamma, \tag{105}$$

that is, the resultant of translation at the origin is equal to the sum of all the impressed forces, and acts along a line which is parallel to the lines of action of the components. Again,

$$\left.\begin{aligned} \mathrm{L} &= \cos c\,\Sigma.\mathrm{P}y - \cos b\,\Sigma.\mathrm{P}z \\ \mathrm{M} &= \cos a\,\Sigma.\mathrm{P}z - \cos c\,\Sigma.\mathrm{P}x \\ \mathrm{N} &= \cos b\,\Sigma.\mathrm{P}x - \cos a\,\Sigma.\mathrm{P}y \end{aligned}\right\}; \tag{106}$$

$$\begin{aligned} \therefore\quad \mathrm{G}^2 = {} & (\Sigma.\mathrm{P}x)^2+(\Sigma.\mathrm{P}y)^2+(\Sigma.\mathrm{P}z)^2 \\ & -\{\cos a\,\Sigma.\mathrm{P}x+\cos b\,\Sigma.\mathrm{P}y+\cos c\,\Sigma.\mathrm{P}z\}^2; \end{aligned} \tag{107}$$

and hereby may $\cos\lambda$, $\cos\mu$, $\cos\nu$, equations (91), be found.

From (106) we have

$$\mathrm{L}\cos a+\mathrm{M}\cos b+\mathrm{N}\cos c = 0,$$

and therefore from (105)

$$\mathrm{LX}+\mathrm{MY}+\mathrm{NZ} = 0; \tag{108}$$

which is the condition requisite that the forces should be capable of reduction to a single force of translation. Let $\mathrm{R}$ be this force, $a\,b\,c$ the direction-angles of its line of action, $\bar{x}\,\bar{y}\,\bar{z}$ the coordinates to its point of application; then introducing at the origin two equal and opposite forces, each of which is equal to $\mathrm{R}$, and acts along a line of action parallel to that of $\mathrm{R}$, we have a force acting at the origin equal to $\mathrm{R}$, and in a parallel direction, and a couple each of whose forces is R, and whose arm is $r$, where

$$r^2 = (\bar{y}\cos c - \bar{z}\cos b)^2 + (\bar{z}\cos a - \bar{x}\cos b)^2 + (\bar{x}\cos b - \bar{y}\cos a)^2, \tag{109}$$

and the direction-cosines of the rotation-axis of which are

$$\frac{\bar{y}\cos c - \bar{z}\cos b}{r}, \quad \frac{\bar{z}\cos a - \bar{x}\cos b}{r}, \quad \frac{\bar{x}\cos b - \bar{y}\cos a}{r}; \tag{110}$$

then, as this system is to produce the same effect as the aggregate of all the impressed parallel forces, we have

$$\left.\begin{aligned} \mathrm{R}\cos a &= \cos\alpha\, \Sigma.\mathrm{P} \\ \mathrm{R}\cos b &= \cos\beta\, \Sigma.\mathrm{P} \\ \mathrm{R}\cos c &= \cos\gamma\, \Sigma.\mathrm{P} \end{aligned}\right\}; \tag{111}$$

therefore squaring and adding,

$$\mathrm{R}^2 = (\Sigma.\mathrm{P})^2, \text{ and therefore } \mathrm{R} = \Sigma.\mathrm{P}. \tag{112}$$

Also $\cos a = \cos\alpha, \quad \cos b = \cos\beta, \quad \cos c = \cos\gamma,$

$$a = \alpha, \quad b = \beta, \quad c = \gamma. \tag{113}$$

Also

$$\left.\begin{aligned} \mathrm{L} &= \mathrm{R}(\bar{y}\cos\gamma - \bar{z}\cos\beta) = \cos\gamma\, \Sigma.\mathrm{P}y - \cos\beta\, \Sigma.\mathrm{P}z \\ \mathrm{M} &= \mathrm{R}(\bar{z}\cos\alpha - \bar{x}\cos\gamma) = \cos\alpha\, \Sigma.\mathrm{P}z - \cos\gamma\, \Sigma.\mathrm{P}x \\ \mathrm{N} &= \mathrm{R}(\bar{x}\cos\beta - \bar{y}\cos\alpha) = \cos\beta\, \Sigma.\mathrm{P}x - \cos\alpha\, \Sigma.\mathrm{P}y \end{aligned}\right\}. \tag{114}$$

From (112) and (113), as well as from general reasoning, it is manifest that the magnitude and direction of the line of action of the resultant of translation is independent of the direction-angles of the lines of action of the impressed forces, and that both remain the same, whatever those direction-angles are. For this reason the point of application of the resultant is called the *centre of parallel forces;* see Article 46. Now from (114),

$$\frac{\mathrm{R}\bar{x} - \Sigma.\mathrm{P}x}{\cos\alpha} = \frac{\mathrm{R}\bar{y} - \Sigma.\mathrm{P}y}{\cos\beta} = \frac{\mathrm{R}\bar{z} - \Sigma.\mathrm{P}z}{\cos\gamma}; \tag{115}$$

but as $\alpha, \beta, \gamma$ are by the preceding reasoning indeterminate, each of the numerators must be equal to zero; therefore

$$\mathrm{R}\bar{x} = \Sigma.\mathrm{P}x, \qquad \mathrm{R}\bar{y} = \Sigma.\mathrm{P}y, \qquad \mathrm{R}\bar{z} = \Sigma.\mathrm{P}z;$$

$$\therefore \quad \bar{x} = \frac{\Sigma.\mathrm{P}x}{\Sigma.\mathrm{P}}, \qquad \bar{y} = \frac{\Sigma.\mathrm{P}y}{\Sigma.\mathrm{P}}, \qquad \bar{z} = \frac{\Sigma.\mathrm{P}z}{\Sigma.\mathrm{P}}; \tag{116}$$

and these are the coordinates to the point of application of the resultant of many parallel impressed forces.

60.] Examples of the preceding formulæ.

Ex. 1. Four parallel forces 2, 4, 6, 8 act at the angles of a square, the length of whose side is $2a$; it is required to determine the magnitude, line of action, and the coordinates to the point of application of the resultant.

Let the plane of the square be taken for the plane of $xy$, and the origin be at the centre of it: let the direction-angles of the parallel lines of action of the forces be $\alpha, \beta, \gamma$: these are therefore the direction-angles of the resultant.

Let R be the resultant; therefore by (104)

$$\begin{aligned} \text{R} = \Sigma.\text{P} &= 2+4+6+8 \\ &= 20. \end{aligned}$$

and let the point of application of the force 2 be $(a, a, 0)$, of 4 $(-a, a, 0)$, and so on: so that

$$\begin{aligned} \Sigma.\text{P}x &= 2a-4a-6a+8a \\ &= 0, \\ \Sigma.\text{P}y &= 2a+4a-6a-8a \\ &= -8a, \\ \Sigma.\text{P}z &= 0; \end{aligned}$$

$$\therefore \quad \bar{x} = 0, \qquad \bar{y} = -\frac{2}{5}a, \qquad \bar{z} = 0.$$

Ex. 2. Three planes, whose equations are

$$\begin{aligned} \text{A}_1 x + \text{B}_1 y + \text{C}_1 z &= 0, \\ \text{A}_2 x + \text{B}_2 y + \text{C}_2 z &= 0, \\ \text{A}_3 x + \text{B}_3 y + \text{C}_3 z &= 0, \end{aligned}$$

meet at the origin, and support between them a heavy sphere of weight W: determine the pressure on each of the planes.

Let the axis of $z$ be taken in a vertical direction; and let the pressures on the planes be $\text{R}_1, \text{R}_2, \text{R}_3$; the lines of action of which are of course normal to the planes; and let the equations of the planes be such as to satisfy in each the condition, $\text{A}^2 + \text{B}^2 + \text{C}^2 = 1$: then (94) become

$$\begin{aligned} \text{R}_1\text{A}_1 + \text{R}_2\text{A}_2 + \text{R}_3\text{A}_3 &= 0 \\ \text{R}_1\text{B}_1 + \text{R}_2\text{B}_2 + \text{R}_3\text{B}_3 &= 0 \\ \text{R}_1\text{C}_1 + \text{R}_2\text{C}_2 + \text{R}_3\text{C}_3 &= \text{W}; \end{aligned}$$

from which we have, using the notation of Art. 150, Vol. II,

$$\text{R}_1 = \text{W}\frac{\Sigma.\pm\text{A}_2\text{B}_3}{\Sigma.\pm\text{A}_1\text{B}_2\text{C}_3}, \quad \text{R}_2 = \text{W}\frac{\Sigma.\pm\text{A}_3\text{B}_1}{\Sigma.\pm\text{A}_2\text{B}_3\text{C}_1}, \quad \text{R}_3 = \text{W}\frac{\Sigma.\pm\text{A}_1\text{B}_2}{\Sigma.\pm\text{A}_3\text{B}_1\text{C}_2},$$

$$\text{and} \qquad \frac{\text{R}_1}{\text{A}_2\text{B}_3 - \text{B}_2\text{A}_3} = \frac{\text{R}_2}{\text{A}_3\text{B}_1 - \text{B}_3\text{A}_1} = \frac{\text{R}_3}{\text{A}_1\text{B}_2 - \text{B}_1\text{A}_2}.$$

61.] The material body or system of material particles, which receives the pressures considered in the preceding Articles, has been supposed to be free from all constraint; let us investigate the modifications required in the general results when the system is subject to certain given constraints.

Firstly, suppose one point of the body to be fixed; let this be taken for the origin: it is evident that, because it is fixed, it will bear any pressure of translation acting on it, and that the body will not move owing to that pressure; but the effects of a pressure of rotation about a rotation-axis passing through that point are not affected by the fixedness of the point; the impressed forces therefore must be so related that $\text{G} = 0$; and therefore that, Art. 57,

$$\text{L} = 0, \qquad \text{M} = 0, \qquad \text{N} = 0; \tag{117}$$

which three conditions are requisite, so that a body, of which one point is fixed, should be at rest. And these three conditions, it will be observed, satisfy equation (108), and therefore indicate that the impressed pressures may be compounded into a single force of translation: that, viz. which passes through the fixed point.

And the pressure on the fixed point, and the direction of its line of action, may thus be found: let $\text{R}$ be the pressure, and $a, b, c$ the direction-angles of its line of action; let the impressed forces be $\text{P}_1, \text{P}_2, \ldots \text{P}_n$, and the direction-angles of their lines of action $\alpha_1 \beta_1 \gamma_1$, &c.; then

$$\left.\begin{aligned} \text{R} \cos a &= \Sigma.\text{P} \cos \alpha \\ \text{R} \cos b &= \Sigma.\text{P} \cos \beta \\ \text{R} \cos c &= \Sigma.\text{P} \cos \gamma \end{aligned}\right\}; \tag{118}$$

$$\therefore \quad \text{R}^2 = (\Sigma.\text{P} \cos \alpha)^2 + (\Sigma.\text{P} \cos \beta)^2 + (\Sigma.\text{P} \cos \gamma)^2; \tag{119}$$

and therefore by (118) $a, b, c$ are known.

62.] Secondly, suppose two points of the body to be fixed; and let the axes of reference be such, that the axis of $z$ may pass through the two points, and the origin may be at the middle point of the line joining them; and let the $z$-ordinates to the points be $+z_1$ and $-z_1$; then it is manifest that the body cannot have any motion of translation, and can have motion of rotation about the axis of $z$ only. The impressed forces therefore must be so related that the rotation-pressure about the axis of $z$ should be equal to zero; therefore the necessary condition is

$$\text{N} = 0. \tag{120}$$

And the pressures on the two points may be determined in the following manner: let them be represented by $R_1$ and $R_2$, and let the direction-angles of their lines of action be $a_1 b_1 c_1$, $a_2 b_2 c_2$; then

$$\left.\begin{array}{l} R_1 \cos a_1 + R_2 \cos a_2 = \Sigma . P \cos \alpha \\ R_1 \cos b_1 + R_2 \cos b_2 = \Sigma . P \cos \beta \\ R_1 \cos c_1 + R_2 \cos c_2 = \Sigma . P \cos \gamma \end{array}\right\}, \tag{121}$$

$$\left.\begin{array}{l} L + R_1 z_1 \cos b_1 - R_2 z_1 \cos b_2 = 0 \\ M - R_1 z_1 \cos a_1 + R_2 z_1 \cos a_2 = 0 \end{array}\right\}. \tag{122}$$

From the first two of (121), and from (122), we have

$$R_1 \cos b_1 = \frac{z_1 \Sigma . P \cos \beta - L}{2 z_1},$$

$$R_2 \cos b_2 = \frac{z_1 \Sigma . P \cos \beta + L}{2 z_1},$$

$$R_1 \cos a_1 = \frac{z_1 \Sigma . P \cos \alpha + M}{2 z_1},$$

$$R_2 \cos a_2 = \frac{z_1 \Sigma . P \cos \alpha - M}{2 z_1};$$

and thus the pressures on the fixed points, which are parallel to the axes of $x$ and $y$, are determined: but the pressures along the axis of $z$ are involved in only the third equation of (121), which shews that the sum of the pressures is equal to $\Sigma . P \cos \gamma$, and therefore that each pressure is indeterminate: now this is, at first sight, a startling fact, and has been urged heretofore as an argument against the truth of our mechanical results and principles; because certainly in nature, when a body is supported in the manner assumed in the problem, say a gate or a door on its two hinges, the vertical pressures are determinate and may be experimentally determined at both hinges; our mechanical formulæ therefore ought to yield a corresponding result. In reply to this objection, I say, that in nature the pressures are determinate, and may be determined by mechanical means: but the bodies which are the subjects of the experiments are more or less compressible and extensible: they are not rigid; and therefore do not satisfy the conditions required in the preceding theory, however nearly they may approach to them; thus if to a door, being in a *horizontal* position, two "eyes" are attached, which correspond to two hooks fixed in a vertical doorpost, and if the distance between the eyes when the door is horizontal is equal to that between the hooks in the vertical doorpost; then doubtless, if the body were perfectly rigid and inextensible, and were attached by the eyes to the hooks, either one or the other hook would be sufficient to bear the vertical pressure; and we should be unable to determine whether one or the other carried the whole weight, and whether it was distributed between them, and in what proportion; yet as such a door is extensible, both hooks would bear a part of the weight, and the respective proportions will depend on the extensibility and the elasticity of the material. Thus if the distance between the eyes is greater than that between the hooks, the pressure will for the most part be on the lower hook, although the compression of the material due to its weight may cause the eyes so to approach each other, that some of the pressure may be brought upon the upper hook; and a similar effect may occur at the lower hook, when the distance between the hooks is greater than that between the eyes. Thus it appears that the determinateness of the pressures is due to the extensibility, compressibility and elasticity of the material which is in nature the subject of the experiment; and the truth of the result which is arrived at in (121) for a rigid body is not affected: for be it remembered that we have nothing in nature of perfect rigidity. We shall see a further example of indeterminateness of the same kind in dynamics.

Again, suppose the circumstances of constraint to be such, that the body is capable of *sliding* along, as well as of turning about, the axis passing through the two fixed points; then the points will be able to bear the pressures arising from the forces, and which are resolved at right-angles to the axis, and parallel to the axes of $x$ and $y$; but will not offer any resistance to those along the axis of $z$: if therefore equilibrium exists, the forces must satisfy the conditions,

$$\Sigma . \mathrm{P} \cos \gamma = 0, \qquad \mathrm{N} = 0.$$

63.] And lastly, if three or more points of the body are fixed, and if all these are not in the same straight line, it is evident that the body is fixed; and therefore whatever are the impressed forces as to intensity, point of application, line of action, and direction, the body is in equilibrium, if we suppose the fixed points of it to be capable of bearing the pressures which are due to the impressed forces.

And it is evident by the following reasoning that, if these points are fixed, the body is also fixed. For suppose the body to consist of $n$ particles, then each of these particles is at rest, if the forces (including the tensions, mutual reaction, &c.) acting on it satisfy the three conditions (69), Article 28: and therefore if all are at rest, $3n$ conditions are required. Now if three points of a body are fixed, the mutual distances of them are also fixed, and hereby we have three conditions; also as the body is rigid, the distances of each of the remaining $n-3$ particles from each of the three fixed points are given, and thus we have $3n-9$ conditions; and as the equations of equilibrium of a rigid body are six, we have six more conditions: and thus altogether we have, as before, $3n$ equations. If the three fixed points are in one and the same straight line, one of the conditions is lost, and the number is insufficient for equilibrium.

64.] Another form in which a body under the action of impressed forces may be in constraint is, when it rests with points of it on a plane, or against any surface.

Let us consider first the more simple case of a smooth plane: and let us suppose the plane to be that of $xy$, and $n$ points of the body to rest on it, which are $(x_1, y_1)$, $(x_2, y_2)$, ... $(x_n, y_n)$, and let the pressures at these points be $\text{R}_1, \text{R}_2, \dots \text{R}_n$; the lines of action of which are parallel to the axis of $z$: thus the equations of equilibrium become

$$\Sigma.\text{P}\cos\alpha = 0, \quad \Sigma.\text{P}\cos\beta = 0, \quad \Sigma.\text{P}\cos\gamma - \Sigma.\text{R} = 0, \tag{123}$$

$$\text{L} - \Sigma.\text{R}y = 0, \quad \text{M} + \Sigma.\text{R}x = 0, \quad \text{N} = 0. \tag{124}$$

Here are six equations, of which only three involve the pressures against the plane and the coordinates of their points of action; there are always therefore three independent conditions to be fulfilled by the impressed forces.

Now if only one point of the body is in contact with the plane, the pressure at that point will be given by the third equation, and the impressed forces must be such as to fulfil the other five.

If two points are in contact, the pressures at them may be determined by either two of the third fourth and fifth equations, and the forces must satisfy the remaining four conditions.

But if three points are in contact, the pressures at them may be determined by means of the three equations which involve the pressures, and the other three equations must be satisfied by the impressed forces.

If more than three points are in contact, the pressures are indeterminate, because there is not a sufficient number of equations whereby to determine them.

In all cases the pressure which the plane has to bear is given by the third equation of (123); and for the existence of equilibrium, if the body only presses against the plane, it is necessary that the $\Sigma.P\cos\gamma$ should act *towards*, and not *from*, the plane; it is also necessary that the line of action of this pressure should pierce the plane of $xy$ at some point within the area determined by straight lines joining the points of contact of the body and the plane: otherwise the rotation-pressure of the $z$-force will cause the body to turn about one of the bounding lines of this area.

And as to the indeterminateness of the several pressures, which act at the points of contact, when more than three points are in contact with the plane, an explanation similar to that of Article 62 is apposite. Suppose a heavy body to rest on a horizontal table, and to be in contact with it at many points; the sum of all the pressures is doubtless equal to the weight of the body; but if the points of contact are more than three, each pressure, so far as the *preceding* theory enables us to determine it, is indeterminate; and so it would be in nature, if the table were accurately plane, and it and the body were perfectly rigid; but such a table and such a body nature does not give us; and so our results when applied to flexible and compressible matter are not true. If however we knew the laws of flexibility and elasticity, and could thus bring into calculation all the conditions of the problem, the result would be determinate and true; and thus it seems that the non-applicability of the mechanical principles is only apparent, and is due to the omission of certain data which the true solution of the problem requires.

65.] Again, suppose the body to be in contact with surfaces whose equations are $F_1 = 0$, $F_2 = 0, \dots F_n = 0$; and the mutual pressures between the body and the several surfaces to be $R_1$, $R_2, \dots R_n$; the direction-angles of the lines of action of these to be $a_1 b_1 c_1$, $a_2 b_2 c_2, \dots a_n b_n c_n$; and the coordinates of the points of contact to be $x_1 y_1 z_1$, $x_2 y_2 z_2, \dots x_n y_n z_n$; then employing our ordinary notation, see Article 29,

$$\cos a = \frac{U}{Q}, \quad \cos b = \frac{V}{Q}, \quad \cos c = \frac{W}{Q};$$

and the equations of equilibrium become

$$\left.\begin{aligned} \Sigma.\mathrm{P}\cos\alpha+\Sigma.\mathrm{R}\cos a &= 0 \\ \Sigma.\mathrm{P}\cos\beta+\Sigma.\mathrm{R}\cos b &= 0 \\ \Sigma.\mathrm{P}\cos\gamma+\Sigma.\mathrm{R}\cos c &= 0 \end{aligned}\right\}, \tag{125}$$

$$\left.\begin{aligned} \mathrm{L}+\Sigma.\mathrm{R}\,(y\cos c-z\cos b) &= 0 \\ \mathrm{M}+\Sigma.\mathrm{R}\,(z\cos a-x\cos c) &= 0 \\ \mathrm{N}+\Sigma.\mathrm{R}\,(x\cos b-y\cos a) &= 0 \end{aligned}\right\}. \tag{126}$$

To which equations, as to the number of points in contact between the body and the surfaces, the remarks of the last three Articles are applicable.

One point however requires further elucidation: suppose that the surface of the body on which the forces act meets $n$ given and fixed points; then the equations (125) and (126) contain $n$ undetermined pressures which act at these points. Now as the equations are six in number, if $n=6$, the six pressures at the points may be determined; and the directions of their lines of action will be along the normals to the surface of the body at the points; if $n$ is greater than 6, $n-6$ of the pressures may be indeterminate, and when they receive given values, the other 6 will be known: and when $n$ is less than 6, the pressures at the given points may be eliminated from the preceding equations, and the remaining $6-n$ conditions must be fulfilled by the impressed forces acting on the body. And hence we infer that generally a body under the action of given forces is in equilibrium and fixed, if the bounding surface of it passes through six given and fixed points*; and that the mobility of it is not taken away, if the surface has to pass through fixed points in number less than six.

66.] And hereby I am led to another subject: viz. to the investigation of the conditions requisite that many bodies subject to given pressures, and in contact with, or under mutual action from, each other, should be in equilibrium.

Let the number of bodies be $n$; let $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n$ be the types of the forces which act on the first, second, ... $n$th body respectively; let $\mathrm{R}$ be the general type of the reacting pressures at the points of contact, and $a, b, c$ the direction angles of its line of action, and $(x, y, z)$ the point of its application; $\mathrm{L}_1\mathrm{M}_1\mathrm{N}_1$, $\mathrm{L}_2\mathrm{M}_2\mathrm{N}_2$, ... the moment-axes of the component couples which act on the several bodies; then the conditions of equilibrium for the several bodies are

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* For various other properties of this kind let me refer the reader to Möbius, Lehrbuch der Statik, Zweiten Theil, Erstes Kapitel; Leipzig, 1837.

$$\left.\begin{aligned}\Sigma.\mathrm{P}_1\cos\alpha_1+\Sigma.\mathrm{R}_1\cos a_1&=0\\ \Sigma.\mathrm{P}_1\cos\beta_1+\Sigma.\mathrm{R}_1\cos b_1&=0\\ \Sigma.\mathrm{P}_1\cos\gamma_1+\Sigma.\mathrm{R}_1\cos c_1&=0\end{aligned}\right\}, \tag{127}$$

$$\left.\begin{aligned}\mathrm{L}_1+\Sigma.\mathrm{R}_1(y_1\cos c_1-z_1\cos b_1)&=0\\ \mathrm{M}_1+\Sigma.\mathrm{R}_1(z_1\cos a_1-x_1\cos c_1)&=0\\ \mathrm{N}_1+\Sigma.\mathrm{R}_1(x_1\cos b_1-y_1\cos a_1)&=0\end{aligned}\right\}, \tag{128}$$

. . . . . . . . . . .

$$\left.\begin{aligned}\Sigma.\mathrm{P}_n\cos\alpha_n+\Sigma.\mathrm{R}_n\cos a_n&=0\\ \Sigma.\mathrm{P}_n\cos\beta_n+\Sigma.\mathrm{R}_n\cos b_n&=0\\ \Sigma.\mathrm{P}_n\cos\gamma_n+\Sigma.\mathrm{R}_n\cos c_n&=0\end{aligned}\right\}, \tag{129}$$

$$\left.\begin{aligned}\mathrm{L}_n+\Sigma.\mathrm{R}_n(y_n\cos c_n-z_n\cos b_n)&=0\\ \mathrm{M}_n+\Sigma.\mathrm{R}_n(z_n\cos a_n-x_n\cos c_n)&=0\\ \mathrm{N}_n+\Sigma.\mathrm{R}_n(x_n\cos b_n-y_n\cos a_n)&=0\end{aligned}\right\}. \tag{130}$$

Now if, of all these groups of equations, all the first of the first sets are added, $\Sigma.\mathrm{R}\cos a$ will disappear, because, the reactions of the several bodies being equal and opposite, the same quantity will appear twice, and with different signs; so that we shall finally obtain $\Sigma.\mathrm{P}\cos a=0$; similarly, by adding all the second equations of the first set in each group, and by adding all the third equations of the first set, we shall have

$$\Sigma.\mathrm{P}\cos\beta=0, \qquad \Sigma.\mathrm{P}\cos\gamma=0.$$

In the same way, by adding the several equations of the second sets of the groups, we shall obtain equations free from the R's, and shall have ultimately

$$\mathrm{L}=0, \quad \mathrm{M}=0, \quad \mathrm{N}=0;$$

and thus the equations of condition necessary for the equilibrium of a system of rigid bodies are of the same form and of the same number as those required for the equilibrium of a single rigid body.

67.] Examples illustrative of the preceding.

Ex. 1. A heavy uniform beam is fixed by a hinge to a given inclined plane: between the beam and the plane a heavy sphere is in equilibrium; determine its position and the several pressures.

Let fig. 37 represent a vertical section of the system made by the plane of the paper: $POB = \alpha$; $POQ = 2\theta$; $OG = GA = a$; $CP = CQ = c$; $W =$ the weight of the beam; $w =$ the weight of the sphere; $R =$ reaction existing between the beam and the sphere; $R' =$ the pressure of the sphere on the inclined plane. And let us consider separately the conditions of equilibrium of the sphere and of the beam.

For the equilibrium of the sphere, resolving the forces along the plane, we have

$$w \sin \alpha = R \sin 2\theta.$$

For the equilibrium of the beam, taking moments about O, we have

$$\begin{aligned} W a \cos(\alpha + 2\theta) &= R \times OQ \\ &= R c \cot \theta, \end{aligned}$$

$$\therefore \quad W a \cos(\alpha + 2\theta) \sin 2\theta = wc \sin \alpha \cot \theta;$$

whence may $\theta$ be determined; and thence $R$; and since

$$R' = w \cos \alpha + R \cos 2\theta,$$

$R'$ may also be found.

Ex. 2. Two heavy beams OA and OA′ of equal lengths are connected, fig. 38, at O by a hinge, and at AA′ by a string of given length; between them a heavy sphere is placed, and the string remains horizontal; determine the tension of the string and the pressure against the beams.

Let length of each beam be $2a$, weight of each beam $= W$; $2c =$ length of string; $T =$ the tension of the string; $b =$ the radius of the sphere; $w =$ the weight of the sphere; $\alpha =$ the angle $AOB = \sin^{-1} \frac{c}{2a}$; then for the equilibrium of either of the beams, taking the moments of the forces about O, we have

$$T\, 2a \cos \alpha = W a \sin \alpha + R b \cot \alpha;$$

and for the equilibrium of the sphere, taking vertical forces, we have

$$w = 2R \sin \alpha;$$

$$\therefore \quad T = \frac{W}{2} \tan \alpha + \frac{b}{a} \frac{w}{4} (\operatorname{cosec} \alpha)^2.$$

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## SECTION 3.—*On Friction.*

68.] All the surfaces, which we have imagined to be in contact in the preceding Articles, are supposed to be smooth, and, as such, to offer no resistance to the motion of the points in contact with them in directions perpendicular to the normal at the points; and therefore the reaction arising from the contact acts along the common normal line only. In nature however we have no surfaces perfectly smooth; the constitution of all bodies is such, that on their bounding surfaces are small elevations and depressions, arising as it seems, from their constituent molecules not being continuous and in perfect contact: so that if the surfaces of two bodies are pressed against each other, the elevations of one fit, at least in a measure, into the depressions of the other, and the surfaces interpenetrate each other; and the mutual penetration is of course greater, if the pressing force is greater; much of this *roughness* may be removed by polishing, and the effect of much of it may be destroyed by lubrication: all however cannot be, and there still remains a resistance due to it, when force is applied so as to cause one body to move or to have a tendency to move on another with which it is in contact. This resistance is called *friction*, and is of two kinds; either of sliding or of rolling; the first is that of a heavy body dragged on a plane or other surface; of an axle turning in a fixed box; of a vertical shaft turning on a horizontal plate, or of a millstone turning upon another concentric stone about a vertical axis. Friction of the second kind is that of a wheel rolling along a plane; the resistance however of which seems to arise from the necessity of the wheel overcoming small obstacles which are successively in its path. It is of friction of the first kind only that I shall state the laws and give examples; and first as to its line of action: it is manifestly along that tangent line of the surfaces at the point of contact which is the line of the tendency to motion; and its direction is opposite to that of the line of motion. Suppose therefore many forces to act on a material particle which is in contact with a rough surface; and, the lines of action of the forces being unaltered, their magnitudes to change, so that motion is on the point of taking place (1) in one direction, and (2) in an opposite direction: the line of action of friction is in both cases the same; but the *direction* of it in the former case is contrary to that of it in the latter. Also the magnitudes of the forces may evidently vary within certain limits, and the particle may still be at rest. Examples of the determination of these limits are given in the following Article.

In our ignorance of the constitution of bodies, and of their molecular action, the laws of friction must be deduced from experiment; and therefore I shall enunciate those only which are necessary for our purpose, and refer the reader to the Treatise by M. Morin*, wherein he will find the subject investigated in all its completeness.

I. Friction is proportional to the normal pressure, when the materials of the surfaces in contact are the same.

II. Friction is independent of the extent of the surfaces in contact.

III. Friction is independent of the velocity of motion.

As to law I; suppose $R$ to be the normal pressure between two surfaces, and $F$ to be the friction, then $F = \mu R$, where $\mu$ is a constant quantity for the same materials and is the value of $F$ when $R = 1$; $\mu$ is called *the coefficient of friction*. And this law, it may be observed, appears to arise out of the preceding theory of friction; because the greater is the pressure, the greater is the interpenetration of the molecules at the surface of the bodies, and the greater is the resistance to be overcome, when motion is just about to take place.

As to law II; it signifies that if the pressure remains the same, and the surface in contact increases, the total resistance is still the same, whilst the pressure on each element and the friction corresponding to that element are diminished in the inverse ratio of the surface in contact.

The treatise of M. Morin will be found to contain a complete account of the modes of determining $\mu$ for different substances; but the following manner of considering the subject is sufficiently general for our purpose, and deserves insertion.

Let a given heavy body rest with a plane face of a finite area on a horizontal plane; and let the plane be turned about a horizontal line in it, so that it becomes inclined to the horizontal plane, that is, becomes tilted: the body will begin to slide when the inclination has reached a certain limit; and this inclination will manifestly depend on the friction which exists between the body and the plane, and may be determined as follows. See fig. 39.

Let $W$ be the weight of the body; $\mu =$ the coefficient of friction; $\alpha =$ the angle between the inclined and the horizontal planes just as motion is beginning to take place; R = the pressure on the plane; so that

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* Nouvelles Expériences sur le frottement faites a Metz, imprimées par ordre de l'Académie des Sciences; 3 voll. in 4to. 1832–1835.

$$F = \mu R; \tag{131}$$

and resolving along, and perpendicular to, the plane,

$$F = W \sin a, \qquad R = W \cos a,$$

$$\therefore \quad \tan a = \mu, \qquad a = \tan^{-1} \mu. \tag{132}$$

$a$ is called *"the angle of friction,"* and *"the angle of repose."* The body will rest on the plane when the angle of inclination is less than the angle of friction, and will slide, if the angle of inclination exceeds that angle.

69.] Various problems involving friction.

Ex. 1. A small ring under the action of known pressures is capable of sliding on a rough curved material line in space; it is required to determine the limits of the forces, so that the ring may be at rest.

Let the resolved parts of the impressed forces along the coordinate axes be X, Y, Z, of which let the resultant be R; so that if $x, y, z$ are the coordinates to the position of the ring on the curve, the whole impressed force along the tangent, which we will call T, is

$$T = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}. \tag{133}$$

Let N = the normal pressure: then

$$N^2 + T^2 = R^2$$
$$= X^2 + Y^2 + Z^2,$$

$$\therefore \quad N^2 = X^2 + Y^2 + Z^2 - \left(X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}\right)^2.$$

Now in order that motion should not take place,

$$T^2 < \mu^2 N^2$$
$$< \mu^2 (R^2 - T^2),$$

$$\therefore \quad \frac{T^2}{R^2} < \frac{\mu^2}{1+\mu^2}$$
$$< (\sin a)^2, \text{ see equation (132);}$$

$$\therefore \quad \left(\frac{X\,dx + Y\,dy + Z\,dz}{R\,ds}\right)^2 < (\sin a)^2; \tag{134}$$

$$\text{and if} \quad \frac{X\,dx + Y\,dy + Z\,dz}{R\,ds} = \pm \sin a, \tag{135}$$

the particle will begin to slide; the ± sign assigning the limits within which the forces are to be confined.

**Ex. 2.** As an example, let us take the helix whose equations are

$$x = a\cos\phi, \qquad y = a\sin\phi, \qquad z = ka\phi;$$

and let the force which acts on the ring be its own weight, and $= w$, and have its line of action parallel to the axis of $z$: then $\text{Z} = \text{R} = w$: and

$$\frac{dz}{ds} = \frac{k}{(1+k^2)^{\frac{1}{2}}} = \pm \sin a, \qquad \therefore \quad k = \pm \tan a;$$

that is, the angle of inclination of the thread of the helix to the horizontal plane is equal to the angle of friction.

**Ex. 3.** To determine the limits of the pressures, so that a particle under the action of them may be at rest on a given rough surface.

Let $\text{F}(x, y, z) = 0$ be the equation to the surface: then employing the ordinary symbols, $\frac{\text{U}}{\text{Q}}\ \frac{\text{V}}{\text{Q}}\ \frac{\text{W}}{\text{Q}}$ are the direction-cosines of the normal at the point $(x, y, z)$: if therefore $\text{N} =$ the normal pressure, $\text{T} =$ the tangential force, and $\text{R} =$ the resultant of the acting forces, of which the resolved parts along the coordinate axes are X, Y, Z,

$$\text{N} = \frac{\text{XU}+\text{YV}+\text{ZW}}{\text{Q}}, \qquad \text{T}^2 = \text{R}^2 - \text{N}^2;$$

therefore that the particle should be at rest

$$\text{T}^2 < \mu^2\text{N}^2, \qquad \text{R}^2 - \text{N}^2 < \mu^2\text{N}^2, \qquad \frac{\text{R}^2}{\text{N}^2} < 1 + \mu^2;$$

$$\therefore \quad \frac{\text{Q}^2\text{R}^2}{(\text{XU}+\text{YV}+\text{ZW})^2} < 1 + \mu^2 \tag{136}$$

$$< (\sec a)^2;$$

and therefore if

$$\frac{\text{QR}}{\text{XU}+\text{YV}+\text{ZW}} = \pm \sec a, \tag{137}$$

the particle will just begin to move; the ± sign assigns the limits of the impressed pressures. As an example let us take the following:

**Ex. 4.** An ellipsoid has its least axis in a vertical direction; determine on the surface the curve, on all points within which a heavy material particle being placed shall remain at rest.

In this case $\text{X} = 0, \qquad \text{Y} = 0, \qquad \text{Z} = \text{R};$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

$$\text{Q}^2 = \text{U}^2 + \text{V}^2 + \text{W}^2$$
$$= \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4};$$

therefore (137) becomes

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = (\sec \alpha)^2 \frac{z^2}{c^4};$$

$$\therefore \quad \frac{x^2}{a^4} + \frac{y^2}{b^4} - (\tan \alpha)^2 \frac{z^2}{c^4} = 0;$$

which is the equation to a cone, whose vertex is at the centre of the ellipsoid; and the line of intersection of which with the ellipsoid is the required bounding curve.

Ex. 5. A heavy particle rests on a rough inclined plane, and is acted on by a given force in a vertical plane which is perpendicular to the inclined plane; determine the limits of the force, and the angle at which the least force capable of drawing the particle up the plane must act.

Let fig. 40 represent a vertical section of the inclined plane, and containing the force P; let the inclination of the plane to the horizontal plane be $i$; and let $\theta$ be the angle between the inclined plane and the line of action of P; $\mu =$ coefficient of friction: and let us first suppose the tendency to motion to be down the plane, so that friction is a force acting up the plane: then resolving along, and perpendicular to, the plane,

$$\text{F} + \text{P} \cos\theta = \text{W} \sin i, \qquad \text{R} + \text{P} \sin\theta = \text{W} \cos i, \qquad \text{F} = \mu \text{R};$$

$$\therefore \quad \text{P} = \text{W} \frac{\sin i - \mu \cos i}{\cos\theta - \mu \sin\theta}. \tag{138}$$

And if P is increased so that motion up the plane is just beginning, F acts in an opposite direction, and therefore the sign of $\mu$ must be changed, and we have

$$\text{P} = \text{W} \frac{\sin i + \mu \cos i}{\cos\theta + \mu \sin\theta}. \tag{139}$$

Now to determine $\theta$ in this latter case, so that P shall be the least,

$$\frac{d\text{P}}{d\theta} = \text{W} (\sin i + \mu \cos i) \frac{\sin\theta - \mu\cos\theta}{(\cos\theta + \mu\sin\theta)^2} = 0,$$
$$\text{if} \quad \tan\theta = \mu;$$

that is, if $\theta$ is equal to the angle of friction. Hence we infer that

A given power acts to the greatest advantage in dragging a weight up a hill, if the angle at which its line of action is inclined to the hill is equal to the angle of friction of the hill. And, similarly, a power acts to the greatest advantage in dragging a weight along a horizontal plane, if its line of action is inclined to the plane at the angle of friction of the plane. Hereby also may we determine the angle at which the "traces" of a drawing horse should be inclined to the plane of traction. The preceding results are those which are *a priori* to be expected, because some part of the power ought to be expended in *lifting* the weight from the plane, that friction may be diminished.

Ex. 6. Also let us consider the case of a rough cylindrical axis, on which given forces act and produce a pressure of rotation, capable of turning within a rough hollow coaxal cylinder.

Let fig. 41 be a section perpendicular to the axis of the cylinder; the smaller and interior circle being a section of the cylindrical axis, and the larger circle of the hollow cylinder; let C be the point of contact of the two cylinders, and at which of course the resultant of all the impressed forces acts: let this force $= \text{P}$, and let $\theta$ be the angle between the lines of action of R and P: then

$$\text{R} = \text{P} \cos \theta, \qquad \text{F} = \text{P} \sin \theta,$$

$$\text{F} = \mu \text{R}; \qquad \therefore \quad \tan \theta = \mu;$$

therefore $\theta$ is equal to the angle of friction. If therefore the angle between R and P is less than the angle of friction, the cylinder will continue at rest; and if it is greater, it will move.

Ex. 7. A heavy circular shaft rests in a vertical position, with its end, which is a circular section, on a horizontal plate; determine the resistance due to friction which is to be overcome, when the shaft begins to revolve about a vertical axis.

Let $a$ be the radius of the circular section of the shaft; and let the plane of $r, \theta$ be the horizontal one of contact between the end of the shaft and the plate; and let the centre of the circular area of contact be the pole; now the vertical pressure on each element of this area manifestly varies as the area, and therefore, if $r\,dr\,d\theta$ is the area-element and $k$ is the coefficient of variation, since, by law III, friction is independent of the velocity of motion,

$$\text{Pressure on element} = kr\,dr\,d\theta;$$

$$\therefore \quad \text{Friction of the element} = \mu kr\,dr\,d\theta;$$

$\therefore$ Moment of friction about vertical axis through the centre

$$= \mu k r^2 \, dr \, d\theta ;$$

$\therefore$ Moment of friction of area of contact

$$= \int_0^{2\pi} \int_0^a \mu k r^2 \, dr \, d\theta$$

$$= \frac{2 \mu k \pi a^3}{3};$$

and therefore varies as the cube of the radius. Hence arises the advantage of reducing to the smallest possible dimensions the area of the base of a vertical shaft revolving upon its end on a horizontal bed.

Similarly may the friction of the upper millstone moving on the nether one be calculated.

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## SECTION 4.—*Further researches into the composition and resolution of pressures acting on a rigid body in space.*

70.] In Section 2 we arrived at the following general results as to many pressures acting in space on a rigid body: Whatever point is taken as the origin, the pressures can generally be reduced to (1) a single force of translation acting at the origin, of a definite magnitude, with a definite line of action, and in a definite direction, and such that if R is its magnitude, and *a*, *b*, *c* the direction-angles of its line of action,

$$R^2 = X^2 + Y^2 + Z^2 ; \tag{140}$$

$$\cos a = \frac{X}{R}, \quad \cos b = \frac{Y}{R}, \quad \cos c = \frac{Z}{R}; \tag{141}$$

and to (2) a couple whose moment-axis and rotation-axis are definite, and such that if G is the moment axis, and $\lambda$, $\mu$, $\nu$ are the direction-angles of the rotation-axis,

$$G^2 = L^2 + M^2 + N^2 ; \tag{142}$$

$$\cos \lambda = \frac{L}{G}, \quad \cos \mu = \frac{M}{G}, \quad \cos \nu = \frac{N}{G}. \tag{143}$$

We proceed to further researches into the properties of these two resultants.

Let it be observed that the position of the origin is arbitrary, the only condition to which it is subject being that it is rigidly connected with the body: and at whatever point it is taken, the magnitude, direction of the line of action, and direction of the force of translation R are always the same; this is evident from the equations (140) and (141), because they do not involve the coordinates of the points of application of the forces.

Imagine now a system of forces acting on a rigid body to be reduced to a single force of translation R acting at the arbitrarily chosen origin, and to a couple whose moment-axis is G, and in which, if $a$ is the length of the arm and $R'$ is the force, $R'a = G$; and let one of these forces act at the origin: also imagine the arm of the couple to be so situated that it may be perpendicular to the line of action of R; then the forces are reduced to (1) R and R′ acting at the origin, with their lines of action inclined to each other at an angle, which is the complement of the angle between the line of action of R, and the rotation-axis of G; let this latter angle $= \phi$, so that

$$\cos\phi = \frac{LX+MY+NZ}{GR}; \tag{144}$$

and to (2) another force R′ which does not meet either of the former forces: now the resultant of the former two forces is equal to

$$\{R^2+2RR'\sin\phi+R'^2\}^{\frac{1}{2}}, \tag{145}$$

and therefore the system is reduced to two forces of translation acting in space and along lines of action which do not meet each other.

And this reduction may in general be arranged so that the lines of action of the two forces shall be perpendicular to each other. Thus, as before, let the arm of the couple be perpendicular to the line of action of R; and let R be resolved into two parts $R\sin\phi$ and $R\cos\phi$ respectively in and perpendicular to the plane of the couple: so that there are, (1) three forces $R'$, $-R'$, $R\sin\phi$ in the plane of the couple, the lines of action of all of which are parallel and are perpendicular to the arm of the couple, and the resultant of which is $R\sin\phi$, which acts in the plane of the couple, at right-angles to its arm, and at a distance $r$ from the origin along the arm, such that $Rr\sin\phi = G$; and (2) the force $R\cos\phi$ whose line of action is perpendicular to the plane of the couple. Of this latter the magnitude is by (144)

$$R\cos\phi = \frac{LX+MY+NZ}{G}; \tag{146}$$

that is, is equal to the sum of the pressures along the coordinate axes resolved along the rotation-axis of the resultant couple.

Now the equations to these lines of action, which are perpendicular to each other, are, to the latter

$$\frac{x}{\text{L}} = \frac{y}{\text{M}} = \frac{z}{\text{N}}; \tag{147}$$

and to the former

$$\frac{x - \frac{\text{G}^2}{\text{T}^2}(\text{NY}-\text{MZ})}{\text{G}^2\text{X} - \text{L}(\text{LX}+\text{MY}+\text{NZ})} = \frac{y - \frac{\text{G}^2}{\text{T}^2}(\text{LZ}-\text{NX})}{\text{G}^2\text{Y} - \text{M}(\text{LX}+\text{MY}+\text{NZ})}$$

$$= \frac{z - \frac{\text{G}^2}{\text{T}^2}(\text{MX}-\text{LY})}{\text{G}^2\text{Z} - \text{N}(\text{LX}+\text{MY}+\text{NZ})}, \tag{148}$$

where

$$\text{T}^2 = (\text{NY}-\text{MZ})^2 + (\text{LZ}-\text{NX})^2 + (\text{MX}-\text{LY})^2 \tag{149}$$

$$= \text{R}^2\text{G}^2(\sin\phi)^2.$$

If $\phi = 90°$, we fall back on the case of Art. 58, and (148) become the equations (102), because $\text{LX}+\text{MY}+\text{NZ} = 0$, and $\text{T}^2 = \text{G}^2\text{R}^2$. And if $\phi = 0$, $r = \infty$, and the forces do not admit of being reduced to two forces of translation whose lines of action are perpendicular to each other.

As subsequent investigations will cast considerable light on the relations between the lines along which two forces, which equivalently replace a system of forces, act, I must reserve other theorems to a future part of the course.

71.] Also the value of G changes when the origin is changed, because L, M, and N are functions of the coordinates of the point of application of each force: this change I propose to investigate.

First however let it be observed, that of all axes passing through a given point that corresponding to G is the one whose moment or moment-axis is the greatest; for the moment of the impressed couples with respect to a rotation-axis inclined at an angle $\theta$ to that of G is $\text{G}\cos\theta$, as is plain from the law of resolution of such moment-axes, which has been investigated in Article 44; and as $\text{G}\cos\theta$ is less than G, it follows that of all lines passing through a given point, the rotation-axis of the resultant couple is that with respect to which the moment-axis is the greatest. For this reason G is called the complete or *principal moment-axis* with reference to the point which is called the *moment-centre*. Hence also we infer that the moment-axis is the same for all axes at a given moment-centre which are inclined at the same angle to the axis of the principal moment;

that is, all axes of equal moment form a right circular cone which has the axis of principal moment for its axis of figure.

Now let the origin be changed to a point $(x_0, y_0, z_0)$, the new axes being parallel to the former ones; and let $\mathrm{L}_0, \mathrm{M}_0, \mathrm{N}_0, \mathrm{G}_0$ be the values of L, M, N, G referred to the new origin; then

$$\mathrm{L}_0 = \Sigma.\mathrm{P}\,\{(y-y_0)\cos\gamma-(z-z_0)\cos\beta\}$$
$$= \Sigma.\mathrm{P}(y\cos\gamma-z\cos\beta)-y_0\,\Sigma.\mathrm{P}\cos\gamma+z_0\,\Sigma.\mathrm{P}\cos\beta,$$

$$\left.\begin{array}{l} \mathrm{L}_0 = \mathrm{L}-\mathrm{Z}y_0+\mathrm{Y}z_0 \\ \mathrm{M}_0 = \mathrm{M}-\mathrm{X}z_0+\mathrm{Z}x_0 \\ \mathrm{N}_0 = \mathrm{N}-\mathrm{Y}x_0+\mathrm{X}y_0 \end{array}\right\}; \qquad (150)$$

$$\therefore\quad \mathrm{L}_0\mathrm{X}+\mathrm{M}_0\mathrm{Y}+\mathrm{N}_0\mathrm{Z} = \mathrm{LX}+\mathrm{MY}+\mathrm{NZ}; \qquad (151)$$

that is, the quantity $\mathrm{LX}+\mathrm{MY}+\mathrm{NZ}$ is the same whatever point is the origin, provided that the axes of coordinates remain parallel to their original directions. And the same quantity is also invariable when the directions of the axes are changed. Let $x'$, $y'$, $z'$ be the new coordinates, and let them be related to $x, y, z$ by the following scheme of direction-cosines:

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x'$ | $a$ | $b$ | $c$ |
| $y'$ | $a_1$ | $b_1$ | $c_1$ |
| $z'$ | $a_2$ | $b_2$ | $c_2$ |

and let $\mathrm{X}', \mathrm{Y}', \mathrm{Z}', \mathrm{L}', \mathrm{M}', \mathrm{N}'$ be the new values of X, Y, Z, L, M, N, so that

$$\left.\begin{array}{l} \mathrm{X} = a\mathrm{X}'+a_1\mathrm{Y}'+a_2\mathrm{Z}' \\ \mathrm{Y} = b\mathrm{X}'+b_1\mathrm{Y}'+b_2\mathrm{Z}' \\ \mathrm{Z} = c\mathrm{X}'+c_1\mathrm{Y}'+c_2\mathrm{Z}' \end{array}\right\}, \qquad (152)$$

$$\left.\begin{array}{l} \mathrm{L}' = a\mathrm{L}+b\mathrm{M}+c\mathrm{N} \\ \mathrm{M}' = a_1\mathrm{L}+b_1\mathrm{M}+c_1\mathrm{N} \\ \mathrm{N}' = a_2\mathrm{L}+b_2\mathrm{M}+c_2\mathrm{N} \end{array}\right\}; \qquad (153)$$

$$\begin{aligned} \therefore\quad \mathrm{LX}+\mathrm{MY}+\mathrm{NZ} &= (a\mathrm{X}'+a_1\mathrm{Y}'+a_2\mathrm{Z}')\,\mathrm{L}+\ldots \\ &= \mathrm{X}'(a\mathrm{L}+b\mathrm{M}+c\mathrm{N})+\ldots \\ &= \mathrm{X}'\mathrm{L}'+\mathrm{Y}'\mathrm{M}'+\mathrm{Z}'\mathrm{N}'. \qquad (154) \end{aligned}$$

This quantity therefore is also invariable whatever are the directions of the coordinate axes. Also since the magnitude, and direction of the line of action of R is constant,

$$\text{L}\frac{\text{X}}{\text{R}}+\text{M}\frac{\text{Y}}{\text{R}}+\text{N}\frac{\text{Z}}{\text{R}} = \text{a constant};$$

$$\therefore \quad \text{L}\cos a+\text{M}\cos b+\text{N}\cos c = \text{a constant}; \qquad (155)$$

that is, whatever point is the origin, and whatever are the coordinate axes, if the moment-axes of the couples whose rotation-axes are the coordinate axes are resolved along the line of action of the resultant of translation, their sum is constant.

And replacing L, M and N from (143), (155) becomes

$$\text{G}\{\cos a\cos\lambda+\cos b\cos\mu+\cos c\cos\nu\} = \text{a constant};$$

so that, if $\phi$ is the angle between the rotation-axis of G and the line of action of R,

$$\text{G}\cos\phi = \text{a constant}; \qquad (156)$$

therefore the resolved part of every principal moment-axis along the line of action of R is constant.

Also from (150) we infer that if $\text{X}=\text{Y}=\text{Z}=0$, that is, if the resultant of translation is equal to zero, $\text{L}_0=\text{L}$, $\text{M}_0=\text{M}$, $\text{N}_0=\text{N}$, and therefore $\text{G}_0=\text{G}$; and thus the principal moment-axis is the same, both in intensity and direction, for all points of space.

72.] Now since, in reference to any given moment-centre, G corresponds to that line whose moment-axis is the greatest of all, but since this maximum moment-axis changes as we pass from one point to another in space, it is desirable to investigate those points the principal moment-axes corresponding to which have a minimum value: the result of the last Article shews (1) that there is such a minimum value, because $\text{G}\cos\phi=$ a constant, and therefore that G is a minimum when $\cos\phi$ is a maximum, that is, when $\phi=0$; and that therefore the rotation-axis of such a minimum principal moment is parallel to the line of action of R; and (2) that G cannot have an absolutely maximum value, because it becomes larger and larger as $\phi$ increases, and ultimately is infinite when $\phi=90°$; and the position of the point, or points, which are moment-centres of the minimum principal moment-axis may be found in the following manner:

Let $x'$, $y'$, $z'$ be the coordinates of the moment-centre for which G is a minimum: then in reference to the origin chosen arbitrarily, if $\text{G}'$, $\text{L}'$, $\text{M}'$, $\text{N}'$ are the moment-axes in reference to the new origin,

$$\text{G}'^2 = \text{L}'^2+\text{M}'^2+\text{N}'^2;$$

and therefore, by reason of (150),

$$G'^2 = (L - Zy' + Yz')^2 + (M - Xz' + Zx')^2 + (N - Yx' + Xy')^2; \quad (157)$$

wherefore, differentiating, we have

$$\begin{aligned} G'.DG' &= \{Z(M - Xz' + Zx') - Y(N - Yx' + Xy')\}\,dx' + \ldots + \ldots \quad (158) \\ &= (ZM' - YN')\,dx' + (XN' - ZL')\,dy' + (YL' - XM')\,dz'; \quad (159) \end{aligned}$$

and as the right-hand member of this equation must vanish, and as no relation is given between $x'$, $y'$, $z'$, the coefficients of $dx'$, $dy'$, $dz'$ must separately vanish; therefore

$$\begin{aligned} NY - MZ &= x'(Y^2 + Z^2) - X(Yy' + Zz') \\ &= x'(X^2 + Y^2 + Z^2) - X(Xx' + Yy' + Zz') \end{aligned}$$

$$\therefore \quad \frac{x' - \dfrac{NY - MZ}{R^2}}{X} = \frac{Xx' + Yy' + Zz'}{R^2};$$

and therefore from the symmetry of the right-hand member

$$\frac{x' - \dfrac{NY - MZ}{R^2}}{X} = \frac{y' - \dfrac{LZ - NX}{R^2}}{Y} = \frac{z' - \dfrac{MX - LY}{R^2}}{Z}; \quad (160)$$

which are the equations to a straight line parallel to the line of action of R. At all points therefore of this straight line, the principal moment-axis is a minimum of all points in space; and as the line is thus the locus of the centres of minima principal moments, it is called the *central axis;* also from (159)

$$\frac{L'}{X} = \frac{M'}{Y} = \frac{N'}{Z}; \quad (161)$$

so that the rotation-axis of the least principal moment is parallel to the line of action of R, and coincides with the central axis.

Let the moment-axis of this least principal moment = K, then as K is equal to the sum of the parts of L, M, N resolved along the direction of R,

$$K = \frac{LX + MY + NZ}{R}, \quad (162)$$

and K is constant for all points in the central axis. Hereby then the forces are reduced to (1) a force of translation R acting along the central axis whose equations are (160), and (2) to a couple whose moment-axis is given by (161) and whose rotation-axis is the central axis; and we have the following results:

(1) If any point of the central axis is taken as the moment-centre, of all axes of rotation passing through that point, that coincident with the central axis has the greatest moment.

(2) And with respect to moment-centres taken at any point in space, the moment of the rotation-axis coincident with the central axis is the least. On account of these theorems K is called the "minimum maximorum" moment.

If it is proposed to find the moment-centre corresponding to the greatest and least principal moments of all points on a given surface, $F(x', y', z') = 0$, then $dx', dy', dz'$ in the right-hand member of (159) are not independent, but are subject to a condition which is determined by the differential of the given function, and we have a problem of relative maxima or minima, which is to be solved in the usual manner.

73.] Of this most important result it is desirable to give further illustration. In fig. 42, let O be the original moment-centre; OR the line of action of the force of translation acting at it; OG the moment-axis of the resultant principal couple at O: let $GOM = \phi$, so that

$$\cos\phi = \frac{LX + MY + NZ}{GR}; \tag{163}$$

resolve OG into two parts, one along, and the other perpendicular to OR; then the part along OR is $G\cos\phi$, and that perpendicular to OR is $G\sin\phi$; the rotation-axis of $G\cos\phi$ is OR, and that of $G\sin\phi$ is a line in the plane containing OG and OR: at O draw OP perpendicular to this plane, and take OP of a length such that $R \times OP = G\sin\phi$; at P introduce two equal and opposite forces, each of which is equal to R, and whose line of action is parallel to that of R: then the couple whose arm is OP, and whose force is R, neutralizes the couple whose moment-axis is ON; and there remain (1) the force R acting at P, and in a line parallel to the original line of action of R, and (2) a couple whose moment-axis is $G\cos\phi$, and whose rotation-axis is along OR. Let the rotation-axis be transferred parallel to itself so as to pass through P, and we have finally a force of translation R acting along PR, and a couple whose rotation-axis is along the line of action of R, and whose moment-axis is K, where

$$\begin{aligned} K &= G\cos\phi \\ &= \frac{LX + MY + NZ}{R}. \end{aligned} \tag{164}$$

Now as OP is perpendicular to OG and OR, if $\alpha, \beta, \gamma$ are its direction-angles, we have

$$\frac{\cos\alpha}{\text{NY}-\text{MZ}} = \frac{\cos\beta}{\text{LZ}-\text{NX}} = \frac{\cos\gamma}{\text{MX}-\text{LY}} = \frac{1}{\Upsilon}, \text{ see (149)}, \tag{165}$$

$$\text{and} \qquad \text{OP} = \frac{\text{G}}{\text{R}}\sin\phi$$

$$= \frac{\Upsilon}{\text{R}^2}; \tag{166}$$

therefore the equations to PR are

$$\frac{x-\dfrac{\text{NY}-\text{MZ}}{\text{R}^2}}{\text{X}} = \frac{y-\dfrac{\text{LZ}-\text{NX}}{\text{R}^2}}{\text{Y}} = \frac{z-\dfrac{\text{MX}-\text{LY}}{\text{R}^2}}{\text{Z}}; \tag{167}$$

which are the equations to the central axis.

74.] Hence if OP $= p$, we have

$$\left.\begin{aligned} \text{R}p &= \text{G}\sin\phi \\ \text{K} &= \text{G}\cos\phi \end{aligned}\right\}, \tag{168}$$

$$\therefore \quad \text{G}^2 = \text{R}^2p^2+\text{K}^2; \tag{169}$$

therefore G, the principal moment at a point, is the same at all points equally distant from the central axis; and therefore the locus of all moment-centres, at which the principal moments are equal, is a circular cylindrical surface, of which the central axis is the axis of figure; and at all points along the same generating line of this cylinder, the rotation-axes of the principal moments are parallel, and all therefore lie in the plane touching the cylinder along the generating line; but the directions of the rotation-axes change as we pass from one generating line to another; for since $\phi$ is the angle between the central axis and the rotation-axis of the principal moment corresponding to a moment-centre at a distance $p$ from the central axis we have from (168)

$$\tan\phi = \frac{\text{R}p}{\text{K}}; \tag{170}$$

and this is therefore constant for all points of the cylindrical surface mentioned above; and as the direction-cosines of the central axis are proportional to X, Y, Z, and those of the rotation axis of the principal moment G to L, M, N, these lines in general do not meet: and therefore if a section is made of the cylindrical surface mentioned above by a plane perpendicular to the central axis, and the principal moment-axes are drawn for the moment-centres situated in the circular section, they will form a hyperboloid of revolution of one sheet, having the central axis for its axis of figure. See Ex. 1, Article 319, Vol. I.

Also if $G_0$ refers to a moment-centre, the perpendicular distance from which on the central axis is equal to $p_0$, then from (169)

$$G_0^2 = R^2 p_0^2 + K^2;$$

therefore subtracting we have

$$G^2 - G_0^2 = R^2 (p^2 - p_0^2).$$

I may observe that if $R = 0$, $G = K$; and therefore the principal moment is the same for all points in space; and as $\tan\phi = 0$, all the principal rotation-axes are parallel.

If the impressed forces admit of a single resultant, then $\phi = 90°$; and therefore from (164) $K = 0$; the "minimum maximorum" therefore of the principal moments is equal to zero.

From (169) it also follows that $G$ increases as $p$ increases, that is, as the moment-centre is removed further from the central axis; and as such increase is infinite, there cannot be a "maximum maximorum." Also from (170) it appears that as the moment-centre is removed further from the central axis, the angle between the central axis and the corresponding principal rotation-axis approaches nearer and nearer to a right-angle.

75.] And the equation to the cylindrical surface which is the locus of all moment-centres of equal moment may thus be found.

Let the origin be taken at a definite point on the central axis, so that the equations to the central axis are

$$\frac{x}{X} = \frac{y}{Y} = \frac{z}{Z}; \tag{171}$$

and let $(x_0, y_0, z_0)$ be the moment-centre at a distance $p$ from the central axis, and in a plane perpendicular to the central axis and passing through the origin; so that the equations to a generating line of the cylinder are

$$\frac{x - x_0}{X} = \frac{y - y_0}{Y} = \frac{z - z_0}{Z}; \tag{172}$$

and we have also

$$\begin{aligned} x_0^2 + y_0^2 + z_0^2 &= p^2 \\ &= \frac{G^2 - K^2}{R^2}, \end{aligned} \tag{173}$$

where $G$ is the principal moment at $(x_0, y_0, z_0)$; also

$$X x_0 + Y y_0 + Z z_0 = 0; \tag{174}$$

therefore from (172)

$$\frac{x-x_0}{\mathrm{X}} = \frac{y-y_0}{\mathrm{Y}} = \frac{z-z_0}{\mathrm{Z}} = \frac{\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z}{\mathrm{R}^2};$$

and substituting in (173),

$$(x^2+y^2+z^2)(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)-(\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z)^2 = \mathrm{G}^2-\mathrm{K}^2,$$

$$(\mathrm{Y}z-\mathrm{Z}y)^2+(\mathrm{Z}x-\mathrm{X}z)^2+(\mathrm{X}y-\mathrm{Y}x)^2 = \mathrm{G}^2-\mathrm{K}^2; \qquad (175)$$

which is the equation to the required cylindrical surface.

We may also in the following manner determine the equation to the surface of the hyperboloid which is the locus-surface of the rotation-axes of the principal moments relative to a section of the preceding cylinder by a plane perpendicular to its axis or to the central axis.

Let $(x_0, y_0, z_0)$ be the moment-centre to which the rotation-axis of the principal moment is referred, and let $p$ be the perpendicular distance from it on the central axis, and suppose the origin to be at the point where $p$ meets the central axis; then the equations to the rotation-axis are

$$\frac{x-x_0}{\mathrm{L}} = \frac{y-y_0}{\mathrm{M}} = \frac{z-z_0}{\mathrm{N}}; \qquad (176)$$

also we have $\quad \mathrm{L}^2+\mathrm{M}^2+\mathrm{N}^2 = \mathrm{G}^2$ (a constant),

$$x_0^2+y_0^2+z_0^2 = p^2 = \frac{\mathrm{G}^2-\mathrm{K}^2}{\mathrm{R}^2},$$

$$\left.\begin{array}{r} \mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0 = 0, \\ \mathrm{L}x_0+\mathrm{M}y_0+\mathrm{N}z_0 = 0, \\ \mathrm{LX}+\mathrm{MY}+\mathrm{NZ} = \mathrm{RG}\cos\phi = \mathrm{RK}; \end{array}\right\} \qquad (177)$$

of the last three equations, the first expresses the condition that $p$ is perpendicular to the central axis, the second that $p$ is perpendicular to the rotation-axis of the principal moment, the third that the rotation-axis is inclined at an angle $\phi$ to the central axis. Now from (176) we have

$$\frac{x-x_0}{\mathrm{L}} = \frac{y-y_0}{\mathrm{M}} = \frac{z-z_0}{\mathrm{N}} = \frac{\{x^2+y^2+z^2+p^2-2(xx_0+yy_0+zz_0)\}^{\frac{1}{2}}}{\mathrm{G}}$$

$$= \frac{\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z}{\mathrm{RK}}. \qquad (178)$$

Also multiplying the terms of (177) severally by those of the equality (176), we have

$$xx_0+yy_0+zz_0 = p^2,$$

therefore from the last terms of (178) we have

$$(x^2+y^2+z^2)\,\mathrm{R}^2\mathrm{K}^2-(\mathrm{G}^2-\mathrm{K}^2)\,\mathrm{K}^2 = (\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z)^2\mathrm{G}^2, \qquad (179)$$

which is the equation to a hyperboloid of revolution of one sheet, the central axis being the axis of figure.

And if the central axis is the coordinate-axis of $z$, $X = 0$, $Y = 0$, $Z = R$, in which case (179) becomes

$$x^2 + y^2 - z^2 (\tan \phi)^2 = p^2. \tag{180}$$

Also for all moment-centres on a straight line perpendicular to the central axis, the corresponding rotation-axes are on the surface of a hyperbolic paraboloid.

Let the central axis be the coordinate axis of $z$; and let the plane in which the given straight line is be that of $xy$; and let the moment-centre be the point $(x_0, y_0)$. Then if $X, Y, Z, L, M, N$ refer to the origin as the moment-centre, and $L_0, M_0, N_0$ to the given moment-centre, $X = 0$, $Y = 0$, $Z = R$, and by reason of (150)

$$L_0 = L - R y_0, \qquad M_0 = M + R x_0, \qquad N_0 = N;$$

so that the equations to the rotation-axis through $(x_0, y_0)$ are

$$\frac{x - x_0}{L - R y_0} = \frac{y - y_0}{M + R x_0} = \frac{z}{N}.$$

Also let the equation to the given straight line on which the moment-centre is be $ax + by = 0$; so that we have

$$\left.\begin{array}{rl} N x_0 - R z y_0 + L z - N x &= 0 \\ R z x_0 + N y_0 + M z - N y &= 0 \\ a x_0 + b y_0 \qquad\qquad &= 0 \end{array}\right\};$$

whence by cross-multiplication, and after reduction, we have

$$R(bL - aM) z^2 + a R N yz - b R N zx + N^2 (ax + by) - (aL + bM) N z = 0;$$

which is the equation to a hyperbolic paraboloid, and on the surface of which therefore all the rotation-axes are found.

76.] And let us further investigate certain properties of moment-centres situated anywhere in space.

Suppose $(x_0, y_0, z_0)$ to be the moment-centre of which $G_0$ is the principal moment, and of which $L_0, M_0, N_0$ are the resolved moments along the coordinate axes; and $L, M, N$ to be the resolved moments along the coordinate axes, of $G$, which is the principal moment with reference to the origin; then, by equations (150)

$$\left.\begin{array}{l} L_0 = L - Z y_0 + Y z_0 \\ M_0 = M - X z_0 + Z x_0 \\ N_0 = N - Y x_0 + X y_0 \end{array}\right\}; \tag{181}$$

and let us call the plane which passes through $(x_0, y_0, z_0)$, and is perpendicular to the rotation-axis of the principal moment at the point *the momental plane* *, so that the equation to the momental plane corresponding to the moment-centre $(x_0, y_0, z_0)$ is

$$(x-x_0)\,L_0+(y-y_0)\,M_0+(z-z_0)\,N_0=0,$$

or, $$(L-Zy_0+Yz_0)\,x+(M-Xz_0+Zx_0)\,y+(N-Yx_0+Xy_0)\,z$$
$$= Lx_0+My_0+Nz_0; \qquad (182)$$

and as this expression is always possible and definite, we conclude that every point in space may be a moment-centre; and have a momental plane which is unique for that point.

Hence also the equation to the momental plane corresponding to the origin as the moment-centre is, as is otherwise evident,

$$Lx+My+Nz=0.$$

Suppose moreover that the equation to a plane is

$$Ax+By+Cz=D, \qquad (183)$$

and that it is required to find in it the point which is its moment-centre; then comparing (183) and (182), we have

$$\frac{L-Zy_0+Yz_0}{A}=\frac{M-Xz_0+Zx_0}{B}=\frac{N-Yx_0+Xy_0}{C}=\frac{Lx_0+My_0+Nz_0}{D} \quad (184)$$
$$=\frac{LX+MY+NZ}{AX+BY+CZ};$$

$$\therefore \quad \left.\begin{aligned} x_0&=\frac{DX+BN-CM}{AX+BY+CZ}\\ y_0&=\frac{DY+CL-AN}{AX+BY+CZ}\\ z_0&=\frac{DZ+AM-BL}{AX+BY+CZ}\end{aligned}\right\}; \qquad (185)$$

and as these quantities are possible and definite, every plane may be a momental plane, and will accordingly have a moment-centre in it and corresponding to it, and whose coordinates are given by (185).

The coordinates of the moment-centres of the three coordinate planes are therefore,

$$\left.\begin{array}{llll} \text{Of the plane } yz, & x=0, & y=-\dfrac{N}{X}, & z=\dfrac{M}{X};\\[2ex] \text{Of the plane } zx, & x=\dfrac{N}{Y}, & y=0, & z=-\dfrac{L}{Y};\\[2ex] \text{Of the plane } xy, & x=-\dfrac{M}{Z}, & y=\dfrac{L}{Z}, & z=0;\end{array}\right\}$$

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* Möbius calls this plane "Null-ebene," as being that with respect to which the moment at the given point vanishes.

all which points evidently lie in the plane whose equation is

$$\mathrm{L}x+\mathrm{M}y+\mathrm{N}z=0,$$

and which is the momental plane of the origin; and hence also we infer that the moment-centres of the three coordinate planes lie in a plane passing through the origin of coordinates.

And the locus of the moment-centres of a system of parallel planes is a straight line which is parallel to the central axis; and of which the equations are

$$\frac{x(\mathrm{AX}+\mathrm{BY}+\mathrm{CZ})+\mathrm{CM}-\mathrm{BN}}{\mathrm{X}}=\frac{y(\mathrm{AX}+\mathrm{BY}+\mathrm{CZ})+\mathrm{AN}-\mathrm{CL}}{\mathrm{Y}}$$
$$=\frac{z(\mathrm{AX}+\mathrm{BY}+\mathrm{CZ})+\mathrm{BL}-\mathrm{AM}}{\mathrm{Z}},$$

these following immediately from (185). This line is of course a generating line of the cylinder (175).

Also if $\mathrm{G}_0$ is the principal moment-axis with reference to the point $(x_0, y_0, z_0)$, equations (185), then

$$\mathrm{G}_0=\frac{(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)^{\frac{1}{2}}}{\mathrm{AX}+\mathrm{BY}+\mathrm{CZ}}\,\mathrm{KR}.$$

77.] Also the perpendicular on the central axis from the moment-centre $(x_0, y_0, z_0)$ is at right angles to the rotation-axis of the principal moment in reference to that moment-centre.

Let us suppose the origin to be on the central axis; then the equations to it are

$$\frac{x}{\mathrm{X}}=\frac{y}{\mathrm{Y}}=\frac{z}{\mathrm{Z}}, \tag{186}$$

and the equations to the line passing through $(x_0, y_0, z_0)$ and perpendicular to and meeting the central axis are

$$\frac{x-x_0}{x_0\mathrm{R}^2-\mathrm{X}(\mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0)}=\frac{y-y_0}{y_0\mathrm{R}^2-\mathrm{Y}(\mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0)}$$
$$=\frac{z-z_0}{z_0\mathrm{R}^2-\mathrm{Z}(\mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0)}; \tag{187}$$

and the equations to the rotation-axis of the principal moment in reference to the moment-centre $(x_0, y_0, z_0)$ are

$$\frac{x-x_0}{\mathrm{L}_0}=\frac{y-y_0}{\mathrm{M}_0}=\frac{z-z_0}{\mathrm{N}_0}; \tag{188}$$

and the sum of the products of the corresponding denominators of (187) and (188) is

$$\mathrm{L}_0\{x_0\mathrm{R}^2-\mathrm{X}(\mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0)\}+\ldots+\ldots$$
$$=\mathrm{R}^2(\mathrm{L}_0x_0+\mathrm{M}_0y_0+\mathrm{N}_0z_0)-(\mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0)(\mathrm{L}_0\mathrm{X}+\mathrm{M}_0\mathrm{Y}+\mathrm{N}_0\mathrm{Z})$$
$$=\mathrm{R}^2(\mathrm{L}x_0+\mathrm{M}y_0+\mathrm{N}z_0)-(\mathrm{X}x_0+\mathrm{Y}y_0+\mathrm{Z}z_0)(\mathrm{LX}+\mathrm{MY}+\mathrm{NZ}) \tag{189}$$

by reason of (150), and wherein L, M, N refer to the central axis; and therefore we have

$$\frac{\mathrm{L}}{\mathrm{X}} = \frac{\mathrm{M}}{\mathrm{Y}} = \frac{\mathrm{N}}{\mathrm{Z}} = \frac{\mathrm{K}}{\mathrm{R}}, \tag{190}$$

so that (189) becomes

$$\mathrm{RK}\,(\mathrm{X}x_0 + \mathrm{Y}y_0 + \mathrm{Z}z_0) - (\mathrm{X}x_0 + \mathrm{Y}y_0 + \mathrm{Z}z_0)\,\mathrm{RK},$$

and which vanishes; and therefore the rotation-axis of the principal moment corresponding to $(x_0, y_0, z_0)$ is perpendicular to the line drawn from the moment-centre at right angles to the central axis; and therefore also the perpendicular from the moment-centre to the central axis is the shortest line between the central axis and the principal rotation-axis at that moment-centre.

Hence if $\phi$ is the angle contained between the central axis and the rotation-axis of the principal moment through the moment-centre $(x_0, y_0, z_0)$,

$$\begin{aligned} \cos\phi &= \frac{\mathrm{L}_0\mathrm{X} + \mathrm{M}_0\mathrm{Y} + \mathrm{N}_0\mathrm{Z}}{\mathrm{G}_0\mathrm{R}} \\ &= \frac{\mathrm{LX} + \mathrm{MY} + \mathrm{NZ}}{\mathrm{G}_0\mathrm{R}} \\ &= \frac{\mathrm{K}}{\mathrm{G}_0}, \end{aligned}$$

the same result as (168).

78.] Now let us consider the relations that exist between principal-moments, their projections on the coordinate axes and their rotation-axes in reference to two different moment-centres $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$. And let us suppose the origin to be on the central axis; and let the symbols carry the subscript 0 or 1, according as they refer to the former or the latter moment-centre. Then we have

$$\left.\begin{aligned} \mathrm{L}_0 &= \mathrm{L} - \mathrm{Z}y_0 + \mathrm{Y}z_0 \qquad & \mathrm{L}_1 &= \mathrm{L} - \mathrm{Z}y_1 + \mathrm{Y}z_1 \\ \mathrm{M}_0 &= \mathrm{M} - \mathrm{X}z_0 + \mathrm{Z}x_0 \qquad & \mathrm{M}_1 &= \mathrm{M} - \mathrm{X}z_1 + \mathrm{Z}x_1 \\ \mathrm{N}_0 &= \mathrm{N} - \mathrm{Y}x_0 + \mathrm{X}y_0 \qquad & \mathrm{N}_1 &= \mathrm{N} - \mathrm{Y}x_1 + \mathrm{X}y_1 \end{aligned}\right\}, \tag{191}$$

$$\therefore\quad \left.\begin{aligned} \mathrm{L}_1 - \mathrm{L}_0 &= \mathrm{Y}(z_1 - z_0) - \mathrm{Z}(y_1 - y_0) \\ \mathrm{M}_1 - \mathrm{M}_0 &= \mathrm{Z}(x_1 - x_0) - \mathrm{X}(z_1 - z_0) \\ \mathrm{N}_1 - \mathrm{N}_0 &= \mathrm{X}(y_1 - y_0) - \mathrm{Y}(x_1 - x_0) \end{aligned}\right\}; \tag{192}$$

$$\therefore\quad (\mathrm{L}_1 - \mathrm{L}_0)(x_1 - x_0) + (\mathrm{M}_1 - \mathrm{M}_0)(y_1 - y_0) + (\mathrm{N}_1 - \mathrm{N}_0)(z_1 - z_0) = 0. \tag{193}$$

Now the momental plane of the moment-centre $(x_0, y_0, z_0)$ is

$$\mathrm{L}_0(x - x_0) + \mathrm{M}_0(y - y_0) + \mathrm{N}_0(z - z_0) = 0; \tag{194}$$

and the momental plane of the moment-centre $(x_1, y_1, z_1)$ is

$$\mathrm{L}_1(x-x_1)+\mathrm{M}_1(y-y_1)+\mathrm{N}_1(z-z_1) = 0. \tag{195}$$

Suppose $(x_1, y_1, z_1)$ to be in the plane (194), then we have

$$\mathrm{L}_0(x_1-x_0)+\mathrm{M}_0(y_1-y_0)+\mathrm{N}_0(z_1-z_0) \quad 0; \tag{196}$$

so that from (193) we have

$$\mathrm{L}_1(x_0-x_1)+\mathrm{M}_1(y_0-y_1)+\mathrm{N}_1(z_0-z_1) = 0, \tag{197}$$

and therefore, by reason of (195), the momental plane of the moment-centre $(x_1, y_1, z_1)$ contains the moment-centre $(x_0, y_0, z_0)$. Hence we infer

The momental planes of all moment-centres in any given plane pass through the moment-centre of that plane:

If a straight line drawn in a given plane through the moment-centre of that plane is called a *ray of the plane*, it follows that the momental planes of all moment-centres in that ray intersect in the ray.

And such a ray may also be considered as lying in another plane which intersects the given momental plane along that ray; and therefore the momental planes of all moment-centres along this line pass through the moment-centre of this other plane, and therefore this latter moment-centre lies in the given ray; and therefore a ray of any plane is the locus of the moment-centres of all planes passing through the ray.

79.] And as to the conditions requisite that a given straight line should be the principal rotation-axis with reference to *some* moment-centre on itself, let the equations to the given straight line be

$$\frac{x-x_0}{\mathrm{A}} = \frac{y-y_0}{\mathrm{B}} = \frac{z-z_0}{\mathrm{C}} = \rho \text{ (say)}; \tag{198}$$

and let the moment-centre on it be $(x, y, z)$: then the direction-cosines of this line must be proportional to $\mathrm{L}$, $\mathrm{M}$, $\mathrm{N}$ which refer to the moment-centre $(x, y, z)$; let $\mathrm{L}_0$, $\mathrm{M}_0$, $\mathrm{N}_0$ refer to the moment-centre $(x_0, y_0, z_0)$; so that

$$\mathrm{L} = \mathrm{L}_0-\mathrm{Z}(y-y_0)+\mathrm{Y}(z-z_0), \quad \mathrm{M} = \ldots, \quad \mathrm{N} = \ldots; \tag{199}$$

$$\therefore \frac{\mathrm{L}_0-\mathrm{Z}(y-y_0)+\mathrm{Y}(z-z_0)}{\mathrm{A}} = \frac{\mathrm{M}_0-\mathrm{X}(z-z_0)+\mathrm{Z}(x-x_0)}{\mathrm{B}}$$

$$= \frac{\mathrm{N}_0-\mathrm{Y}(x-x_0)+\mathrm{X}(y-y_0)}{\mathrm{C}}; \tag{200}$$

therefore by means of (198),

$$\begin{aligned}\frac{\text{L}_0-\rho(\text{BZ}-\text{CY})}{\text{A}} &= \frac{\text{M}_0-\rho(\text{CX}-\text{AZ})}{\text{B}} = \frac{\text{N}_0-\rho(\text{AY}-\text{BX})}{\text{C}} \\ &= \frac{\text{AL}_0+\text{BM}_0+\text{CN}_0}{\text{A}^2+\text{B}^2+\text{C}^2} = \frac{\text{XL}_0+\text{YM}_0+\text{ZN}_0}{\text{AX}+\text{BY}+\text{CZ}} \\ &= \frac{\text{RK}}{\text{AX}+\text{BY}+\text{CZ}}\,;\end{aligned} \tag{201}$$

$$\therefore\quad (\text{AL}_0+\text{BM}_0+\text{CN}_0)(\text{AX}+\text{BY}+\text{CZ}) = \text{RK}(\text{A}^2+\text{B}^2+\text{C}^2), \tag{202}$$

which is the required condition; and if $\theta$ is the angle between the rotation-axis of the principal moment $\text{G}_0$ and the given straight line, and if $\phi$ is the angle between the given straight line and the central axis, (202) becomes

$$\text{G}_0 \cos\theta \cos\phi = \text{K}. \tag{203}$$

One property of this equation is worth notice; if the line which is to be the rotation-axis is perpendicular to the central axis, $\phi = 90°$, and therefore $\text{K} = 0$: that is, the system of forces is reducible to a single resultant of translation. Hence, conversely, if a system of forces is reducible to a single resultant, the only straight lines which can be rotation-axes with respect to any moment-centres on themselves are those which intersect at right angles the line of action of the single resultant; and if the forces do not admit of a single resultant, no straight line which is perpendicular to the central axis can be a rotation-axis with respect to a moment-centre on itself.

Also the coordinates of the moment-centre are thus determined: operate on the successive terms of the equality (201) with the factors $\text{BZ}-\text{CY}$, $\text{CX}-\text{AZ}$, $\text{AY}-\text{BX}$, then the sum of the denominators vanishes, and therefore the sum of the numerators also vanishes; whence we have

$$\rho = \frac{\text{L}_0(\text{BZ}-\text{CY})+\text{M}_0(\text{CX}-\text{AZ})+\text{N}_0(\text{AY}-\text{BX})}{(\text{BZ}-\text{CY})^2+(\text{CX}-\text{AZ})^2+(\text{AY}-\text{BX})^2}\,; \tag{204}$$

and therefore from (198)

$$x = x_0+\rho\text{A}, \quad y = y_0+\rho\text{B}, \quad z = z_0+\rho\text{C}, \tag{205}$$

and these are the coordinates to the moment-centre. If these values are substituted in (199), the principal moment with respect to the determined moment-centre may be found in terms of the principal moment at the given point $(x_0, y_0, z_0)$ and other known quantities.

In application of these results, if the axis of $x$ is a rotation-axis with respect to a moment-centre on itself, that is, if $\text{B} = \text{C} = 0$, equation (202) becomes

$$\text{M}_0\text{Y}+\text{N}_0\text{Z}=0,$$

and if the point $(x_0, y_0, z_0)$ is the origin, then from (205)

$$x=\frac{\text{N}_0}{\text{Y}}=-\frac{\text{M}_0}{\text{Z}},$$

and therefore the principal moment at the point is $\text{L}_0$.

By a similar process we may determine the conditions that the other two rectangular coordinate-axes may be rotation-axes with respect to moment-centres on themselves; and also the coordinates to the required moment-centres. It is evident by reason of the required condition that all three cannot simultaneously be rotation-axes of the given nature.

80.] And another inquiry suggests itself; to what relations are all momental planes subject, whose moment-centres are on a given straight line?

Let $\text{L}_1, \text{M}_1, \text{N}_1$ refer to the point $(x_1, y_1, z_1)$ as a moment-centre; so that the equation to the corresponding momental plane is

$$\text{L}_1(x-x_1)+\text{M}_1(y-y_1)+\text{N}_1(z-z_1)=0; \qquad (206)$$

and suppose the moment-centre to be on a straight line whose equations are

$$\frac{x_1-x_0}{\text{A}}=\frac{y_1-y_0}{\text{B}}=\frac{z_1-z_0}{\text{C}}=\rho\ \text{(say)}, \qquad (207)$$

where $(x_0, y_0, z_0)$ is a fixed point on the line: then

$$x_1=x_0+\text{A}\rho, \quad y_1=y_0+\text{B}\rho, \quad z_1=z_0+\text{C}\rho; \qquad (208)$$

and if $\text{L}_0, \text{M}_0, \text{N}_0$ refer to the fixed point $(x_0, y_0, z_0)$,

$$\left.\begin{aligned}\text{L}_1&=\text{L}_0-\text{Z}(y_1-y_0)+\text{Y}(z_1-z_0)\\ \text{M}_1&=\text{M}_0-\text{X}(z_1-z_0)+\text{Z}(x_1-x_0)\\ \text{N}_1&=\text{N}_0-\text{Y}(x_1-x_0)+\text{X}(y_1-y_0)\end{aligned}\right\}; \qquad (209)$$

and if $\quad \text{BZ}-\text{CY}=\alpha, \quad \text{CX}-\text{AZ}=\beta, \quad \text{AY}-\text{BX}=\gamma, \qquad (210)$

we have from (208) and (209)

$$\text{L}_1=\text{L}_0-\rho\alpha, \quad \text{M}_1=\text{M}_0-\rho\beta, \quad \text{N}_1=\text{N}_0-\rho\gamma; \qquad (211)$$

so that (206) becomes

$$(\text{L}_0-\alpha\rho)(x-x_0-\text{A}\rho)+(\text{M}_0-\beta\rho)(y-y_0-\text{B}\rho)$$
$$+(\text{N}_0-\gamma\rho)(z-z_0-\text{C}\rho)=0; \qquad (212)$$

and therefore since $\quad \text{A}\alpha+\text{B}\beta+\text{C}\gamma=0, \qquad (213)$

we have

$$\text{L}_0(x-x_0)+\text{M}_0(y-y_0)+\text{N}_0(z-z_0)$$
$$-\rho\{\alpha(x-x_0)+\beta(y-y_0)+\gamma(z-z_0)+\text{AL}_0+\text{BM}_0+\text{CN}_0\}=0; \qquad (214)$$

which equation, it will be observed, consists of two linear expressions of the first degree, the second of which is multiplied by the variable factor $\rho$: and therefore we conclude that all the momental planes, whose moment-centres are on the straight line (207), pass through a line which is the intersection of the two planes

$$\left.\begin{array}{r} \text{L}_0(x-x_0)+\text{M}_0(y-y_0)+\text{N}_0(z-z_0)=0 \\ a(x-x_0)+\beta(y-y_0)+\gamma(z-z_0)+\text{AL}_0+\text{BM}_0+\text{CN}_0=0 \end{array}\right\}. \qquad (215)$$

The direction-cosines of this last line are proportional to

$$\text{M}_0\gamma-\text{N}_0\beta, \qquad \text{N}_0 a-\text{L}_0\gamma, \qquad \text{L}_0\beta-\text{M}_0 a,$$

that is, if $\text{Q}=\text{AL}_0+\text{BM}_0+\text{CN}_0,$ (216)

to $\text{ARK}-\text{XQ}, \quad \text{BRK}-\text{YQ}, \quad \text{CRK}-\text{ZQ};$

and if we substitute

$$\begin{aligned} \omega &= \text{L}_0(\beta-\gamma)+\text{M}_0(\gamma-a)+\text{N}_0(a-\beta) \\ &= \text{RK}(\text{A}+\text{B}+\text{C})-\text{Q}(\text{X}+\text{Y}+\text{Z}), \end{aligned}$$

the equations (215) may be put into the form

$$\frac{x-x_0-\frac{\text{Q}}{\omega}(\text{M}_0-\text{N}_0)}{\text{ARK}-\text{XQ}}=\frac{y-y_0-\frac{\text{Q}}{\omega}(\text{N}_0-\text{L}_0)}{\text{BRK}-\text{YQ}}=\frac{z-z_0-\frac{\text{Q}}{\omega}(\text{L}_0-\text{M}_0)}{\text{CRK}-\text{ZQ}}, \quad (217)$$

or,

$$\frac{x-x_0-\frac{\text{Q}}{\omega}(\text{M}_0-\text{N}_0)}{\text{M}_0\gamma-\text{N}_0\beta}=\frac{y-y_0-\frac{\text{Q}}{\omega}(\text{N}_0-\text{L}_0)}{\text{N}_0 a-\text{L}_0\gamma}=\frac{z-z_0-\frac{\text{Q}}{\omega}(\text{L}_0-\text{M}_0)}{\text{L}_0\beta-\text{M}_0 a};$$

and this line therefore is that through which pass all the momental planes whose moment-centres are on the line (207).

81.] Let us however consider the converse problem of the preceding: viz. to find the locus of the moment-centres of a series of planes passing through the line whose equations are (207).

The equation to any such plane is evidently of the form

$$l(x-x_0)+m(y-y_0)+n(z-z_0) \\ +\rho\{l'(x-x_0)+m'(y-y_0)+n'(z-z_0)\}=0, \quad (218)$$

where $\rho$ is an arbitrary constant, and where

$$\left.\begin{array}{r} l\text{A}+m\text{B}+n\text{C}=0 \\ l'\text{A}+m'\text{B}+n'\text{C}=0 \end{array}\right\}; \qquad (219)$$

and if $(\xi,\eta,\zeta)$ is the moment-centre, the direction-cosines of the rotation-axis of its principal moment are proportional to

$$\text{L}_0-\text{Z}(\eta-y_0)+\text{Y}(\zeta-z_0), \quad \dots, \quad \dots;$$

and to $\quad l+\rho l', \quad m+\rho m', \quad n+\rho n'$; therefore

$$\frac{\text{L}_0-\text{Z}(\eta-y_0)+\text{Y}(\zeta-z_0)}{l+\rho l'}=\frac{\text{M}_0-\text{X}(\zeta-z_0)+\text{Z}(\xi-x_0)}{m+\rho m'}$$
$$=\frac{\text{N}_0-\text{Y}(\xi-x_0)+\text{X}(\eta-y_0)}{n+\rho n'}; \quad (220)$$

and since $(\xi, \eta, \zeta)$ is in the plane (218), we have

$$(l+\rho l')(\xi-x_0)+(m+\rho m')(\eta-y_0)+(n+\rho n')(\zeta-z_0)=0; \quad (221)$$

and from (219)

$$(l+\rho l')\text{A}+(m+\rho m')\text{B}+(n+\rho n')\text{C}=0: \quad (222)$$

so that operating on the terms of the equality (220), we have

$$\text{L}_0(\xi-x_0)+\text{M}_0(\eta-y_0)+\text{N}_0(\zeta-z_0)=0, \quad (223)$$

$$\alpha(\xi-x_0)+\beta(\eta-y_0)+\gamma(\zeta-z_0)+\text{AL}_0+\text{BM}_0+\text{CN}_0=0, \quad (224)$$

and which are the same equations as (215); and therefore the locus of the moment-centres of all momental-planes passing through the line (207) is the line of intersection of the two planes whose equations are (223) and (224). Hence, to recapitulate the results,

If a moment-centre moves along a line whose equations are

$$\frac{x-x_0}{\text{A}}=\frac{y-y_0}{\text{B}}=\frac{z-z_0}{\text{C}}, \quad (225)$$

all the momental-planes pass through the line (217), which is formed by the intersection of the two planes

$$\left.\begin{array}{r}\text{L}_0(x-x_0)+\text{M}_0(y-y_0)+\text{N}_0(z-z_0)=0,\\(\text{BZ}-\text{CY})(x-x_0)+(\text{CX}-\text{AZ})(y-y_0)+(\text{AY}-\text{BX})(z-z_0)\\+\text{AL}_0+\text{BM}_0+\text{CN}_0=0\end{array}\right\}; \quad (226)$$

and if a moment-centre moves along the line of intersection of these two planes, all the momental planes pass through the line whose equations are (225). These lines are for an obvious reason called *reciprocal lines*, (Gegen-linien)*; (207) and (217) are therefore reciprocal lines.

I may observe that, if $\text{K}=0$, that is, if the system of forces is reducible to a single resultant of translation, (217) are independent of A, B, and C; and that the line is parallel to the line of action of the resultant of translation: hence we infer that under these circumstances this latter line is reciprocal to every line in space.

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* See Schweins, Crelle Vol. XXXVIII, pp. 40–76; Möbius, Statik, § 86, Erster Theil, Leipzig, 1837; and Steichen, Crelle Vol. XXXVIII, pp. 277–330; and Vol. XLIV, pp. 181–219.

Let us briefly consider the geometrical position of two reciprocal lines; let the origin be placed at the point $(x_0, y_0, z_0)$ on the line (225): then the first plane of (226) is the momental-plane with respect to that point as the moment-centre; and the second plane is parallel to the plane containing the first reciprocal line and the line of the resultant of translation at the origin; also if $\phi$ is the angle between the line (225) and the line of action of the resultant of translation, then

$$(\sin\phi)^2 = \frac{(\mathrm{BZ}-\mathrm{CY})^2+(\mathrm{CX}-\mathrm{AZ})^2+(\mathrm{AY}-\mathrm{BX})^2}{(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)\,\mathrm{R}^2},$$

and the perpendicular from the origin on the second plane of (226) is equal to

$$\frac{\mathrm{AL}_0+\mathrm{BM}_0+\mathrm{CN}_0}{\mathrm{R}\sin\phi\,\{\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2\}^{\frac{1}{2}}}.$$

82.] The two reciprocal lines are perpendicular to each other, if

$$\mathrm{A}\,\{\mathrm{M}_0(\mathrm{AY}-\mathrm{BX})-\mathrm{N}_0(\mathrm{CX}-\mathrm{AZ})\}+\ldots+\ldots=0, \qquad (227)$$

$$(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)(\mathrm{LX}+\mathrm{MY}+\mathrm{NZ})-(\mathrm{AX}+\mathrm{BY}+\mathrm{CZ})(\mathrm{AL}_0+\mathrm{BM}_0+\mathrm{CN}_0)=0, \quad (228)$$

which is the same as (202), and expresses the condition that the line (225) should be the rotation-axis of the principal moment with reference to some moment-centre on itself. And when this condition is fulfilled, (226) is also the rotation-axis of the principal moment with reference to some moment-centre on itself.

For comparing the direction-cosines of this latter line as derived from (199) with those derived from (226), we have

$$\frac{\mathrm{L}_0-\mathrm{Z}(y-y_0)+\mathrm{Y}(z-z_0)}{\mathrm{M}_0(\mathrm{AY}-\mathrm{BX})-\mathrm{N}_0(\mathrm{CX}-\mathrm{AZ})}=\ldots=\ldots \qquad (229)$$

of which, when operated on with the factors A, B, C, the sum of the numerators vanishes by reason of (226); and therefore the sum of the denominators also vanishes: and this latter equation is that marked (227), so that the proposition is proved.

And as reciprocal lines which are perpendicular to each other are rotation-axes of the principal moments with reference to moment-centres on themselves, so if $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$ are the moment-centres, the equations to the lines are

$$\frac{x-x_0}{\mathrm{L}_0}=\frac{y-y_0}{\mathrm{M}_0}=\frac{z-z_0}{\mathrm{N}_0},$$

$$\frac{x-x_1}{\mathrm{L}_1}=\frac{y-y_1}{\mathrm{M}_1}=\frac{z-z_1}{\mathrm{N}_1}.$$

Now $\text{K} = \text{G} \cos \phi$, see equation (168): therefore if $\phi_0$ and $\phi_1$ are the angles between the central axis and the reciprocal lines

$$\text{K} = \text{G}_0 \cos \phi_0 = \text{G}_1 \cos \phi_1,$$

$$\therefore \quad \frac{1}{\text{G}_0^2} + \frac{1}{\text{G}_1^2} = \frac{(\cos \phi_0)^2 + (\cos \phi_1)^2}{\text{K}^2};$$

but as the reciprocal lines are perpendicular to each other, $\phi_0 + \phi_1 = 90°$, therefore

$$\frac{1}{\text{G}_0^2} + \frac{1}{\text{G}_1^2} = \frac{1}{\text{K}^2}.$$

Also two reciprocal lines are parallel, if

$$\frac{\text{ARK}-\text{XQ}}{\text{A}} = \frac{\text{BRK}-\text{YQ}}{\text{B}} = \frac{\text{CRK}-\text{ZQ}}{\text{C}},$$

that is, if $\quad \frac{\text{X}}{\text{A}} = \frac{\text{Y}}{\text{B}} = \frac{\text{Z}}{\text{C}};$

that is, when each line is parallel to the central axis; and also when $\text{Q} = 0$, in which case the reciprocal lines coincide with each other, and with the central axis, every point of which is of course a moment-centre with respect to the line which is the principal rotation-axis with respect to it.

83.] The equations to the line of shortest distance between two reciprocal lines are found in the following way:

For convenience of symbolization let

$$\text{M}_0\gamma - \text{N}_0\beta = \text{A}', \qquad \text{N}_0\alpha - \text{L}_0\gamma = \text{B}', \qquad \text{L}_0\beta - \text{M}_0\alpha = \text{C}', \tag{230}$$

so that the equations to the reciprocal lines are, from (207) and (217),

$$\frac{x-x_0}{\text{A}} = \frac{y-y_0}{\text{B}} = \frac{z-z_0}{\text{C}} = r, \tag{231}$$

$$\frac{x-x_0-\frac{\text{Q}}{\omega}(\text{M}_0-\text{N}_0)}{\text{A}'} = \frac{y-y_0-\frac{\text{Q}}{\omega}(\text{N}_0-\text{L}_0)}{\text{B}'} = \frac{z-z_0-\frac{\text{Q}}{\omega}(\text{L}_0-\text{M}_0)}{\text{C}'} = r'; \tag{232}$$

and let $\cos\lambda$, $\cos\mu$, $\cos\nu$ be the direction-cosines of the shortest line; then, as we know that the shortest line intersects both the above lines at right angles, we have

$$\left.\begin{array}{l} \text{A}\cos\lambda + \text{B}\cos\mu + \text{C}\cos\nu = 0 \\ \text{A}'\cos\lambda + \text{B}'\cos\mu + \text{C}'\cos\nu = 0 \end{array}\right\}; \tag{233}$$

$$\therefore \quad \frac{\cos\lambda}{\mathrm{BC}'-\mathrm{B}'\mathrm{C}} = \frac{\cos\mu}{\mathrm{CA}'-\mathrm{AC}'} = \frac{\cos\nu}{\mathrm{AB}'-\mathrm{A}'\mathrm{B}},$$

$$\text{or,} \quad \frac{\cos\lambda}{\mathrm{BZ}-\mathrm{CY}} = \frac{\cos\mu}{\mathrm{CX}-\mathrm{AZ}} = \frac{\cos\nu}{\mathrm{AY}-\mathrm{BX}}. \tag{234}$$

Let $\xi, \eta, \zeta$ be the current coordinates of the shortest line; and $(x, y, z)$, $(x', y', z')$ the points where it intersects severally (231) and (232); then, by reason of its being perpendicular to (231) and to (232), we have

$$(\xi-x)\,\mathrm{A} + (\eta-y)\,\mathrm{B} + (\zeta-z)\,\mathrm{C} = 0,$$
$$(\xi-x)\,\mathrm{A}' + (\eta-y)\,\mathrm{B}' + (\zeta-z)\,\mathrm{C}' = 0,$$
$$\therefore \quad (\xi-x_0)\,\mathrm{A} + (\eta-y_0)\,\mathrm{B} + (\zeta-z_0)\,\mathrm{C} = (\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)\,r,$$
$$(\xi-x_0)\,\mathrm{A}' + (\eta-y_0)\,\mathrm{B}' + (\zeta-z_0)\,\mathrm{C}' = (\mathrm{AA}'+\mathrm{BB}'+\mathrm{CC}')\,r;$$

whence eliminating $r$, we have

$$(\xi-x_0)\,\{\mathrm{A}\,(\mathrm{AA}'+\mathrm{BB}'+\mathrm{CC}')-\mathrm{A}'(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)\}+\ldots+\ldots=0; \tag{235}$$

and if we operate in a similar manner on (232), we shall have

$$\{\xi-x_0-\frac{\mathrm{Q}}{\omega}(\mathrm{M}_0-\mathrm{N}_0)\}\,\{\mathrm{A}'(\mathrm{AA}'+\mathrm{BB}'+\mathrm{CC}')-\mathrm{A}\,(\mathrm{A}'^2+\mathrm{B}'^2+\mathrm{C}'^2)\}$$
$$+\ldots+\ldots=0; \tag{236}$$

and if we substitute

$$\left.\begin{aligned} \mathrm{L}_0\alpha+\mathrm{M}_0\beta+\mathrm{N}_0\gamma &= k \\ \alpha^2+\beta^2+\gamma^2 &= \rho^2 \end{aligned}\right\}, \tag{237}$$

after obvious reductions these equations become respectively

$$(\xi-x_0)\,(\mathrm{C}\beta-\mathrm{B}\gamma)+(\eta-y_0)\,(\mathrm{A}\gamma-\mathrm{C}\alpha)+(\zeta-z_0)\,(\mathrm{B}\alpha-\mathrm{A}\beta)=0,$$
$$(\xi-x_0)\,(\alpha k-\mathrm{L}_0\rho^2)+(\eta-y_0)\,(\beta k-\mathrm{M}_0\rho^2)$$
$$+(\zeta-z_0)\,(\gamma k-\mathrm{N}_0\rho^2)+\mathrm{Q}k=0; \tag{238}$$

and the shortest line is the line of intersection of these two planes.

To find its equations; from the preceding equations let $\xi-x_0$, $\eta-y_0$, $\zeta-z_0$ be successively eliminated; whereby we have

$$\left.\begin{aligned} (\eta-y_0)\,\gamma-(\zeta-z_0)\,\beta &= -\frac{k}{\rho^2}(\mathrm{C}\beta-\mathrm{B}\gamma) \\ (\zeta-z_0)\,\alpha-(\xi-x_0)\,\gamma &= -\frac{k}{\rho^2}(\mathrm{A}\gamma-\mathrm{C}\alpha) \\ (\xi-x_0)\,\beta-(\eta-y_0)\,\alpha &= -\frac{k}{\rho^2}(\mathrm{B}\alpha-\mathrm{A}\beta) \end{aligned}\right\}; \tag{239}$$

whence we have

$$\begin{aligned}\{(\xi-x_0)\alpha+(\eta-y_0)\beta+(\zeta-z_0)\gamma\}\,\alpha-(\xi-x_0)\,\rho^2 &= -\frac{k}{\rho^2}\{\mathrm{A}\beta^2+\mathrm{A}\gamma^2-\mathrm{B}\alpha\beta-\mathrm{C}\alpha\gamma\}\\ &= -\frac{k}{\rho^2}\{\rho^2\mathrm{A}-\alpha(\mathrm{A}\alpha+\mathrm{B}\beta+\mathrm{C}\gamma)\}\\ &= -k\mathrm{A}.\end{aligned}$$

Therefore from the symmetry we have

$$\frac{\xi-x_0-\dfrac{k\mathrm{A}}{\rho^2}}{\alpha}=\frac{\eta-y_0-\dfrac{k\mathrm{B}}{\rho^2}}{\beta}=\frac{\zeta-z_0-\dfrac{k\mathrm{C}}{\rho^2}}{\gamma}; \qquad (240)$$

and these are the equations to the shortest line between a pair of reciprocal lines.

Now with regard to the central axis, it is in the first place plain that the line whose equations are (240) is perpendicular to it: for the direction-cosines of the central axis are (see equations (167)) proportional to X, Y, Z; and

$$\begin{aligned}\alpha\mathrm{X}+\beta\mathrm{Y}+\gamma\mathrm{Z} &= \mathrm{X}(\mathrm{BZ}-\mathrm{CY})+\mathrm{Y}(\mathrm{CX}-\mathrm{AZ})+\mathrm{Z}(\mathrm{AY}-\mathrm{BX})\\ &= 0;\end{aligned}$$

therefore the shortest distance between two reciprocal lines is perpendicular to the central axis. And it also meets the central axis; for writing the equations (167) of the central axis in the form

$$\frac{\xi-x_0-\dfrac{\mathrm{N}_0\mathrm{Y}-\mathrm{M}_0\mathrm{Z}}{\mathrm{R}^2}}{\mathrm{X}}=\frac{\eta-y_0-\dfrac{\mathrm{L}_0\mathrm{Z}-\mathrm{N}_0\mathrm{X}}{\mathrm{R}^2}}{\mathrm{Y}}=\frac{\zeta-z_0-\dfrac{\mathrm{M}_0\mathrm{X}-\mathrm{L}_0\mathrm{Y}}{\mathrm{R}^2}}{\mathrm{Z}}, \qquad (241)$$

and applying to them, and to the equations (240), the ordinary conditions that two straight lines should be in the same plane, we have

$$\begin{aligned}&\left(\frac{k\mathrm{A}}{\rho^2}-\frac{\mathrm{N}_0\mathrm{Y}-\mathrm{M}_0\mathrm{Z}}{\mathrm{R}^2}\right)(\beta\mathrm{Z}-\gamma\mathrm{Y})+\left(\frac{k\mathrm{B}}{\rho^2}-\frac{\mathrm{L}_0\mathrm{Z}-\mathrm{N}_0\mathrm{X}}{\mathrm{R}^2}\right)(\gamma\mathrm{X}-\alpha\mathrm{Z})\\ &\qquad\qquad+\left(\frac{k\mathrm{C}}{\rho^2}-\frac{\mathrm{M}_0\mathrm{X}-\mathrm{L}_0\mathrm{Y}}{\mathrm{R}^2}\right)(\alpha\mathrm{Y}-\beta\mathrm{X})=\\ &\left(\frac{k\mathrm{A}}{\rho^2}-\frac{\mathrm{N}_0\mathrm{Y}-\mathrm{M}_0\mathrm{Z}}{\mathrm{R}^2}\right)\{\mathrm{X}(\mathrm{AX}+\mathrm{BY}+\mathrm{CZ})-\mathrm{AR}^2\}+\ldots+\ldots=\\ &\frac{k}{\rho^2}\{(\mathrm{AX}+\mathrm{BY}+\mathrm{CZ})^2-(\mathrm{A}^2+\mathrm{B}^2+\mathrm{C}^2)\mathrm{R}^2\}+\mathrm{A}(\mathrm{N}_0\mathrm{Y}-\mathrm{M}_0\mathrm{Z})\\ &\qquad\qquad+\mathrm{B}(\mathrm{L}_0\mathrm{Z}-\mathrm{N}_0\mathrm{X})+\mathrm{C}(\mathrm{M}_0\mathrm{X}-\mathrm{L}_0\mathrm{Y})=\\ &-\frac{k}{\rho^2}\{\alpha^2+\beta^2+\gamma^2\}+\mathrm{L}_0(\mathrm{BZ}-\mathrm{CY})+\mathrm{M}_0(\mathrm{CX}-\mathrm{AZ})+\mathrm{N}_0(\mathrm{AY}-\mathrm{BX})=\\ &\qquad\qquad-k+\mathrm{L}_0\alpha+\mathrm{M}_0\beta+\mathrm{N}_0\gamma=0;\end{aligned}$$

whence we conclude that the shortest distance also meets the central axis.

84.] We are now able to resume the discussions of Art. 70, and to investigate other properties of two forces, which, applied at definite points, along definite lines of action, and in definite directions, equivalently replace a system of forces acting on a rigid body in space; and the preceding properties of reciprocal lines will aid us much in the investigation.

Let the two forces be $\mathrm{P}_0$ and $\mathrm{P}_1$, of which let the points of application be $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$; and which may also be considered to be the current coordinates of their lines of action: and let the components along the coordinate axes be $\mathrm{X}_0\,\mathrm{Y}_0\,\mathrm{Z}_0$, $\mathrm{X}_1\,\mathrm{Y}_1\,\mathrm{Z}_1$; so that we have the following equations: viz.

$$\mathrm{X} = \mathrm{X}_0 + \mathrm{X}_1, \qquad \mathrm{Y} = \mathrm{Y}_0 + \mathrm{Y}_1, \qquad \mathrm{Z} = \mathrm{Z}_0 + \mathrm{Z}_1, \tag{242}$$

$$\left.\begin{aligned} \mathrm{L} &= y_1\mathrm{Z}_1 - z_1\mathrm{Y}_1 + y_0\mathrm{Z}_0 - z_0\mathrm{Y}_0 \\ \mathrm{M} &= z_1\mathrm{X}_1 - x_1\mathrm{Z}_1 + z_0\mathrm{X}_0 - x_0\mathrm{Z}_0 \\ \mathrm{N} &= x_1\mathrm{Y}_1 - y_1\mathrm{X}_1 + x_0\mathrm{Y}_0 - y_0\mathrm{X}_0 \end{aligned}\right\}; \tag{243}$$

and let $\mathrm{L}_0$, $\mathrm{M}_0$, $\mathrm{N}_0$ be the moment-axes of the original impressed couples about the coordinate axes at the point $(x_0, y_0, z_0)$; so that

$$\left.\begin{aligned} \mathrm{L} &= \mathrm{L}_0 + \mathrm{Z}y_0 - \mathrm{Y}z_0 \\ \mathrm{M} &= \mathrm{M}_0 + \mathrm{X}z_0 - \mathrm{Z}x_0 \\ \mathrm{N} &= \mathrm{N}_0 + \mathrm{Y}x_0 - \mathrm{X}y_0 \end{aligned}\right\}; \tag{244}$$

and substituting these in (243), and replacing $\mathrm{X}_0, \mathrm{Y}_0, \mathrm{Z}_0$ from (242), we have

$$\left.\begin{aligned} \mathrm{L}_0 &= \mathrm{Z}_1(y_1 - y_0) - \mathrm{Y}_1(z_1 - z_0) \\ \mathrm{M}_0 &= \mathrm{X}_1(z_1 - z_0) - \mathrm{Z}_1(x_1 - x_0) \\ \mathrm{N}_0 &= \mathrm{Y}_1(x_1 - x_0) - \mathrm{X}_1(y_1 - y_0) \end{aligned}\right\}; \tag{245}$$

from which equations $x_1, y_1, z_1$, the coordinates of the point of application of $\mathrm{P}_1$, are to be determined; but as the equations involve the equation of condition

$$\begin{aligned} \mathrm{L}_0\mathrm{X}_1 + \mathrm{M}_0\mathrm{Y}_1 + \mathrm{N}_0\mathrm{Z}_1 &= 0, \\ \text{or,} \quad \mathrm{L}_0\mathrm{X}_0 + \mathrm{M}_0\mathrm{Y}_0 + \mathrm{N}_0\mathrm{Z}_0 &= \mathrm{LX} + \mathrm{MY} + \mathrm{NZ} \\ &= \mathrm{RK}, \end{aligned} \tag{246}$$

they are equivalent to only two independent equations, and are therefore the equations to a straight line; and may be put into the form

$$\frac{x_1 - x_0 + \dfrac{\mathrm{M}_0\mathrm{Z}_1 - \mathrm{N}_0\mathrm{Y}_1}{\mathrm{P}_1^{\,2}}}{\mathrm{X}_1} = \frac{y_1 - y_0 + \dfrac{\mathrm{N}_0\mathrm{X}_1 - \mathrm{L}_0\mathrm{Z}_1}{\mathrm{P}_1^{\,2}}}{\mathrm{Y}_1} = \frac{z_1 - z_0 + \dfrac{\mathrm{L}_0\mathrm{Y}_1 - \mathrm{M}_0\mathrm{X}_1}{\mathrm{P}_1^{\,2}}}{\mathrm{Z}_1}, \tag{247}$$

or,

$$\frac{x_1-x_0-\dfrac{\mathrm{M}_0-\mathrm{N}_0}{\mathrm{X}_1+\mathrm{Y}_1+\mathrm{Z}_1}}{\mathrm{X}_1}=\frac{y_1-y_0-\dfrac{\mathrm{N}_0-\mathrm{L}_0}{\mathrm{X}_1+\mathrm{Y}_1+\mathrm{Z}_1}}{\mathrm{Y}_1}=\frac{z_1-z_0-\dfrac{\mathrm{L}_0-\mathrm{M}_0}{\mathrm{X}_1+\mathrm{Y}_1+\mathrm{Z}_1}}{\mathrm{Z}_1}; \quad (248)$$

and therefore the point of application of $\mathrm{P}_1$ may be anywhere in this line; and as the direction-cosines of the line are proportional to the components, along the coordinate axes, of $\mathrm{P}_1$, we infer that the line of action of $\mathrm{P}_1$ is coincident with the line whose equations are (247) or (248).

Now from (245) we have

$$\mathrm{L}_0(x_1-x_0)+\mathrm{M}_0(y_1-y_0)+\mathrm{N}_0(z_1-z_0)=0, \quad (249)$$

and also

$$(\mathrm{Y}_0\mathrm{Z}_1-\mathrm{Z}_0\mathrm{Y}_1)(x_1-x_0)+(\mathrm{Z}_0\mathrm{X}_1-\mathrm{X}_0\mathrm{Z}_1)(y_1-y_0)$$
$$+(\mathrm{X}_0\mathrm{Y}_1-\mathrm{Y}_0\mathrm{X}_1)(z_1-z_0)+\mathrm{X}_0\mathrm{L}_0+\mathrm{Y}_0\mathrm{M}_0+\mathrm{Z}_0\mathrm{N}_0=0, \quad (250)$$

but $\mathrm{Y}_0\mathrm{Z}_1-\mathrm{Z}_0\mathrm{Y}_1=\mathrm{Y}_0(\mathrm{Z}-\mathrm{Z}_0)-\mathrm{Z}_0(\mathrm{Y}-\mathrm{Y}_0)$

$$=\mathrm{Y}_0\mathrm{Z}-\mathrm{Z}_0\mathrm{Y};$$

similarly $\mathrm{Z}_0\mathrm{X}_1-\mathrm{X}_0\mathrm{Z}_1=\mathrm{Z}_0\mathrm{X}-\mathrm{X}_0\mathrm{Z}$; and $\mathrm{X}_0\mathrm{Y}_1-\mathrm{Y}_0\mathrm{X}_1=\mathrm{X}_0\mathrm{Y}-\mathrm{Y}_0\mathrm{X}$; therefore (250) becomes

$$(\mathrm{Y}_0\mathrm{Z}-\mathrm{Z}_0\mathrm{Y})(x_1-x_0)+(\mathrm{Z}_0\mathrm{X}-\mathrm{X}_0\mathrm{Z})(y_1-y_0)+(\mathrm{X}_0\mathrm{Y}-\mathrm{Y}_0\mathrm{X})(z_1-z_0)$$
$$+\mathrm{X}_0\mathrm{L}_0+\mathrm{Y}_0\mathrm{M}_0+\mathrm{Z}_0\mathrm{N}_0=0. \quad (251)$$

If therefore the equation to the line of action of $\mathrm{P}_0$ is

$$\frac{x-x_0}{\mathrm{X}_0}=\frac{y-y_0}{\mathrm{Y}_0}=\frac{z-z_0}{\mathrm{Z}_0}, \quad (252)$$

the line of action of $\mathrm{P}_1$, which is the line of intersection of the two planes (249) and (251), is reciprocal to it, as is manifest on comparing the equations with (215) in Article 80. Hence also the line (247), which is the line of intersection of the planes (249) and (251), is the reciprocal of (252). And it will be also found on reduction that the general form of the reciprocal line given in equations (217) assumes in this case the form (248).

85.] Hence we have the following Theorem:

If a system of forces acting on a rigid body is equivalently replaced by two forces, one of which acts along the line whose equations are

$$\frac{x-x_0}{\mathrm{A}}=\frac{y-y_0}{\mathrm{B}}=\frac{z-z_0}{\mathrm{C}}, \quad (253)$$

then the line of the other force is the reciprocal of this.

And therefore by the properties of reciprocal lines, proved above, we infer that the lines of action of two such forces are perpendicular to a line which passes through, and is perpendicular to, the central axis of the system.

And the forces acting along the lines may thus be found. Let $P_0$ and $P_1$ be the forces and $X_0Y_0Z_0$, $X_1Y_1Z_1$ their components along the coordinate axes; let (253) be the equations to the line of action of $P_0$; then we have

$$\frac{X_0}{A} = \frac{Y_0}{B} = \frac{Z_0}{C} = \frac{X_0L_0+Y_0M_0+Z_0N_0}{AL_0+BM_0+CN_0}$$

$$= \frac{RK}{AL_0+BM_0+CN_0}, \qquad (254)$$

whence, as $L_0$, $M_0$ and $N_0$ are given, $X_0$, $Y_0$ and $Z_0$ are given in known quantities. Also from (242)

$$X_1 = X-X_0, \qquad Y_1 = Y-Y_0, \qquad Z_1 = Z-Z_0, \qquad (255)$$

so that $X_1Y_1Z_1$ are also determined.

If the lines of action of the two forces are in one and the same plane, then substituting in the usual equation of condition, that two straight lines may be in the same plane, from (247) and (252) we have

$$(M_0Z_1-N_0Y_1)(Y_0Z_1-Z_0Y_1)+(N_0X_1-L_0Z_1)(Z_0X_1-X_0Z_1)$$
$$+(L_0Y_1-M_0Z_1)(X_0Y_1-Y_0X_1) = 0;$$

and therefore

$$(L_0X_0+M_0Y_0+N_0Z_0)(X_1^2+Y_1^2+Z_1^2)$$
$$-(X_0X_1+Y_0Y_1+Z_0Z_1)(L_0X_1+M_0Y_1+N_0Z_1) = 0,$$

the second part of which expression vanishes by reason of (245), and therefore

$$L_0X_0+M_0Y_0+N_0Z_0 = 0,$$

$$\text{or,} \qquad LX+MY+NZ = 0,$$

which is the condition required; and is the same as that already determined, so that the forces should be reducible to a single force of translation.

86.] And other properties of this system of double forces may be thus investigated. Let the forces, as above, be $P_0, P_1$, and their points of application be $(x_0, y_0, z_0)$, $(x_1, y_1, z_1)$; let $r_0$ and $r_1$ be the lengths on the shortest distance between the lines of action of the two forces, from the central axis to the respective lines of action; and let $\theta_0, \theta_1$ be the angles between the central axis and the lines of action; then

$$P_0 \cos\theta_0 + P_1 \cos\theta_1 = R, \qquad (256)$$

since $R$ is the resultant of translation along the central axis.

$$\text{P}_0 \sin\theta_0 = \text{P}_1 \sin\theta_1 = \frac{\text{K}}{r_0+r_1}, \tag{257}$$

because K is the principal moment-axis about the central axis, and because the whole resultant of translation acts along that axis; and therefore also

$$\text{P}_0 r_0 \cos\theta_0 = \text{P}_1 r_1 \cos\theta_1, \tag{258}$$

$$\frac{\tan\theta_0}{r_0} = \frac{\tan\theta_1}{r_1} = \frac{\text{K}}{\text{R}\, r_0 r_1}. \tag{259}$$

If the lines of action of $\text{P}_0$ and of $\text{P}_1$ are perpendicular to each other, then $\theta_0+\theta_1 = 90°$, and

$$\tan\theta_0 \tan\theta_1 = 1, \tag{260}$$

$$\therefore \quad r_0 r_1 = \frac{\text{K}^2}{\text{R}^2}, \qquad \tan\theta_0 = r_0\frac{\text{R}}{\text{K}}, \qquad \tan\theta_1 = r_1\frac{\text{R}}{\text{K}}. \tag{261}$$

Also let $\text{G}_0$ and $\text{G}_1$ be the principal moment-axes of all the impressed forces at the points $(x_0, y_0, z_0)$ and $(x_1, y_1, z_1)$; then by (168)

$$\text{G}_0 = \frac{\text{K}}{\cos\theta_0}, \qquad \text{G}_1 = \frac{\text{K}}{\cos\theta_1},$$

and since $\sin\theta_0 = \cos\theta_1$, $\sin\theta_1 = \cos\theta_0$, therefore from (257)

$$\frac{\text{G}_0}{\text{P}_1} = \frac{\text{G}_1}{\text{P}_0} = r_0+r_1, \tag{262}$$

$$\frac{\text{P}_0}{\text{P}_1} + \frac{\text{P}_1}{\text{P}_0} = \frac{\text{R}}{\text{K}}(r_0+r_1), \tag{263}$$

$$\therefore \quad \text{RK} = \text{P}_0\text{P}_1(r_0+r_1),$$

$$\therefore \quad \text{G}_0\text{G}_1 = \text{RK}(r_0+r_1);$$

and from (262),

$$\begin{aligned}\text{G}_0^2+\text{G}_1^2 &= (\text{P}_0^2+\text{P}_1^2)(r_0+r_1)^2\\ &= \text{R}^2(r_0+r_1)^2,\end{aligned}$$

$$\text{since} \qquad \text{R}^2 = \text{P}_0^2+\text{P}_1^2.$$

Also on the same supposition of the lines of action of the two forces being perpendicular to each other, it is plain, that consistently with the condition (261), $r_0+r_1$ will be a minimum when

$$r_0 = r_1 = \frac{\text{K}}{\text{R}}, \tag{264}$$

in which case $\tan\theta_0 = \tan\theta_1$; $\theta_0 = \theta_1 = 45°$; $\text{P}_0 = \text{P}_1 = \frac{\text{R}}{2^{\frac{1}{2}}}$; and therefore we have the theorem: A given system of forces acting on a rigid body may be replaced by two equal forces whose lines of action are perpendicular to each other, and each of which has a line of action inclined at 45° to the central axis; and the forces act perpendicularly at the ends of an arm which is bisected at right angles by that axis; the magnitude of each force is equal to $\frac{R}{2^{\frac{1}{2}}}$, and the length of the arm is $\frac{2K}{R}$; and no other pair of forces, having lines of action perpendicular to each other, which can replace the system, can be at a distance less than this.

And similarly, if the lines of action of the forces are inclined to each other at a given definite angle, say $2a$, may the minimum distance between their lines of action be determined; and we shall find $\theta_0 = \theta_1 = a$, $P_0 = P_1 = \frac{R}{2} \sec a$, $r_0 = r_1$.

And if the lines of action of the two forces are parallel, then the system of impressed forces is reducible to a single resultant of translation; $P_0 = P_1 = \frac{R}{2}$; $r_0 = r_1 =$ an indeterminate quantity; and each increases infinitely, when $P_0 = P_1$ decreases infinitesimally; and the line of action of each force is parallel to the central axis.

If the forces have not lines of action inclined to each other at a given angle, the distance does not admit of a minimum; the limiting case is that of two forces of infinite magnitude, but differing by a finite quantity, which act along one line in opposite directions. This result is also evident from the mode by which in Art. 70 it is shewn that a system of forces acting on a rigid body may be reduced to two forces of translation.

In all cases the two forces act perpendicularly at the ends of an arm which intersects the central axis at right angles.

87.] The following is a curious geometrical property of the two forces $P_0$ and $P_1$, to which a system of forces acting on a rigid body may be reduced. Along the lines of action of $P_0$ and $P_1$ let lines be taken severally proportional to the forces, then the content of the tetrahedron of which these lines are opposite edges is constant, whatever is the position of the lines on the lines of action of the forces.

Let one end of the line-symbol of $P_0$ be $(x_0, y_0, z_0)$, then the other end is $(x_0 + X_0,\ y_0 + Y_0,\ z_0 + Z_0)$; and let one extremity of the line-symbol of $P_1$ be, in virtue of equations (247),

$$\left(x_0 + \frac{N_0 Y_1 - M_0 Z_1}{P_1^2},\quad y_0 + \frac{L_0 Z_1 - N_0 X_1}{P_1^2},\quad z_0 + \frac{M_0 X_1 - L_0 Y_1}{P_1^2}\right),$$

then the coordinates of the other end are the same quantities increased severally by $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1$; and placing the origin at the point $(x_0, y_0, z_0)$, the volume of the tetrahedron is, according to the notation of Article 150, Vol. II, equal to $\frac{\Sigma . \pm x_1 y_2 z_3}{6}$,

$$= \frac{\mathrm{N}_0\mathrm{Y}_1 - \mathrm{M}_0\mathrm{Z}_1 + \mathrm{X}_1\mathrm{P}_1^2}{6\,\mathrm{P}_1^4}\{\mathrm{Y}_0(\mathrm{M}_0\mathrm{X}_1 - \mathrm{L}_0\mathrm{Y}_1) - \mathrm{Z}_0(\mathrm{L}_0\mathrm{Z}_1 - \mathrm{N}_0\mathrm{X}_1)\} + \ldots + \ldots$$

$$= \frac{\mathrm{L}_0\mathrm{X}_0 + \mathrm{M}_0\mathrm{Y}_0 + \mathrm{N}_0\mathrm{Z}_0}{6}$$

$$= \frac{\mathrm{R}\mathrm{K}}{6}, \tag{265}$$

and this is a constant quantity; and therefore the content of the above-mentioned tetrahedron is constant.

If the volume of the tetrahedron vanishes, the two forces act in the same plane, and the system is reducible, either to a single force of translation, or to a couple: that is, either $\mathrm{K} = 0$, or $\mathrm{R} = 0$.

Hence also it is evident that if four forces are in equilibrium, the volume of the tetrahedron constructed on the line-symbols of any two is equal to that of the tetrahedron constructed on the line-symbols of the other two.

Other properties of a similar character will be found in a memoir by M. Chasles, Liouville's Journal, Vol. XII, p. 213.

88.] It remains for us to investigate for forces acting in space relations analogous to that which was discovered in Art. 52 for forces acting in one plane; and therefore we must determine (1) whether at all, and (2) under what circumstances, a body which is in equilibrium under the action of many forces acting in given lines and at given points, is also in equilibrium, when the position of the body is changed in the most general manner; and when the forces are applied to the body in its displaced position at the same points and in the same directions as before, and along lines of action parallel to the original ones.

Let the body be referred to a system of rectangular axes, in respect of which $\mathrm{P}$ is the type-symbol of a force, $(x, y, z)$ is its point of application, $\alpha, \beta, \gamma$ are the direction-angles of its line of action; so that the conditions of equilibrium are

$$\left.\begin{aligned} \Sigma.\mathrm{P}\cos\alpha &= \mathrm{X} = 0 \\ \Sigma.\mathrm{P}\cos\beta &= \mathrm{Y} = 0 \\ \Sigma.\mathrm{P}\cos\gamma &= \mathrm{Z} = 0 \end{aligned}\right\}, \tag{266}$$

$$\left.\begin{aligned} \Sigma.\mathrm{P}(y\cos\gamma - z\cos\beta) &= \mathrm{L} = 0 \\ \Sigma.\mathrm{P}(z\cos\alpha - x\cos\gamma) &= \mathrm{M} = 0 \\ \Sigma.\mathrm{P}(x\cos\beta - y\cos\alpha) &= \mathrm{N} = 0 \end{aligned}\right\}. \tag{267}$$

Now let the body be displaced, so that (1) every point of it receives an equal lineal displacement; and which is compounded of three linear displacements parallel to the three coordinate axes: viz. those of $x_0, y_0, z_0$ parallel to the three axes of $x, y, z$ respectively; so that hereby the former origin becomes shifted to the point $(x_0, y_0, z_0)$: and (2) let the body be turned about its new origin, with reference to which let the coordinate symbols be distinguished by an accent, and let the direction-cosines of the two systems of axes be related by the following scheme:

| | $x$ | $y$ | $z$ |
|---|---|---|---|
| $x'$ | $a$ | $a_1$ | $a_2$ |
| $y'$ | $b$ | $b_1$ | $b_2$ |
| $z'$ | $c$ | $c_1$ | $c_2$ |

; (268)

so that finally we have

$$\left.\begin{aligned} x' &= x_0 + ax + a_1 y + a_2 z \\ y' &= y_0 + bx + b_1 y + b_2 z \\ z' &= z_0 + cx + c_1 y + c_2 z \end{aligned}\right\}. \tag{269}$$

Then because the forces after the displacement of the body act on the same points of the body, and in lines of action parallel to their former ones, the following conditions of equilibrium must hold true: viz.

$$\left.\begin{aligned} \Sigma.\mathrm{P}(y'\cos\gamma - z'\cos\beta) = 0, \quad \Sigma.\mathrm{P}(z'\cos\alpha - x'\cos\gamma) = 0, \\ \Sigma.\mathrm{P}(x'\cos\beta - y'\cos\alpha) = 0; \end{aligned}\right\} \tag{270}$$

and therefore substituting for $x'\, y'\, z'$ from the preceding formulæ, we have, for the first of these conditions,

$$\begin{aligned} 0 = y_0\, \Sigma.\mathrm{P}\cos\gamma - z_0\, \Sigma.\mathrm{P}\cos\beta \\ + b\, \Sigma.\mathrm{P}x\cos\gamma + b_1\, \Sigma.\mathrm{P}y\cos\gamma + b_2\, \Sigma.\mathrm{P}z\cos\gamma \\ - c\, \Sigma.\mathrm{P}x\cos\beta - c_1\, \Sigma.\mathrm{P}y\cos\beta - c_2\, \Sigma.\mathrm{P}z\cos\beta = 0. \end{aligned} \tag{271}$$

Of the right-hand member, the first two terms vanish by reason of (266): and to abbreviate the expression, let

$$\left.\begin{aligned} \Sigma.\mathrm{P}y\cos\gamma = \Sigma.\mathrm{P}z\cos\beta = \mathrm{D} \\ \Sigma.\mathrm{P}z\cos\alpha = \Sigma.\mathrm{P}x\cos\gamma = \mathrm{E} \\ \Sigma.\mathrm{P}x\cos\beta = \Sigma.\mathrm{P}y\cos\alpha = \mathrm{F} \end{aligned}\right\} \qquad \left.\begin{aligned} \Sigma.\mathrm{P}x\cos\alpha = l \\ \Sigma.\mathrm{P}y\cos\beta = m \\ \Sigma.\mathrm{P}z\cos\gamma = n \end{aligned}\right\}, \tag{272}$$

the double equivalents of $\mathrm{D}$, $\mathrm{E}$, $\mathrm{F}$ following from (267); so that (271) becomes

$$(b_1-c_2)\,\mathrm{D}+b\,\mathrm{E}-c\,\mathrm{F}-c_1m+b_2n = 0. \tag{273}$$

Similarly from the second and third of (270) we have

$$(c_2-a)\,\mathrm{E}+c_1\,\mathrm{F}-a_1\,\mathrm{D}-a_2n+cl = 0, \tag{274}$$

$$(a-b_1)\,\mathrm{F}+a_2\,\mathrm{D}-b_2\,\mathrm{E}-bl+a_1m = 0. \tag{275}$$

As both the systems of coordinates are rectangular, their direction-cosines are subject to the following relations: viz.

$$\left.\begin{aligned} b_1c_2-b_2c_1&=a\\ b_2c-bc_2&=a_1\\ bc_1-b_1c&=a_2 \end{aligned}\right\}\quad \left.\begin{aligned} c_1a_2-c_2a_1&=b\\ c_2a-ca_2&=b_1\\ ca_1-c_1a&=b_2 \end{aligned}\right\}\quad \left.\begin{aligned} a_1b_2-b_1a_2&=c\\ a_2b-b_2a&=c_1\\ ab_1-ba_1&=c_2 \end{aligned}\right\}; \tag{276}$$

therefore multiplying (274) by $b$, and (275) by $c$, and adding, we have

$$(a_2c-a_1b)\,\mathrm{D}-(a_1+ab)\,\mathrm{E}+(a_2+ac)\,\mathrm{F}+a_1cm-a_2bn = 0; \tag{277}$$

and by reason of (276)

$$\begin{aligned} a_2c-a_1b &= c_2a-b_1-ab_1+c_2\\ &= (1+a)(c_2-b_1), \end{aligned}$$

so that (277) becomes

$$(1+a)(c_2-b_1)\,\mathrm{D}-(a_1+ab)\,\mathrm{E}+(a_2+ac)\,\mathrm{F}+a_1cm-a_2bn = 0;$$

and multiplying (273) by $1+a$, and adding to this last equation, we have

$$\left.\begin{aligned} &(b-a_1)\,\mathrm{E}+(a_2-c)\,\mathrm{F}+(b_2-c_1)(m+n) = 0;\\ \text{similarly}\quad &(c_1-b_2)\,\mathrm{F}+(b-a_1)\,\mathrm{D}+(c-a_2)(n+l) = 0,\\ &(a_2-c)\,\mathrm{D}+(c_1-b_2)\,\mathrm{E}+(a_1-b)(l+m) = 0, \end{aligned}\right\} \tag{278}$$

and if we substitute as follows,

$$\left.\begin{aligned} m+n &= \Sigma.\mathrm{P}\,(y\cos\beta+z\cos\gamma) = \mathrm{U}\\ n+l &= \Sigma.\mathrm{P}\,(z\cos\gamma+x\cos\alpha) = \mathrm{V}\\ l+m &= \Sigma.\mathrm{P}\,(x\cos\alpha+y\cos\beta) = \mathrm{W} \end{aligned}\right\}, \tag{279}$$

we have

$$\left.\begin{aligned} \mathrm{U}(c_1-b_2)-\mathrm{F}(a_2-c)-\mathrm{E}(b-a_1) &= 0\\ -\mathrm{F}(c_1-b_2)+\mathrm{V}(a_2-c)-\mathrm{D}(b-a_1) &= 0\\ -\mathrm{E}(c_1-b_2)-\mathrm{D}(a_2-c)+\mathrm{W}(b-a_1) &= 0 \end{aligned}\right\}; \tag{280}$$

and from these we have

$$\mathrm{UVW}-\mathrm{D}^2\mathrm{U}-\mathrm{E}^2\mathrm{V}-\mathrm{F}^2\mathrm{W}-2\,\mathrm{DEF} = 0, \tag{281}$$

which equation expresses the relation between the forces, the direction-cosines of their lines of action, and their points of application, which is requisite so that the body should be in equilibrium in both positions.

It will be observed that $x_0\, y_0\, z_0$ have disappeared, because they enter only as coefficients of $\Sigma.\text{P}\cos\alpha$, $\Sigma.\text{P}\cos\beta$, $\Sigma.\text{P}\cos\gamma$; and these three vanish; and therefore we infer that no lateral displacement of the body affects the condition of equilibrium, so long as the forces are applied at their former points, in their original directions, and along parallel lines of action: (280) and (281) require and deserve closer consideration.

From (280) we have

$$\frac{(c_1-b_2)^2}{\text{D}^2-\text{VW}} = \frac{(a_2-c)^2}{\text{E}^2-\text{WU}} = \frac{(b-a_1)^2}{\text{F}^2-\text{UV}}; \tag{282}$$

and these two equations involve only six out of the nine direction-cosines, and therefore give only *two* conditions of relation amongst them; and *six* relations have been given heretofore; one other condition is therefore required ere the new coordinate axes become fixed, and if this cannot be determined, the position of them is indeterminate: we must therefore consider this subject; and in the course of it we shall introduce a new process of transformation of coordinates; and which is due, so far as I know, to M. Olinde Rodrigues, and was published by him in Liouville's Journal, Vol. V, 1840.

89.] Imagine a body referred to a system of rectangular coordinate axes to rotate through an angle $\theta$ about an axis passing through the origin, and of which the direction-angles are $f, g, h$: let $(x, y, z)$ be any point of the body, and let this point after the rotation be $(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$: let $\Delta s$ be the distance between the two positions of this point, so that

$$(\Delta s)^2 = (\Delta x)^2+(\Delta y)^2+(\Delta z)^2; \tag{283}$$

and let $p$ be the perpendicular distance from $(x, y, z)$ to the axis of rotation; so that $\Delta s$ is the chord of a circular arc, of radius $p$ and angle $\theta$, described by $(x, y, z)$ revolving about the axis of rotation; and therefore

$$\Delta s = 2p\sin\frac{\theta}{2}. \tag{284}$$

The coordinates to the middle point of $\Delta s$ are

$$x+\frac{\Delta x}{2}, \quad y+\frac{\Delta y}{2}, \quad z+\frac{\Delta z}{2},$$

and as $\Delta x$, $\Delta y$, $\Delta z$ are the projections of $\Delta s$ on the coordinate axes, they are proportional to the direction-cosines of $\Delta s$; and therefore as $\Delta s$ is perpendicular to the rotation-axis, and also to the line through the origin and its own middle point, we have

$$\left.\begin{aligned} \Delta x\cos f + \Delta y\cos g + \Delta z\cos h &= 0 \\ \Delta x\left(x+\frac{\Delta x}{2}\right) + \Delta y\left(y+\frac{\Delta y}{2}\right) + \Delta z\left(z+\frac{\Delta z}{2}\right) &= 0 \end{aligned}\right\}; \tag{285}$$

$$\therefore \quad \frac{\Delta x}{\left(y+\frac{\Delta y}{2}\right)\cos h - \left(z+\frac{\Delta z}{2}\right)\cos g} = \frac{\Delta y}{\left(z+\frac{\Delta z}{2}\right)\cos f - \left(x+\frac{\Delta x}{2}\right)\cos h}$$

$$= \frac{\Delta z}{\left(x+\frac{\Delta x}{2}\right)\cos g - \left(y+\frac{\Delta y}{2}\right)\cos f} = \frac{\Delta s}{p\cos\frac{\theta}{2}} = 2\tan\frac{\theta}{2}, \tag{286}$$

because the square root of the sum of the squares of the denominators is the length of the perpendicular on the rotation-axis from the point $\left(x+\frac{\Delta x}{2},\ y+\frac{\Delta y}{2},\ z+\frac{\Delta z}{2}\right)$; therefore

$$\left.\begin{aligned} \Delta x &= 2\tan\frac{\theta}{2}\left\{\left(y+\frac{\Delta y}{2}\right)\cos h - \left(z+\frac{\Delta z}{2}\right)\cos g\right\} \\ \Delta y &= 2\tan\frac{\theta}{2}\left\{\left(z+\frac{\Delta z}{2}\right)\cos f - \left(x+\frac{\Delta x}{2}\right)\cos h\right\} \\ \Delta z &= 2\tan\frac{\theta}{2}\left\{\left(x+\frac{\Delta x}{2}\right)\cos g - \left(y+\frac{\Delta y}{2}\right)\cos f\right\} \end{aligned}\right\}. \tag{287}$$

In these expressions let

$$2\tan\frac{\theta}{2}\cos f = \lambda, \quad 2\tan\frac{\theta}{2}\cos g = \mu, \quad 2\tan\frac{\theta}{2}\cos h = \nu, \tag{288}$$

$$\therefore \quad 4\left(\tan\frac{\theta}{2}\right)^2 = \lambda^2+\mu^2+\nu^2; \tag{289}$$

so that (287) become

$$\left.\begin{aligned} 2\Delta x - \nu\Delta y + \mu\Delta z &= 2(\nu y - \mu z) \\ \nu\Delta x + 2\Delta y - \lambda\Delta z &= 2(\lambda z - \nu x) \\ -\mu\Delta x + \lambda\Delta y + 2\Delta z &= 2(\mu x - \lambda y) \end{aligned}\right\}; \tag{290}$$

whence we have

$$\left.\begin{aligned} \Delta x(4+\lambda^2+\mu^2+\nu^2) &= -2(\mu^2+\nu^2)x + 2(\lambda\mu+2\nu)y + 2(\lambda\nu-2\mu)z \\ \Delta y(4+\lambda^2+\mu^2+\nu^2) &= 2(\mu\lambda-2\nu)x - 2(\nu^2+\lambda^2)y + 2(\mu\nu+2\lambda)z \\ \Delta z(4+\lambda^2+\mu^2+\nu^2) &= 2(\nu\lambda+2\mu)x + 2(\nu\mu-2\lambda)y - 2(\lambda^2+\mu^2)z \end{aligned}\right\}; \tag{291}$$

and these are the quantities by which $x$, $y$, $z$ are increased by means of the rotation of the body through an angle $\theta$ about an axis, of which the direction-angles are $f$, $g$, $h$. And if we replace $\lambda$, $\mu$, $\nu$ by their values from (288), we have

$$\left.\begin{aligned}
\Delta x &= \sin\theta\,(y\cos h - z\cos g) \\
&\qquad + 2\left(\sin\frac{\theta}{2}\right)^2\{\cos f\,(x\cos f + y\cos g + z\cos h) - x\} \\
\Delta y &= \sin\theta\,(z\cos f - x\cos h) \\
&\qquad + 2\left(\sin\frac{\theta}{2}\right)^2\{\cos g\,(x\cos f + y\cos g + z\cos h) - y\} \\
\Delta z &= \sin\theta\,(x\cos g - y\cos f) \\
&\qquad + 2\left(\sin\frac{\theta}{2}\right)^2\{\cos h\,(x\cos f + y\cos g + z\cos h) - z\}
\end{aligned}\right\}. \quad (292)$$

But we shall find (291) more useful to our present purpose.

Also I would observe that if the body is turned about the axis of rotation through an infinitesimal angle, then from (292) $\Delta x$, $\Delta y$, $\Delta z$ become infinitesimal; and for them putting $dx$, $dy$, $dz$; and neglecting the terms in (292) involving $\theta^2$ we have

$$dx = \theta\,(y\cos h - z\cos g), \qquad dy = \theta\,(z\cos f - x\cos h),$$
$$dz = \theta\,(x\cos g - y\cos f). \quad (293)$$

90.] Imagine now two systems of rectangular coordinate axes to originate at the origin, and to be coincident ere the rotation about the axis has taken place; and imagine the second system to be fixed in the body and to move with it during the rotation; then if $x, y, z$ refer to the coordinate axes, as they are in their original position, so they do also refer to the system which has shifted with the body. Let the scheme of direction-cosines of this system in reference to the system in the original position be that indicated in Article 88, (268), so that we have

$$\left.\begin{aligned}
x + \Delta x &= ax + a_1 y + a_2 z \\
y + \Delta y &= bx + b_1 y + b_2 z \\
z + \Delta z &= cx + c_1 y + c_2 z
\end{aligned}\right\};$$

therefore

$$\left.\begin{aligned}
\Delta x &= -(1-a)\,x + a_1 y + a_2 z \\
\Delta y &= bx - (1-b_1)\,y + b_2 z \\
\Delta z &= cx + c_1 y - (1-c_2)\,z
\end{aligned}\right\}, \quad (294)$$

and these are *identical* with (291); therefore

$$\left.\begin{aligned}
a &= \frac{4+\lambda^2-\mu^2-\nu^2}{4+\lambda^2+\mu^2+\nu^2}, & a_1 &= \frac{2\,(\lambda\mu+2\nu)}{4+\lambda^2+\mu^2+\nu^2}, & a_2 &= \frac{2\,(\lambda\nu-2\mu)}{4+\lambda^2+\mu^2+\nu^2} \\
b &= \frac{2\,(\mu\lambda-2\nu)}{4+\lambda^2+\mu^2+\nu^2}, & b_1 &= \frac{4+\mu^2-\nu^2-\lambda^2}{4+\lambda^2+\mu^2+\nu^2}, & b_2 &= \frac{2\,(\mu\nu+2\lambda)}{4+\lambda^2+\mu^2+\nu^2} \\
c &= \frac{2\,(\nu\lambda+2\mu)}{4+\lambda^2+\mu^2+\nu^2}, & c_1 &= \frac{2\,(\nu\mu-2\lambda)}{4+\lambda^2+\mu^2+\nu^2}, & c_2 &= \frac{4+\nu^2-\lambda^2-\mu^2}{4+\lambda^2+\mu^2+\nu^2}
\end{aligned}\right\}, \quad (295)$$

whereby the direction-cosines of the new axes are expressed in terms of the direction-cosines of the axis of rotation, and of the angle through which the body has been turned about that axis; and as rational functions of these quantities. And also inversely, $\lambda$, $\mu$, $\nu$ can be determined in terms of the direction-cosines of the new axes, and we have

$$\lambda = \frac{2(b_2-c_1)}{1+a+b_1+c_2}, \quad \mu = \frac{2(c-a_2)}{1+a+b_1+c_2}, \quad \nu = \frac{2(a_1-b)}{1+a+b_1+c_2}. \qquad (296)$$

The changes in position which the coordinate axes have undergone are represented in fig. 65, wherein XYZ, X′Y′Z′ are two equilateral right-angled and quadrantal triangles described on the surface of a sphere, whose centre is supposed to be at the origin; and R is the point on the surface of the sphere, where the axis, about which the rotation takes place, intersects the surface: about this line therefore the body has been turned through an angle $\theta$, and the triangle X′Y′Z′, which was coincident with XYZ, has been brought into its present position: therefore the arc RX = RX′ = $f$, RY = RY′ = $g$, RZ = RZ′ = $h$; XRX′ = YRY′ = ZRZ′ = $\theta$. And I may observe that the relations in (295) and (296) may be deduced from the geometrical figure, by the ordinary theorems of spherical trigonometry; see a paper by Mr. Cayley, of Trinity College, Cambridge, in the Cambridge Mathematical Journal, Vol. III, p. 225.

If the nine direction-cosines connecting two systems of rectangular axes are given, we can by means of (296) (288) and (289) determine the direction-cosines with respect to the fixed system of the axis about which the rotation, to which the displacement is due, has taken place; and also the *amplitude* (or the angle through which the body has been turned) of that rotation: and as these quantities are always possible, whatever is the position of the new axes, the change may always be produced by a displacement of the former axes about a determinate straight line. We proceed now to apply these results to the mechanical problem from which we started.

91.] In reference to the equations (282), we have from (295),

$$\left.\begin{aligned} c_1-b_2 &= -\frac{8\lambda}{4+\lambda^2+\mu^2+\nu^2} \\ &= -2\sin\theta\cos f \\ a_2-c &= -2\sin\theta\cos g \\ b-a_1 &= -2\sin\theta\cos h \end{aligned}\right\}, \qquad (297)$$

and (280) become

$$\left.\begin{aligned} -\mathrm{U}\cos f+\mathrm{F}\cos g+\mathrm{E}\cos h &= 0 \\ \mathrm{F}\cos f-\mathrm{V}\cos g+\mathrm{D}\cos h &= 0 \\ \mathrm{E}\cos f+\mathrm{D}\cos g-\mathrm{W}\cos h &= 0 \end{aligned}\right\}; \tag{298}$$

and these are the relations which the direction-cosines of an equilibrium-axis must satisfy; and therefore from (282)

$$\frac{(\cos f)^2}{\mathrm{D}^2-\mathrm{VW}}=\frac{(\cos g)^2}{\mathrm{E}^2-\mathrm{WU}}=\frac{(\cos h)^2}{\mathrm{F}^2-\mathrm{UV}}=\frac{1}{\mathrm{D}^2+\mathrm{E}^2+\mathrm{F}^2-(\mathrm{VW}+\mathrm{WU}+\mathrm{UV})}, \tag{299}$$

because $(\sin\theta)^2$ is a common factor in each numerator, and divides out. We have then the following results: the direction-cosines of the axis about which the body may be turned, so that it may be in equilibrium in its new, as well as in its original position, are given by (299); and as this axis possesses some remarkable properties in reference to the system of forces, it is convenient that it should bear a distinctive name, and so it has been called *the equilibrium-axis;* but as $\theta$ is not involved in these last expressions, and as there is no other condition for determining it, it remains undetermined, and therefore the nine direction-cosines are still undetermined, because they are subject to only eight conditions; and therefore if there is an equilibrium axis, the equilibrium subsists, whatever is the amplitude through which the system is turned about the equilibrium-axis; and therefore, combining this result with that which arises from a parallel and lateral displacement of the body, we infer, that

If a body is changed from one position into another by means of a lateral displacement and of a rotation about a certain axis, and is in equilibrium in both positions under the action of forces which act on the same points and in invariable directions in both positions; then is the body also in equilibrium in any third position into which it can be brought by a lateral displacement and by a further rotation about the same or a parallel axis.

If $\mathrm{D}^2=\mathrm{VW}$, $\mathrm{E}^2=\mathrm{WU}$, $\mathrm{F}^2=\mathrm{UV}$, $f, g, h$ are indeterminate, and the body is in equilibrium, whatever is the position of the axis about which it is turned. On this subject more is said hereafter.

If $c_1-b_2=a_2-c=b-a_1=0$, then from (297), $\sin\theta=0$; and therefore either $\theta=0$, or $\theta=\pi$: in the former case the body undergoes no displacement by means of rotation; in the latter, it is turned through 180°, and the position of the equilibrium-axis is still determined by means of the equations (299).

92.] The relation between the forces, their points of application and their lines of action, which is involved in the equation of condition (281), will be understood more easily by the following process than it would be if interpreted directly.

Suppose the axis of $z$ to be the equilibrium-axis, so that $\cos f = \cos g = 0$; then

$$\text{D} = 0, \quad \text{E} = 0, \quad \text{W} = 0, \tag{300}$$

that is,

$$\Sigma.\text{P}y\cos\gamma = 0, \quad \Sigma.\text{P}x\cos\gamma = 0, \quad \Sigma.\text{P}(x\cos\alpha + y\cos\beta) = 0; \tag{301}$$

from the first two of which taken in combination with $\Sigma.\text{P}\cos\gamma = 0$, we infer that, if the forces are at their points of application resolved in directions parallel to the coordinate axes, those parallel to the axis of $z$ are in equilibrium; and from the last, combined with the first two of (266) and the last of (267), we infer that the forces whose lines of action are parallel to the plane of $xy$ satisfy the conditions required for *a centre*, see Art. 52, and are therefore in equilibrium when the body is turned through an angle about the axis of $z$. Hence the meaning of the condition (281) is,

If the forces acting on a body are resolved along a certain straight line, and in planes perpendicular to that line; then the forces parallel to the straight line are in equilibrium, and those in the planes perpendicular to the straight line satisfy the conditions required for a centre.

Again, let us suppose the forces to be such that the coordinate axes of $x$ and $y$ are both equilibrium-axes: then from the equations (298) we have

$$\text{D} = \text{E} = \text{F} = 0, \quad \text{U} = \text{V} = 0,$$

and therefore also, from the last of (280), $b - a_1 = 0$; and thus from (297) $\cos h = 0$; and therefore any line parallel to the plane of $xy$ may also be an equilibrium-axis.

And to investigate generally the conditions requisite that any two lines inclined at any angle to each other should be equilibrium-axes; let the direction-angles of the two be $f, g, h$, $f', g', h'$; then from (298) we have

$$\left.\begin{aligned} -\text{U}\cos f + \text{F}\cos g + \text{E}\cos h &= 0 \\ \text{F}\cos f - \text{V}\cos g + \text{D}\cos h &= 0 \\ \text{E}\cos f + \text{D}\cos g - \text{W}\cos h &= 0 \end{aligned}\right\}, \tag{302}$$

$$\left.\begin{aligned} -\text{U}\cos f' + \text{F}\cos g' + \text{E}\cos h' &= 0 \\ \text{F}\cos f' - \text{V}\cos g' + \text{D}\cos h' &= 0 \\ \text{E}\cos f' + \text{D}\cos g' - \text{W}\cos h' &= 0 \end{aligned}\right\}; \tag{303}$$

and let
$$\left.\begin{aligned} \cos g\cos h' - \cos h\cos g' &= p \\ \cos h\cos f' - \cos f\cos h' &= q \\ \cos f\cos g' - \cos g\cos f' &= r \end{aligned}\right\}, \tag{304}$$

so that from the first of (302) and from the first of (303) we have

and similarly,
$$\left.\begin{aligned} \frac{-\mathrm{U}}{p} = \frac{\mathrm{F}}{q} &= \frac{\mathrm{E}}{r} \\ \frac{\mathrm{F}}{p} = \frac{-\mathrm{V}}{q} &= \frac{\mathrm{D}}{r} \\ \frac{\mathrm{E}}{p} = \frac{\mathrm{D}}{q} &= \frac{-\mathrm{W}}{r} \end{aligned}\right\}; \tag{305}$$

therefore $\mathrm{DU} + \mathrm{EF} = 0, \quad \mathrm{EV} + \mathrm{FD} = 0, \quad \mathrm{FW} + \mathrm{DE} = 0,$ (306)

and also $\mathrm{D}^2 = \mathrm{VW}, \quad \mathrm{E}^2 = \mathrm{WU}, \quad \mathrm{F}^2 = \mathrm{UV},$ (307)

and therefore $\cos f$, $\cos g$, $\cos h$ in (299) are indeterminate; and substituting from (306) and (307) in any of the equations (302) we have
$$\frac{\cos f}{(\mathrm{VW})^{\frac{1}{2}}} + \frac{\cos g}{(\mathrm{WU})^{\frac{1}{2}}} + \frac{\cos h}{(\mathrm{UV})^{\frac{1}{2}}} = 0; \tag{308}$$

and similarly from any of (303) we have
$$\frac{\cos f'}{(\mathrm{VW})^{\frac{1}{2}}} + \frac{\cos g'}{(\mathrm{WU})^{\frac{1}{2}}} + \frac{\cos h'}{(\mathrm{UV})^{\frac{1}{2}}} = 0; \tag{309}$$

and therefore a condition of the same kind has to be fulfilled by each of two axes, if they are equilibrium-axes; and these conditions are satisfied for not only the two preceding axes, but also for all axes which are in, or parallel to, a plane whose equation is
$$\frac{x}{(\mathrm{VW})^{\frac{1}{2}}} + \frac{y}{(\mathrm{WU})^{\frac{1}{2}}} + \frac{z}{(\mathrm{UV})^{\frac{1}{2}}} = 0. \tag{310}$$

Hence we infer that a body which is in equilibrium for two equilibrium-axes which meet and are not parallel to each other, is also in equilibrium for all axes parallel to the plane which contains the two above-mentioned equilibrium-axes. And hence

If a body has three equilibrium-axes which are not parallel to one and the same plane, so has it also any fourth equilibrium-axis. And therefore

If a body is in equilibrium in four different and not parallel positions, so is it also generally in every fifth position.

And when this last case occurs, $\mathrm{D} = \mathrm{E} = \mathrm{F} = 0$, $\mathrm{U} = \mathrm{V} = \mathrm{W} = 0$; so that the position of the plane (310) becomes indeterminate.

93.] Although a system of forces acting on a rigid body and being in equilibrium admits of an equilibrium-axis, only when (281) is fulfilled, and therefore not generally; yet if a system is in equilibrium, two new equal forces acting at certain definite points, along the same line of action and in opposite directions, may be introduced in such a manner that the system thus modified may have an equilibrium-axis in a given direction. The new forces, it will be observed, as introduced in the first position of the body, being equal and opposite, neutralize each other, and do not disturb equilibrium, and in the other positions form a couple which equilibrates with the impressed forces of the system in their new positions.

Let us, as in the last Article, suppose $f, g, h$ to be the direction-angles of the given equilibrium-axis; $\text{P}'$ and $\text{P}''$ to be the two new forces, equal and opposite to each other; $(x', y', z')$, $(x'', y'', z'')$ their points of application; $l, m, n$ the direction-cosines of their common line of action; $r$ the distance between their points of application; let $x''-x'$, $y''-y'$, $z''-z'$ be positive quantities; then for the equilibrium of these two forces we have

$$\text{P}''-\text{P}' = 0; \tag{311}$$

also we have

$$\frac{x''-x'}{l} = \frac{y''-y'}{m} = \frac{z''-z'}{n} = r; \tag{312}$$

and if the accented letters refer to the system when increased by the two new forces, and the unaccented letters to the original system,

$$\begin{aligned} \text{D}' &= \text{D}+y''\text{P}''n+y'\text{P}'n \\ &= \text{D}+(y''-y')\,\text{P}''n, \end{aligned}$$

$$\left.\begin{aligned} \therefore\quad \text{D}' &= \text{D}+\text{P}''r\,mn; \\ \text{similarly}\quad \text{E}' &= \text{E}+\text{P}''r\,nl, \\ \text{F}' &= \text{F}+\text{P}''r\,lm, \end{aligned}\right\} \tag{313}$$

$$\text{U}'=\text{U}+\text{P}''r(m^2+n^2),\quad \text{V}'=\text{V}+\text{P}''r(n^2+l^2),\quad \text{W}'=\text{W}+\text{P}''r(l^2+m^2); \tag{314}$$

and substituting these values in the conditions (298) which are requisite for an equilibrium-axis, we have from the first of them

$$\begin{aligned} -\text{U}\cos f+\text{F}\cos g+\text{E}\cos h \\ &= \text{P}''r\,\{(m^2+n^2)\cos f-lm\cos g-ln\cos h\} \\ &= \text{P}''r\,\{\cos f-l\,(l\cos f+m\cos g+n\cos h)\}. \end{aligned} \tag{315}$$

Let $\phi$ be the angle between the line of action of $\text{P}''$ and the equilibrium-axis; then

$$\cos\phi = l\cos f + m\cos g + n\cos h; \tag{316}$$

and therefore we have

$$\left.\begin{aligned} -\mathrm{U}\cos f + \mathrm{F}\cos g + \mathrm{E}\cos h &= \mathrm{P}''r\{\cos f - l\cos\phi\} = u \\ \mathrm{F}\cos f - \mathrm{V}\cos g + \mathrm{D}\cos h &= \mathrm{P}''r\{\cos g - m\cos\phi\} = v \\ \mathrm{E}\cos f + \mathrm{D}\cos g - \mathrm{W}\cos h &= \mathrm{P}''r\{\cos h - n\cos\phi\} = w \end{aligned}\right\}, \tag{317}$$

employing $u, v, w$ as abbreviated symbols for the left-hand members of the equations, which are known quantities.

Hence we have

$$\begin{aligned} u\cos f + v\cos g + w\cos h &= \mathrm{P}''r\{1-(\cos\phi)^2\} \\ &= \mathrm{P}''r(\sin\phi)^2. \end{aligned}$$

Also

$$\begin{aligned} u^2+v^2+w^2 &= \mathrm{P}''^2r^2\{1-(\cos\phi)^2\}, \\ &= \mathrm{P}''^2r^2(\sin\phi)^2, \end{aligned}$$

$$\therefore\quad \mathrm{P}''r = \frac{u^2+v^2+w^2}{u\cos f + v\cos g + w\cos h}, \tag{318}$$

$$(\sin\phi)^2 = \frac{(u\cos f + v\cos g + w\cos h)^2}{u^2+v^2+w^2}; \tag{319}$$

and therefore from (317) we are enabled to determine $l, m, n$; and thus the direction of the line of action of $\mathrm{P}''$ is completely determined. The intensity of $\mathrm{P}''$ and the position of its point of application is involved in only (318); and therefore we may take any two points on the line defined by $(l, m, n)$ at a distance $r$ apart, and at them apply two equal and opposite forces $\mathrm{P}'$ and $\mathrm{P}''$ of such magnitude that $\mathrm{P}''r$ is equal to the constant in the right-hand member of (318).

From the preceding it appears that two equal forces, acting originally in opposite directions along the same line of action, will, when the body is turned about a definite axis, equilibrate with the forces of the system: but as the two forces in this displaced position form a couple, we infer that

If a rigid body, on which a system of forces in equilibrium acts, is turned about any axis, and if the forces act on the same points of the body as before and in the same directions, they are generally reducible to a couple; and in the particular case when the condition (281) is fulfilled, the moment of the couple vanishes.

94.] If a system of forces acting on a rigid body is not in equilibrium, it can always be reduced to either a single force of translation, or two forces (see Art. 70); and this last reduction may take place in various ways. I will now shew that into a system of forces not in equilibrium it is possible to introduce two new forces, which with the original forces shall not only produce equilibrium, but shall also make the system capable of an equilibrium-axis of a given direction: or, in other words, it is always possible to reduce a system of forces to two forces of translation, such that they with two other new forces shall be in equilibrium, and also shall have a given equilibrium-axis.

Let the two new forces be $P'$ and $P''$; let $\alpha'\beta'\gamma'$, $\alpha''\beta''\gamma''$ be the direction-angles of their lines of action; $(x', y', z')$, $(x'', y'', z'')$ their points of application; then for the condition of equilibrium of these two new forces, with the former forces of the system, we have

$$\left.\begin{aligned} P'\cos\alpha' + P''\cos\alpha'' + X &= 0 \\ P'\cos\beta' + P''\cos\beta'' + Y &= 0 \\ P'\cos\gamma' + P''\cos\gamma'' + Z &= 0 \end{aligned}\right\}. \qquad (320)$$

Also let

$$\left.\begin{aligned} \Sigma.Py\cos\gamma &= D' \\ \Sigma.Pz\cos\alpha &= E' \\ \Sigma.Px\cos\beta &= F' \end{aligned}\right\} \qquad \left.\begin{aligned} \Sigma.Pz\cos\beta &= D'' \\ \Sigma.Px\cos\gamma &= E'' \\ \Sigma.Py\cos\alpha &= F'' \end{aligned}\right\}, \qquad (321)$$

$$\Sigma.P\,(y\cos\beta + z\cos\gamma) = U, \qquad \Sigma.P\,(z\cos\gamma + x\cos\alpha) = V,$$
$$\Sigma.P\,(x\cos\alpha + y\cos\beta) = W; \qquad (322)$$

then, as the three expressions for the moment-axes of the couples about the coordinate-axes are to vanish, we have

$$\left.\begin{aligned} &P'y'\cos\gamma' + P''y''\cos\gamma'' + D' \\ &\qquad = P'z'\cos\beta' + P''z''\cos\beta'' + D'' = D \text{ (say)} \\ &P'z'\cos\alpha' + P''z''\cos\alpha'' + E' \\ &\qquad = P'x'\cos\gamma' + P''x''\cos\gamma'' + E'' = E \text{ (say)} \\ &P'x'\cos\beta' + P''x''\cos\beta'' + F' \\ &\qquad = P'y'\cos\alpha' + P''y''\cos\alpha'' + F'' = F \text{ (say)} \end{aligned}\right\}; \qquad (323)$$

also let

$$\left.\begin{aligned} P'(y'\cos\beta' + z'\cos\gamma') + P''(y''\cos\beta'' + z''\cos\gamma'') + U &= U' \\ P'(z'\cos\gamma' + x'\cos\alpha') + P''(z''\cos\gamma'' + x''\cos\alpha'') + V &= V' \\ P'(x'\cos\alpha' + y'\cos\beta') + P''(x''\cos\alpha'' + y''\cos\beta'') + W &= W' \end{aligned}\right\}; \qquad (324)$$

and therefore, if the direction-angles of the given equilibrium-axis are $f, g, h$, the conditions required are, see (298),

$$\left.\begin{aligned} -U'\cos f + F\cos g + E\cos h &= 0 \\ F\cos f - V'\cos g + D\cos h &= 0 \\ E\cos f + D\cos g - W'\cos h &= 0 \end{aligned}\right\}; \qquad (325)$$

and these are all the conditions which are requisite for the existence of an equilibrium-axis: viz. the equations severally of (320), (323), and (325), and of which the whole number is nine; and they contain twelve undetermined quantities: viz. $\text{P}'\cos\alpha'$, $\text{P}'\cos\beta'$, ... $\text{P}''\cos\gamma''$, $x'$, $y'$, ... $z''$; of these therefore nine may be eliminated, and there will remain a condition involving the other three: the elimination however is so long and tedious that I shall only state results. If we eliminate the forces $\text{P}'$, $\text{P}''$, the direction-angles of their lines of action, and the coordinates of the point of application of one of them (say $x''y''z''$), it will be found that the resulting equation is of the second order in terms of $x'y'z'$; and will therefore represent a surface of the second order: and it will also be found that the point of application of the other force is also upon the same surface, and also that every point in the line joining the two points is on the same surface: the surface is therefore an hyperboloid of one sheet, the line joining the points of application of the forces being one of the generating straight lines of the surface; and the equilibrium axis is the imaginary axis of the surface. And hence we conclude that to a system of forces which is not in equilibrium two forces may be introduced, so that the system thus modified may be in equilibrium and may also have an equilibrium-axis; and the points of application of these two forces may be at such points on the surface of a certain hyperboloid of one sheet, that the line joining them lies wholly on the surface; and when these points of application are given the lines of action of the forces are also determined.

95.] The two forces, to which a system of forces not in equilibrium and acting on a rigid body may be reduced, generally undergo a change when the body is turned about a given axis; and therefore the two forces, which may be introduced into the system, and which so modify it, that it is both in equilibrium, and also has an equilibrium-axis, also undergo a corresponding change. If however the two forces and the axis about which the rotation takes place are so related that their points of application are on the axis, then their effects continue the same, whatever is the angle through which the body is turned: this axis, which has a definite relation to the system of forces, it is convenient to call by a specific name, and therefore I shall call it the *principal axis of rotation*.

Let $f g h$ be the direction-angles of this definite axis; $(x', y', z')$,

$(x'', y'', z'')$ the points at which the forces are applied; then, if $r$ is the distance between these points,

$$\frac{x''-x'}{\cos f} = \frac{y''-y'}{\cos g} = \frac{z''-z'}{\cos h} = r, \tag{326}$$

and we have from the last Article, and from this equation

$$\left.\begin{array}{ll} \mathrm{P}'\cos\alpha' = -\{\mathrm{X}+\mathrm{P}''\cos\alpha''\} & x' = x''-r\cos f \\ \mathrm{P}'\cos\beta' = -\{\mathrm{Y}+\mathrm{P}''\cos\beta''\} & y' = y''-r\cos g \\ \mathrm{P}'\cos\gamma' = -\{\mathrm{Z}+\mathrm{P}''\cos\gamma''\} & z' = z''-r\cos h \end{array}\right\}, \tag{327}$$

and substituting these in D, E, F, and in U′, V′, W′, we have

$$\left.\begin{array}{l} \mathrm{D} = \mathrm{Z}(r\cos g-y'')+r\mathrm{P}''\cos\gamma''\cos g+\mathrm{D}' \\ \qquad = \mathrm{Y}(r\cos h-z'')+r\mathrm{P}''\cos\beta''\cos h+\mathrm{D}'' \\ \mathrm{E} = \mathrm{X}(r\cos h-z'')+r\mathrm{P}''\cos\alpha''\cos h+\mathrm{E}' \\ \qquad = \mathrm{Z}(r\cos f-x'')+r\mathrm{P}''\cos\gamma''\cos f+\mathrm{E}'' \\ \mathrm{F} = \mathrm{Y}(r\cos f-x'')+r\mathrm{P}''\cos\beta''\cos f+\mathrm{F}' \\ \qquad = \mathrm{X}(r\cos g-y'')+r\mathrm{P}''\cos\alpha''\cos g+\mathrm{F}'' \end{array}\right\}, \tag{328}$$

$$\left.\begin{array}{l} \mathrm{U}' = \mathrm{Y}(r\cos g-y'')+\mathrm{Z}(r\cos h-z'') \\ \qquad +\mathrm{P}''r(\cos\beta''\cos g+\cos\gamma''\cos h)+\mathrm{U} \\ \mathrm{V}' = \mathrm{Z}(r\cos h-z'')+\mathrm{X}(r\cos f-x'') \\ \qquad +\mathrm{P}''r(\cos\gamma''\cos h+\cos\alpha''\cos f)+\mathrm{V} \\ \mathrm{W}' = \mathrm{X}(r\cos f-x'')+\mathrm{Y}(r\cos g-y'') \\ \qquad +\mathrm{P}''r(\cos\alpha''\cos f+\cos\beta''\cos g)+\mathrm{W} \end{array}\right\}; \tag{329}$$

and substituting these in (325) we have

$$\left.\begin{array}{l} (\mathrm{Y}y''+\mathrm{Z}z''-\mathrm{U})\cos f+(\mathrm{F}'-x''\mathrm{Y})\cos g+(\mathrm{E}''-x''\mathrm{Z})\cos h = 0 \\ (\mathrm{F}''-y''\mathrm{X})\cos f+(\mathrm{Z}z''+\mathrm{X}x''-\mathrm{V})\cos g+(\mathrm{D}'-y''\mathrm{Z})\cos h = 0 \\ (\mathrm{E}'-z''\mathrm{X})\cos f+(\mathrm{D}''-z''\mathrm{Y})\cos g+(\mathrm{X}x''+\mathrm{Y}y''-\mathrm{W})\cos h = 0 \end{array}\right\}. \tag{330}$$

Also from (328) we have

$$(\mathrm{D}'-\mathrm{D}''+\mathrm{Y}z''-\mathrm{Z}y'')\cos f+(\mathrm{E}'-\mathrm{E}''+\mathrm{Z}x''-\mathrm{X}z'')\cos g$$
$$+(\mathrm{F}'-\mathrm{F}''+\mathrm{X}y''-\mathrm{Y}x'')\cos h = 0. \tag{331}$$

To abbreviate the preceding expressions, let

$$\left.\begin{array}{r} -\mathrm{U}\cos f+\mathrm{F}'\cos g+\mathrm{E}''\cos h = u \\ \mathrm{F}''\cos f-\mathrm{V}\cos g+\mathrm{D}'\cos h = v \\ \mathrm{E}'\cos f+\mathrm{D}''\cos g-\mathrm{W}\cos h = w \end{array}\right\}; \tag{332}$$

also let $(\mathrm{D}'-\mathrm{D}'')\cos f+(\mathrm{E}'-\mathrm{E}'')\cos g+(\mathrm{F}'-\mathrm{F}'')\cos h = s,$ (333)

$z''\cos g-y''\cos h=\xi,\ \ x''\cos h-z''\cos f=\eta,\ \ y''\cos f-x''\cos g=\zeta;$ (334)

whence we have $\xi\cos f+\eta\cos g+\zeta\cos h = 0;$ (335)

and therefore from (330)

$$\left.\begin{aligned} u+\text{Y}\zeta-\text{Z}\eta &= 0 \\ v+\text{Z}\xi-\text{X}\zeta &= 0 \\ w+\text{X}\eta-\text{Y}\xi &= 0 \end{aligned}\right\}; \tag{336}$$

and therefore $$u\text{X}+v\text{Y}+w\text{Z} = 0. \tag{337}$$

Also (331) becomes $$\text{X}\xi+\text{Y}\eta+\text{Z}\zeta = s; \tag{338}$$

and from (336) we have

$$\left.\begin{aligned} w\cos g-v\cos h &= (\text{X}\cos f+\text{Y}\cos g+\text{Z}\cos h)\,\xi \\ u\cos h-w\cos f &= (\text{X}\cos f+\text{Y}\cos g+\text{Z}\cos h)\,\eta \\ v\cos f-u\cos g &= (\text{X}\cos f+\text{Y}\cos g+\text{Z}\cos h)\,\zeta \end{aligned}\right\}; \tag{339}$$

and multiplying these severally by X, Y, Z, we have

$$\text{X}(w\cos g-v\cos h)+\text{Y}(u\cos h-w\cos f)+\text{Z}(v\cos f-u\cos g)$$
$$= (\text{X}\cos f+\text{Y}\cos g+\text{Z}\cos h)\,s; \tag{340}$$

and these equations may be interpreted in the following manner: $u$, $v$, $w$ and $s$ are, as appears from (332) and (333), linear functions of $\cos f$, $\cos g$, $\cos h$; so that (337) is a homogeneous equation of the first degree, and (340) of the second degree in terms of $\cos f$, $\cos g$, $\cos h$. Imagine then, for an instant, a line to be drawn through the origin, whose direction-angles are $f, g, h$; and a point to be taken on this line at a distance $r$ from the origin, and of which the coordinates are $x, y, z$; so that

$$\frac{\cos f}{x} = \frac{\cos g}{y} = \frac{\cos h}{z} = \frac{1}{r}; \tag{341}$$

then replacing the direction-cosines in (337) and (340) by their proportional values given by (341), (337) will become the equation of a plane passing through the origin, and (340) will be that of a cone of the second degree whose vertex is at the origin; and such a plane may cut the cone in two straight lines which intersect at the origin, or may touch it along its surface, or may not intersect it, save at the origin. Now the lines common to the cone and the plane will be the principal axes of rotation; there may therefore be, corresponding to a system of forces, two such axes, or only one, or none.

In the most general case, therefore, there are two principal axes of rotation, and thus two several values of $f, g, h$: corresponding to these, there are from (339) two different values of $\xi\,\eta\,\zeta$, and therefore from (334), of $x''y''z''$; but as these values are subject to the condition (335), they are not independent, and are therefore the equations to a straight line, along which $\text{P}''$ acts, and of which the point of application is undetermined;

and these coordinates together with the values of $f$, $g$, $h$ are sufficient to determine the position of the principal axes of rotation.

Also let the preceding equations be applied to a system consisting of only two forces; then it will be found that of the principal axes of rotation, one is perpendicular to the plane which is parallel to the lines of action of the two forces, and pierces that plane at the centre of the orthographic projections of the two forces on that plane: and the second is the line joining the points of application of the forces.

Hence if the line joining the points of application of the forces is perpendicular to each of the lines of action of the forces, the two principal axes coincide; such is the case of the system of forces in Article 86.

If the two forces are applied at the same point, any line passing through that point is a principal axis of rotation: and so if the lines of action of the forces are parallel, any line passing through the centre of the two parallel forces is a principal rotation-axis.

96.] There is also another theorem of centres of forces not parallel, which it is expedient to investigate; and first we will take the more simple case of forces acting in one plane. If many forces act on a rigid body in one plane, and if at their points of application straight lines are drawn in an arbitrary direction, all parallel to each other, and along which the forces are resolved; then the locus of the centres of all the systems of the parallel resolved forces is a straight line.

Let us use the same notation as that of Article 46; and let $\theta$ be the angle at which the lines, along which the forces are resolved, are inclined to the axis of $x$; then the resolved parts of the forces, and all of which are parallel to each other, are

$$\text{P}_1 \cos(\alpha_1-\theta), \quad \text{P}_2 \cos(\alpha_2-\theta), \ \dots \ \text{P}_n \cos(\alpha_n-\theta);$$

and therefore if $\bar{x}\,\bar{y}$ are the coordinates to the centre of this system of parallel forces,

$$\left.\begin{aligned} \bar{x}\,\Sigma.\text{P}\cos(\alpha-\theta) &= \Sigma.\text{P}x\cos(\alpha-\theta) \\ \bar{y}\,\Sigma.\text{P}\cos(\alpha-\theta) &= \Sigma.\text{P}y\cos(\alpha-\theta) \end{aligned}\right\}; \qquad (342)$$

$$\therefore \quad \cos\theta\{\Sigma.\text{P}x\cos\alpha-\bar{x}\,\Sigma.\text{P}\cos\alpha\}+\sin\theta\,\{\Sigma.\text{P}x\sin\alpha-\bar{x}\,\Sigma.\text{P}\sin\alpha\} = 0$$
$$\cos\theta\{\Sigma.\text{P}y\cos\alpha-\bar{y}\,\Sigma.\text{P}\cos\alpha\}+\sin\theta\,\{\Sigma.\text{P}y\sin\alpha-\bar{y}\,\Sigma.\text{P}\sin\alpha\} = 0;$$

whence eliminating $\theta$, which is arbitrary, we have

$$\bar{x}\,\text{Y}\,\Sigma.\text{P}y\cos a - \bar{y}\,\text{Y}\,\Sigma.\text{P}x\cos a + \Sigma.\text{P}x\cos a \times \Sigma.\text{P}y\sin a - \Sigma.\text{P}y\cos a \times \Sigma.\text{P}x\sin a = 0; \quad (343)$$

which is the equation to a straight line; and which is called the *central line* of the system; and passes through the origin if

$$\frac{\Sigma.\text{P}y\sin a}{\Sigma.\text{P}y\cos a} = \frac{\Sigma.\text{P}x\sin a}{\Sigma.\text{P}x\cos a}. \quad (344)$$

Also a similar theorem is true of a system of forces applied to a rigid body in space. At the points of application of each force let a straight line be drawn parallel to a straight line passing through the origin, and of which the direction-angles are $\lambda\,\mu\,\nu$; and let the resolved parts of the forces along these lines be considered; they constitute a system of parallel forces, of each one of which the type is

$$\text{P}\,\{\cos a\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu\};$$

and therefore if $\bar{x}, \bar{y}, \bar{z}$ are the coordinates to the centre of these parallel forces, we have

$$\bar{x}\,\Sigma.\text{P}\,\{\cos a\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu\} = \Sigma.\text{P}x\,\{\cos a\cos\lambda + \cos\beta\cos\mu + \cos\gamma\cos\nu\},$$

and we have similar values for $\bar{y}$ and $\bar{z}$; therefore

$$\left.\begin{aligned}
\{\bar{x}\text{X} - \Sigma.\text{P}x\cos a\}\cos\lambda + \{\bar{x}\text{Y} - \Sigma.\text{P}x\cos\beta\}\cos\mu & \\
+ \{\bar{x}\text{Z} - \Sigma.\text{P}x\cos\gamma\}\cos\nu &= 0 \\
\{\bar{y}\text{X} - \Sigma.\text{P}y\cos a\}\cos\lambda + \{\bar{y}\text{Y} - \Sigma.\text{P}y\cos\beta\}\cos\mu & \\
+ \{\bar{y}\text{Z} - \Sigma.\text{P}y\cos\gamma\}\cos\nu &= 0 \\
\{\bar{z}\text{X} - \Sigma.\text{P}z\cos a\}\cos\lambda + \{\bar{z}\text{Y} - \Sigma.\text{P}z\cos\beta\}\cos\mu & \\
+ \{\bar{z}\text{Z} - \Sigma.\text{P}z\cos\gamma\}\cos\nu &= 0
\end{aligned}\right\}; \quad (345)$$

and therefore eliminating $\cos\lambda$, $\cos\mu$, $\cos\nu$, there will result an equation of the first degree in terms of $\bar{x}\,\bar{y}\,\bar{z}$, and which is therefore the equation of a plane; and as this plane is the same whatever are the values of $\lambda$, $\mu$, and $\nu$, it follows that whatever are the direction-cosines of the system of parallel lines along which the forces are resolved, the centres of the corresponding systems of parallel forces are in a plane. This plane is called the *central plane* of the system.

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## SECTION 5.—*Stability and Instability of Equilibrium.*

97.] Suppose a rigid body to be at rest under the action of many forces; and also suppose the body to receive a small displacement, of such an infinitesimal nature that the forces, when applied at the same points as before, act in the same directions, and along lines of action parallel to, and infinitesimally distant from, their former lines of action ; then the body in its new position (1) may not be in equilibrium, (2) may be in equilibrium: if it is not in equilibrium, the forces acting may tend either to bring it back again into its former position or to remove it farther from it; the previous equilibrium is said in the former state to be *stable*, in the latter to be *unstable;* and if the body in its displaced state is in equilibrium, it may be so, either for that axis about which its displacement of rotation has taken place, and for certain other axes infinitesimally near to that, in which case the equilibrium is said to be *neutral;* or the equilibrium may still exist for the axis about which the rotation has taken place, and for all others, in which case the equilibrium is said to be *continuous*. I say, for the axis about which the displacement of rotation has taken place: for every displacement of a body consists generally of two displacements; one of translation, in which all the parts of the body pass over equal and parallel straight lines; and another of rotation, when the body revolves through a certain angle about a given axis. Now for an infinitesimal displacement of translation, a body which under the action of certain forces is in equilibrium before the displacement, is also in equilibrium after, if the forces act at the same points, and in the same directions, and along lines of action parallel to their former lines; but after an infinitesimal displacement of rotation about a certain axis, a system of forces which was in equilibrium may, under the same incidents, either be or not be in equilibrium; and therefore it is for a displacement of rotation only about a certain axis that I shall have to investigate analytical criteria of the four states mentioned above. And in reference to equilibrium-axes which have been the subjects of discussion in the preceding Articles I must observe, that if a body in equilibrium under the action of certain forces has no equilibrium-axis, its equilibrium is either stable or unstable; if it has one or two equilibrium-axes, its equilibrium is neutral, when the displacement of rotation takes place about one of them; and if the system of forces is such, see Article 92, that every axis is an equilibrium-axis, then the equilibrium is continuous.

98.] And with the view of removing any obscurity of conception, let me take the most simple case of a body held in equilibrium under the action of two forces only: these of course are equal to each other, and act along the same line, and in opposite directions: but these conditions may be satisfied in two ways: the forces may act to draw their points of application either nearer to, or farther from, each other. Let $P_1$, $P_2$, see fig. 66, be the two forces; $A_1$, $A_2$ their respective points of application. Let the body receive an infinitesimal displacement of rotation about an axis perpendicular to the line of action of the forces: so that the line $A_1 A_2$, which before the displacement was in the same line with the line of action of the forces, is now in the positions, relatively to them, indicated in the figures ($\alpha$) and ($\beta$): ($\alpha$) is evidently the state in which the forces applied at $A_1$ and $A_2$ tend to bring the points nearer to each other; and in which, now that the displacement has taken place, the action of the forces tends to remove the system farther and farther from its original position, and in which therefore the original equilibrium was unstable; ($\beta$) is the state in which the forces act to separate their points of application, and in which the forces act after the displacement to bring the body back to its original position; and in which therefore the equilibrium is stable. If the two forces act at the same point, equilibrium is continuous for every displacement of the body about an axis perpendicular to the line of action of the forces; and also because the point at which they act is their *centre*.

And let us investigate an analytical criterion of these several states of equilibrium. Let $(x_1, y_1)$ $(x_2, y_2)$ be the points of application of $P_1$ and of $P_2$ respectively; then the conditions of equilibrium of these two forces are

$$P_1 + P_2 = 0,$$

$$G = \sin\alpha\, \Sigma.Px - \cos\alpha\, \Sigma.Py = 0. \tag{346}$$

Let the system be turned about an axis perpendicular to the plane of $xy$ through an angle $d\theta$; so that

$$dx = -y\,d\theta, \qquad dy = x\,d\theta, \tag{347}$$

$$\therefore \quad \frac{dG}{d\theta} = \sin\alpha\, \Sigma.P\frac{dx}{d\theta} - \cos\alpha\, \Sigma.P\frac{dy}{d\theta}$$

$$= -\{\sin\alpha\, \Sigma.Py + \cos\alpha\, \Sigma.Px\}; \tag{348}$$

but as $\frac{dG}{d\theta}$ is positive or negative, so does the couple arising from the action of the forces in the displaced position of the body tend to remove the body farther from, or to bring it back nearer to, its former position: that is, so is the equilibrium of the body unstable or stable. And therefore is equilibrium stable or unstable, according as

$$\Sigma.\text{P}x \cos a + \Sigma.\text{P}y \sin a \tag{349}$$

is positive or negative.

And because $a$ is the same for both the forces, and is also generally indeterminate, since the directions of the axes are arbitrary, so does the criterion (349) reduce itself to either $\Sigma.\text{P}x$ or $\Sigma.\text{P}y$, and thus the stability depends on the sign of either of these.

If $(349) = 0$, then, since $\text{P}_1 + \text{P}_2 = 0$, $x_1 = x_2 = 0$, $y_1 = y_2 = 0$; that is, the forces are applied at the same point, viz. the origin, and the equilibrium is continuous.

The rotation has hitherto taken place about an axis perpendicular to the line of action of the forces. I would only further observe, that if it takes place about the line of action of the two forces, their points of application undergo no displacement, and no criterion of stability is therefrom discernible.

99.] To deduce generally the criterion of stability for a system of forces in equilibrium and acting on a rigid body in one plane, I will apply the process of resolution of Article 52, considering the infinitesimal rotation to take place about an axis perpendicular to the plane of the forces.

Let the forces be $\text{P}_1, \text{P}_2, \ldots \text{P}_n$, and let their points of application and lines of action be expressed as heretofore; let P, fig. 67, be the type-force; let A be its point of application; join OA; resolve P along, and perpendicular to, OA; and operate similarly on all the forces: then we shall have two expressions, viz.

$$\left.\begin{array}{rl} & \Sigma.\text{P}\,(x \sin a - y \cos a) \\ \text{and} & \Sigma.\text{P}\,(x \cos a + y \sin a) \end{array}\right\}, \tag{350}$$

of which the first is evidently the moment of the resultant couple, and vanishes because the system is in equilibrium; and the second is the central moment of Article 52, and is unaltered whatever is the origin and whatever are the coordinate axes, because the system is in equilibrium. Now let the body receive a slight displacement about an axis passing through O, and perpendicular to the plane of the forces; and let the amplitude of displacement be $d\theta$: then (350) become

$$\left.\begin{array}{r} -d\theta\, \Sigma.\text{P}\,(x \cos a + y \sin a) \\ \Sigma.\text{P}\,(x \cos a + y \sin a) \end{array}\right\}; \tag{351}$$

and therefore from the first of these it appears, that the moment of the couple which arises from the displacement is negative or positive, and therefore tends to bring back the body to its former position or remove it farther from it according as $\Sigma . \text{P} (x \cos a + y \sin a)$ is positive or negative; and the case in which it is zero, the resultant couple vanishes in the displaced as well as in the original state, and therefore the equilibrium is neutral. Also the central moment is unaltered by the displacement. Hence we have the following test:

Of a system of forces acting on a rigid body in a plane, and being in equilibrium, the equilibrium is stable, neutral, or unstable, according as $\Sigma . \text{P} (x \cos a + y \sin a)$, that is, the central moment, is positive, zero, or negative.

It is also manifest that the preceding criterion is true only for a motion of the body about an axis perpendicular to the plane in which the forces act; and to illustrate this fact let us suppose four forces to act on a body in one plane and to be in equilibrium; and suppose them to be such, that a pair of them is in equilibrium; and that therefore the other pair also equilibrates; let the body be turned about an axis coinciding with the line of action of the latter pair, the equilibrium of the other pair may evidently be either stable or unstable: and if the rotation takes place about the line of action of the former pair, the equilibrium of the latter pair may be either stable or unstable; and evidently there is no necessity that it should be of the same character as the other; hence in this case we are unable to state *a priori* the axes of stable or of unstable equilibrium.

And the preceding test is applicable to the case of forces whose lines of action are parallel to a given plane when the displacement takes place about a line perpendicular to that plane.

100.] I proceed now to the case of a rigid body in equilibrium under the action of many forces acting along any lines of action in space; and here again, for the reasons stated in Art. 97, we have to consider only the effects of a displacement of rotation as to the kind of equilibrium which the body is in.

Let the direction-angles of the axis of rotation be $f, g, h$; and let the moment-axes of the impressed couples along the three coordinate axes be L, M, N; then, if G is the moment of the couple tending to turn the body about the rotation-axis, by reason of the law of resolution of couples,

$$\begin{aligned} \mathrm{G} &= \mathrm{L}\cos f + \mathrm{M}\cos g + \mathrm{N}\cos h \\ &= \cos f\,\Sigma.\mathrm{P}\,(y\cos\gamma - z\cos\beta) + \ldots + \ldots \end{aligned} \qquad (352)$$

Now in equations (293), Art. 89, the expressions for $dx$, $dy$, $dz$, which are there given in terms of the direction-angles of the rotation-axis and the amplitude, have ambiguous signs by reason of the square root which enters in the equality (286); let us therefore fix our thoughts, and suppose the rotation to be positive, if, the rotation-axes coinciding with the axes severally of $x, y, z$, the directions of rotation are from $y$ to $z$, from $z$ to $x$, and from $x$ to $y$: so that, if the amplitude of rotation is $d\theta$,

$$\left.\begin{aligned} dx &= d\theta\,(z\cos g - y\cos h) \\ dy &= d\theta\,(x\cos h - z\cos f) \\ dz &= d\theta\,(y\cos f - x\cos g) \end{aligned}\right\}, \qquad (353)$$

$$\begin{aligned} \frac{d\mathrm{G}}{d\theta} &= \cos f\,\Sigma.\mathrm{P}\left(\cos\gamma\frac{dy}{d\theta} - \cos\beta\frac{dz}{d\theta}\right) + \ldots + \ldots \\ &= -(\cos f)^2\,\Sigma.\mathrm{P}\,(y\cos\beta + z\cos\gamma) + \cos g\cos h\,\Sigma.\mathrm{P}\,(y\cos\gamma + z\cos\beta) \\ &\quad -(\cos g)^2\,\Sigma.\mathrm{P}\,(z\cos\gamma + x\cos\alpha) + \cos h\cos f\,\Sigma.\mathrm{P}\,(z\cos\alpha + x\cos\gamma) \\ &\quad -(\cos h)^2\,\Sigma.\mathrm{P}\,(x\cos\alpha + y\cos\beta) + \cos f\cos g\,\Sigma.\mathrm{P}\,(x\cos\beta + y\cos\alpha); \end{aligned} \qquad (354)$$

and employing the abbreviated notation of Article 88,

$$\begin{aligned} \frac{d\mathrm{G}}{d\theta} &= -\mathrm{U}\,(\cos f)^2 - \mathrm{V}\,(\cos g)^2 - \mathrm{W}\,(\cos h)^2 \\ &\quad + 2\,\mathrm{D}\cos g\cos h + 2\,\mathrm{E}\cos h\cos f + 2\,\mathrm{F}\cos f\cos g; \end{aligned} \qquad (355)$$

and since the effect of G due to a small variation of $\theta$ is to bring back the system to its former position, or to remove it farther therefrom, according as $\frac{d\mathrm{G}}{d\theta}$ is negative or positive, so is the equilibrium stable or unstable according as the right-hand member of (355) is negative or positive. And I may observe in passing, that if the axis of rotation is the $z$-coordinate axis, (355) requires that $\mathrm{W} = \Sigma.\mathrm{P}\,(x\cos\alpha + y\cos\beta)$ should be positive for stable, and negative for unstable equilibrium: which is the same result as that of the last Article.

And if all the forces are parallel to the axis of $z$, so that $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$, then

$$\frac{d\mathrm{G}}{d\theta} = -(\sin h)^2\,\Sigma.\mathrm{P}z + \cos h\,\{\cos f\,\Sigma.\mathrm{P}x + \cos g\,\Sigma.\mathrm{P}y\}; \qquad (356)$$

and if the axis about which the infinitesimal rotation takes place is at right-angles to the lines of action of the forces, then $h = 90°$, and we have

$$\frac{d\mathrm{G}}{d\theta} = -\Sigma.\mathrm{P}z; \qquad (357)$$

and therefore equilibrium is stable or unstable according as $\Sigma . \mathrm{P} z$ is positive or negative.

101.] For convenience of reference let us symbolize the right-hand member of (355) by $\mathrm{S}$; so that

$$\begin{aligned}\mathrm{S} = & -\mathrm{U} (\cos f)^2 - \mathrm{V} (\cos g)^2 - \mathrm{W} (\cos h)^2 \\ & + 2\mathrm{D} \cos g \cos h + 2\mathrm{E} \cos h \cos f + 2\mathrm{F} \cos f \cos g;\end{aligned}$$

so that equilibrium is stable or unstable, according as $\mathrm{S}$ is negative or positive: and the sign evidently depends, not only on the impressed forces and their incidents, but also on the direction-angles of the rotation-axis; and therefore the equilibrium for a given system of forces may be stable for one rotation-axis, unstable for another, and neither one nor the other for a third; that is, in the third case $\mathrm{S}$ may be equal to zero.

Suppose now that $\mathrm{S}$ is arranged in the form

$$\begin{aligned}\{-\mathrm{U} \cos f + \mathrm{F} \cos g + \mathrm{E} \cos h\} \cos f + \{\mathrm{F} \cos f - \mathrm{V} \cos g + \mathrm{D} \cos h\} \cos g & \\ + \{\mathrm{E} \cos f + \mathrm{D} \cos g - \mathrm{W} \cos h\} \cos h, & \quad (358)\end{aligned}$$

and that we have also

$$\left.\begin{aligned}-\mathrm{U} \cos f + \mathrm{F} \cos g + \mathrm{E} \cos h &= 0 \\ \mathrm{F} \cos f - \mathrm{V} \cos g + \mathrm{D} \cos h &= 0 \\ \mathrm{E} \cos f + \mathrm{D} \cos g - \mathrm{W} \cos h &= 0\end{aligned}\right\}, \qquad (359)$$

so that
$$\mathrm{UVW} - \mathrm{D}^2\mathrm{U} - \mathrm{E}^2\mathrm{V} - \mathrm{F}^2\mathrm{W} - 2\mathrm{DEF} = 0; \qquad (360)$$

and this is the condition requisite for the existence of an equilibrium-axis; in this case $\mathrm{S} = 0$, and the equilibrium is neutral.

If also, according to Article 92, equations (307),

$$\mathrm{D}^2 = \mathrm{VW}, \qquad \mathrm{E}^2 = \mathrm{WU}, \qquad \mathrm{F}^2 = \mathrm{UV}, \qquad (361)$$

and if the axis about which the rotation takes place is parallel to the plane whose equation is

$$\frac{x}{(\mathrm{VW})^{\frac{1}{2}}} + \frac{y}{(\mathrm{WU})^{\frac{1}{2}}} + \frac{z}{(\mathrm{UV})^{\frac{1}{2}}} = 0, \qquad (362)$$

then equilibrium is neutral for all such axes; and is continuous, if the change of axis is from any one line to any other line lying in the plane.

And if in addition,
$$\left.\begin{aligned}\mathrm{D} = \mathrm{E} = \mathrm{F} = 0 \\ \mathrm{U} = \mathrm{V} = \mathrm{W} = 0\end{aligned}\right\}, \qquad (363)$$

so that any axis about which the body is turned is an equilibrium-axis, then the equilibrium is continuous for all axes.

Also the right-hand member of (355) admits of being put in the following form:

$$\text{S} = \Sigma.\text{P}\{(x\cos f+y\cos g+z\cos h)(\cos a\cos f+\cos\beta\cos g+\cos\gamma\cos h)\} - \Sigma.\text{P}\{x\cos a+y\cos\beta+z\cos\gamma\}. \quad (364)$$

Hence we may deduce another criterion of its sign; and in reference to this I would observe that, for a given rotation-axis, $x\cos f+y\cos g+z\cos h$ is the projection on the rotation-axis of the distance from the origin of the point of application of P, and $\text{P}\{\cos a\cos f+\cos\beta\cos g+\cos\gamma\cos h\}$ is the resolved part of P along the rotation-axis; and therefore both these quantities are constant independently of the rotation; it is therefore only the latter part of (364) whose value will be altered by an infinitesimal rotation.

I may also observe that, if the directions of action of all the forces are reversed, the signs of U, V, W, D, E, F are changed, and therefore the sign of S is changed; and thus the nature of the equilibrium is changed: in the case however of neutral equilibrium no alteration takes place.

102.] And the right-hand member of (355) admits of the following geometrical interpretation: on the straight line drawn through the origin, and whose direction-angles are $f, g, h$, let a point $(x, y, z)$ be taken: then $x, y, z$ are proportional to $\cos f$, $\cos g$, $\cos h$, and (355) becomes

$$-\text{U}x^2-\text{V}y^2-\text{W}z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy, \quad (365)$$

which, when equated to zero, is the equation to a cone of the second degree; and therefore for all lines passing through the origin, and lying within this cone, and employed as rotation-axes, the above expression has a different sign to that which it has for all lines lying outside of the cone; and for all lines on the surface of the cone it vanishes; so that for all the generating lines of the cone, equilibrium is neutral; and the cone divides space into two parts such, that for all axes within its surface, the equilibrium is the opposite to that which it is for axes outside the surface.

I may however observe that if lines are drawn through the vertex of the cone, and if these are called *interior* or *exterior* lines according as from points on them real tangent planes cannot, or can, be drawn to the cone; then will interior lines be axes of stable, and exterior lines axes of unstable, equilibrium, if

$$\text{UVW}-\text{D}^2\text{U}-\text{E}^2\text{V}-\text{F}^2\text{W}-2\text{DEF} = \nabla \text{ (say)}$$

is positive; and if $\nabla$ is negative, the converse is the case.

If $\text{V} = 0$, we have the following circumstances. If we reduce the expression (365) equated to zero, so as to deprive it of the terms containing the products of the variables, we get the (commonly so called) discriminating cubic, of which the constant term is V, and therefore if $\text{V} = 0$, one of the roots of this cubic is zero, and the reduced equation becomes of the form

$$\text{U}' x^2 \pm \text{V}' y^2 = 0,$$

which, if the upper sign is taken, represents the axis of $z$; and, if the lower sign is taken, two planes. In the former of these two cases the axis of $z$ is an axis of neutral equilibrium, and all other lines are axes either all of unstable, or all of stable, equilibrium: in the latter case, any line in either of the planes is an axis of neutral equilibrium, and the other lines are either all axes of stable, or all of unstable, equilibrium.

103.] There are also some remarkable properties of the preceding criteria for the stability of a system of forces in reference to maxima and minima.

First let us take the case of a system of forces acting in one plane; that, viz., of $xy$: then the conditions of equilibrium are

$$\Sigma.\text{P} \cos a = 0, \qquad \Sigma.\text{P} \sin a = 0,$$
$$\Sigma.\text{P}\,(x \sin a - y \cos a) = 0. \tag{366}$$

Let the system be turned about an axis perpendicular to the plane of the forces, through an infinitesimal angle $d\theta$: then the increments of $x$ and $y$ are

$$dx = -y\,d\theta, \qquad dy = x\,d\theta, \tag{367}$$

$$\therefore \quad y = -\frac{dx}{d\theta}, \qquad x = \frac{dy}{d\theta}; \tag{368}$$

and therefore (366) becomes

$$\Sigma.\text{P}\left(\cos a \frac{dx}{d\theta} + \sin a \frac{dy}{d\theta}\right) = 0 = \frac{du}{d\theta} \text{ (say)}, \tag{369}$$

$$\therefore \quad u = \Sigma.\text{P}\,(x \cos a + y \sin a); \tag{370}$$

and therefore $u$ is a maximum, or a minimum, or a constant: also since

$$\begin{aligned} \frac{d^2u}{d\theta^2} &= \frac{d}{d\theta}\Sigma.\text{P}\,(x \sin a - y \cos a) \\ &= \Sigma.\text{P}\left(\frac{dx}{d\theta} \sin a - \frac{dy}{d\theta} \cos a\right) \\ &= -\Sigma.\text{P}\,(x \cos a + y \sin a) \qquad (371) \\ &= -u, \end{aligned}$$

$u$ is a maximum or a minimum according as $u$ is positive or negative; and is neither one nor the other when it is equal to zero. Hence

$$\Sigma.\mathrm{P}\,(x\cos a+y\sin a)$$

is a maximum when equilibrium is stable, a minimum when equilibrium is unstable, and zero when it is either neutral or continuous.

For a system of forces acting on a rigid body in space, let us suppose the infinitesimal rotation to take place about an axis whose direction-angles are $f, g, h$, and let the moment-axes of the couples, whose rotation-axes are the coordinate-axes, be L, M, N; then for equilibrium we have

$$\mathrm{L}\cos f+\mathrm{M}\cos g+\mathrm{N}\cos h=0; \qquad (372)$$

that is, replacing L, M, N by their values, and introducing $d\theta$,

$$\Sigma.\mathrm{P}\,\{\cos a\,(z\cos g-y\cos h)+\cos\beta\,(x\cos h-z\cos f)$$
$$+\cos\gamma\,(y\cos f-x\cos\gamma)\}\,d\theta=0;$$

and by means of (353)

$$\Sigma.\mathrm{P}\,\{\cos a\,dx+\cos\beta\,dy+\cos\gamma\,dz\}=0=du \text{ (say)}; \qquad (373)$$

therefore by integration

$$u=\Sigma.\mathrm{P}\,(x\cos a+y\cos\beta+z\cos\gamma); \qquad (374)$$

and therefore $u$ is a maximum, a minimum, or a constant: and since, see equation (352),

$$\frac{du}{d\theta}=\mathrm{G},$$

$$\therefore\ \frac{d^2u}{d\theta^2}=\frac{d\mathrm{G}}{d\theta}$$
$$=\mathrm{S},$$

see equation (355); therefore $u=\Sigma.\mathrm{P}\,(x\cos a+y\cos\beta+z\cos\gamma)$ is a maximum or minimum, according as S is negative or positive, that is, according as equilibrium is stable or unstable; and therefore, by reason of equation (364), according as $u$ is greater than or less than

$$\Sigma.\mathrm{P}\,\{(x\cos f+y\cos g+z\cos h)\,(\cos a\cos f+\cos\beta\cos g+\cos\gamma\cos h)\};$$

and if $\mathrm{S}=0$, equilibrium is either neutral or continuous.

In Article 52, the forces are resolved along, and perpendicular to, the radius vector of the point of application; and $\Sigma.\mathrm{P}(x\cos a+y\sin a)$ has been called the central moment of the system, because it is the product of the radius vector of the point of application, and of the radial component. Similarly in space, if we resolve P along the radius vector of its point of application, and call U its radial component,

$$\text{U} = \frac{\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma)}{r}, \qquad (375)$$

where $r$ is the radius vector of the point of application of P: therefore

$$\Sigma.\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma) = \Sigma.\text{U}r, \qquad (376)$$

and $\Sigma.\text{U}r$ is called *the central moment of the system*. Hence we have the following theorem:

The equilibrium of a system of forces is stable or unstable according as the central moment is a maximum or a minimum.

Some examples of a particular form of these problems will be given hereafter.

And the central moment also possesses one or two other properties which it is desirable to insert. Let us symbolize the central moment by $\Pi$: and let us suppose the body or system of particles on which the forces act to receive a small displacement, and all the forces to act at their points of application, along lines of action parallel to the former ones, and in the same directions. And (1) let us suppose the motion of the body to be constrained in translation along a given line, and let $s$ be the space described; then $\frac{d\Pi}{ds}$ is the sum of the components of the forces estimated along that line; and (2) let the motion be one of rotation about a given axis; and let $\theta$ be the amplitude of rotation; then $\frac{d\Pi}{d\theta}$ is, in any position, the moment-axis of the couple arising out of the system of forces about that axis.

104.] Again, let $p$ be the projection of $r$ on the line of action of P, then

$$p = x\cos\alpha + y\cos\beta + z\cos\gamma,$$

$$\therefore \quad \Sigma.\text{P}\,(x\cos\alpha + y\cos\beta + z\cos\gamma) = \Sigma.\text{P}p; \qquad (377)$$

and since for equilibrium this quantity is such that its differential vanishes, therefore, from the right-hand member of (377),

$$\Sigma.\text{P}\,dp = 0; \qquad (378)$$

the change of position of the system being infinitesimal, and such that whilst the points of application of the forces undergo infinitesimal variations, the forces and the directions of their lines of action are unaltered.

And this equation is the mathematical expression of the *principle of virtual velocities;* a principle which holds in its grasp all statics, and to which all dynamics may be reduced; we shall enter on the full elucidation of it hereafter, and so shall at present only enuntiate it, and as follows:

If a system of forces is in equilibrium, and the system receives an infinitesimal displacement of the most general kind possible, whereby the points of application of the forces are displaced; but the forces act along lines parallel to, and infinitesimally distant from, their former lines of action; and if the relative distances of their points of application are unaltered, then the sum of the products of each force, and the projection on its line of action of the displacement of its point of application, is equal to zero.

The projection on the line of action of a force of the infinitesimal displacement of its point of application is called the virtual velocity of the force: and is to be affected with a different sign, by virtue of the law of projection, according as it is projected on the line of action in the direction towards which the force acts, or in that direction produced backwards.

And from the equation (378) we can immediately deduce the six equations of equilibrium given in Article 57; let the system receive the most general infinitesimal displacement which is possible; that is, let it be moved (1) through distances $\xi$, $\eta$, $\zeta$ parallel to the three coordinate axes, and (2) through an angle $d\theta$ about an axis whose direction-angles are $f, g, h$: then

$$\begin{aligned} dp = \{\xi + d\theta\,(z\cos g - y\cos h)\}\cos\alpha + \{\eta + d\theta\,(x\cos h - z\cos f)\}\cos\beta \\ + \{\zeta + d\theta\,(y\cos f - x\cos g)\}\cos\gamma, \quad (379) \end{aligned}$$

so that (377) becomes

$$\begin{aligned} \Sigma.\mathrm{P}\,dp = \xi\,\Sigma.\mathrm{P}\cos\alpha + \eta\,\Sigma.\mathrm{P}\cos\beta + \zeta\,\Sigma.\mathrm{P}\cos\gamma \\ + d\theta\cos f\,\Sigma.\mathrm{P}\,(y\cos\gamma - z\cos\beta) + d\theta\cos g\,\Sigma.\mathrm{P}\,(z\cos\alpha - x\cos\gamma) \\ + d\theta\cos h\,\Sigma.\mathrm{P}\,(x\cos\beta - y\cos\alpha) = 0; \quad (380) \end{aligned}$$

and as $\xi, \eta, \zeta, f, g, h$ are indeterminate, we have

$$\Sigma.\mathrm{P}\cos\alpha = 0, \qquad \Sigma.\mathrm{P}\cos\beta = 0, \qquad \Sigma.\mathrm{P}\cos\gamma = 0,$$

$$\left.\begin{aligned} \Sigma.\mathrm{P}\,(y\cos\gamma - z\cos\beta) = 0 \\ \Sigma.\mathrm{P}\,(z\cos\alpha - x\cos\gamma) = 0 \\ \Sigma.\mathrm{P}\,(x\cos\beta - y\cos\alpha) = 0 \end{aligned}\right\}, \qquad (381)$$

which are the well known conditions.

105.] Connected with, and easily deducible from, the above is a theorem of Gauss, published for the first time, so far as I know, in the IVth Volume of Crelle's Journal; and inserted in Möbius' Statik, and subsequently in a note appended to the edition of Lagrange's Mécanique Analytique, 3rd edition, by M. Bertrand, Vol. II, Paris. 1855.

For a system of forces in equilibrium we have

$$\Sigma . \mathrm{P} \{\cos \alpha \, dx + \cos \beta \, dy + \cos \gamma \, dz\} = 0. \tag{382}$$

Let the forces be replaced by line-representatives, and let $(x, y, z)$ be the point of application of the type-force P, and $(\xi, \eta, \zeta)$ the other extremity of the representative, then replacing $\mathrm{P} \cos \alpha$, $\mathrm{P} \cos \beta$, $\mathrm{P} \cos \gamma$ respectively by $\xi - x$, $\eta - y$, $\zeta - z$, (382) becomes

$$\Sigma . \{(\xi - x)\, dx + (\eta - y)\, dy + (\zeta - z)\, dz\} = 0 = -\frac{\mathrm{D}\Omega}{2} \text{ (say)}; \tag{383}$$

and if the displacement of the system is such that the extremity $(\xi, \eta, \zeta)$ of the line-representative of the type-force is fixed, while the other extremity $(x, y, z)$ receives an infinitesimal displacement, then integrating (383) we have

$$\Sigma . \{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\} = \Omega, \tag{384}$$

and thus $\Omega$, which is the sum of the squares of the line-representatives of the forces, is a maximum, a minimum, or a constant.

Hence we have the following theorem:

If there are $n$ points, at invariable distances apart, the system of which is however moveable, and also if there is a system of $n$ points wholly fixed, each of which corresponds to a point of the former system, then if the sum of the squares of the distances between each of the moveable points and its corresponding fixed point has a critical value, the system of forces represented as to intensity and line of action by these distances, and acting severally at the moveable points, is in equilibrium; and the equilibrium is stable or unstable, according as the sum of the squares of the distances is a minimum or a maximum, and is neutral if it is constant.

Also differentiating again (383) we have

$$\mathrm{D}^2\Omega = 2\Sigma . (dx^2 + dy^2 + dz^2) \\ - 2\Sigma . \{(\xi - x)\, d^2x + (\eta - y)\, d^2y + (\zeta - z)\, d^2z\}; \tag{385}$$

and if the motion, to which the variations of the coordinates of the points of application of the forces are due, is such that $d^2x = d^2y = d^2z = 0$, then $\mathrm{D}^2\Omega$ is necessarily positive, and $\Omega$ is a minimum; also if $\Sigma . \{(\xi - x)\, d^2x + (\eta - y)\, d^2y + (\zeta - z)\, d^2z\}$ is negative, that is, when equilibrium is stable, as appears by the differentiation of (373), $\Omega$ is a minimum.

The line-representatives of the forces however can always be taken so small that $\xi-x$, $\eta-y$, $\zeta-z$ shall be infinitesimal; whereby the second part of (385) being infinitesimal, and of the third order, must be neglected; and as the first part is positive, $\Omega$ is a minimum; that is, the sum of the squares of the line-representatives is a minimum.

To this subject however we shall return hereafter, and in a more general way. And in respect of the preceding it is also to be observed that, in the displaced position of the body on which the forces act, $\frac{\xi-x}{\mathrm{P}\cos\alpha} = \ldots = \ldots = \frac{r}{\mathrm{P}}$ are supposed to act along lines parallel to their original lines of action; whereas, in the most general case, the new lines of action would be functions of the original points of application.

# CHAPTER IV.

## ON GRAVITY AND CENTRES OF GRAVITY.

### SECTION 1.—*Preliminary considerations on Gravity.*

106.] All terrestrial, as well as most other matter, the properties of which we are acquainted with, is attracted towards the earth by a force called *gravity*; the cause and the mode of operation of which we are ignorant of. This property of matter is called *heaviness* (*pesanteur* = fallingness). Gravity is a force which penetrates matter; and acts with equal effect on a particle of matter, whether it is within a body, or on its surface, or whether it is separate: amount of heaviness is called *weight*: and the weight of many material particles is the same, whether they are separated, or whether they are collected into one body.

The intensity of gravity on the same matter is not the same at all places of the earth's surface: it increases as we go from the equator, where it has its least value, towards the pole, where it has its greatest value: this is due to two causes: (1) the statical attraction of the earth, and (2) the dynamical action of centrifugal force: to the consideration of both causes we shall return hereafter. And it also changes, as we pass further from the centre of the earth: it decreases in the ratio of the inverse square of the distance from the centre of the earth; as, for instance, at a distance of two radii of the earth, the intensity of gravity is one-fourth of its value at the surface of the earth; and as we pass from the surface of the earth towards the centre, as e. g. down a mine, its intensity also decreases, and varies directly as the distance from the centre of the earth. A proof of these propositions will be given hereafter. Considerable change of place however is required before the effects become sensible; and as the bodies, which will be the subjects of investigation in the present chapter are of dimensions very small in comparison of distances required to indicate variations of gravity, it is unnecessary for us to enter further into the inquiry at present. Gravity also varies according to the nature of the materials of the earth in the neighbourhood of the place where it is considered: its value on an island is different to that on a continent: it is also affected by neighbouring mountains.

The line of action of it is *vertical*, that is, is perpendicular to the surface of still water. Now although the earth is nearly spherical, so that all verticals meet, with a small error, at the centre; yet its radius, about 4000 miles, is so large, compared with the dimensions of any bodies which we shall at present consider to be subject to gravity, that all vertical lines may, without sensible error, be reckoned parallel; and therefore all the particles of material bodies may be considered to be acted on by forces whose lines of action are parallel; and thus the forces are such as to be subject to the results of Article 59.

107.] Now were all bodies *homogeneous*, that is, did all bodies throughout their extent contain equal quantities of matter in equal spaces, their weights would doubtless be proportional to their volumes: but all bodies are not homogeneous; and therefore we are obliged to have recourse to another mode of estimating their weight. *Weight*, be it remembered, is the amount of the earth's attraction on a body, and depends on, not the space occupied by the body, but the quantity of matter which it contains, or its mass: this fact is the result of experiment: and although hereafter we shall be obliged to introduce another method of measuring mass, yet at present we measure mass by weight, and we say that the mass of one body is twice that of another, if the weight of the former is twice the weight of the latter; and the balance affords a convenient mode of measuring weight. Suppose then a heterogeneous body to be the subject of consideration, equal volumes at different points of which contain unequal quantities of matter; and let the quantity of matter contained in an unit-volume of the substance be called the *density* of the substance, the matter being supposed to be homogeneous throughout that unit-volume; and let density be measured by means of comparison with some substance which is assumed to be of unit-density; and let $\rho$ be the ratio of the density of the given substance to that of unit-density; that is, the quantity of matter contained in an unit-volume of the substance is $\rho$ times that in an unit-volume of the substance of the unit-density: so that $\rho$ is a number. And let us suppose the density of the substance to vary according to a *continuous* law. Then, if $dv$ is an infinitesimal volume-element at a given point of a body, and $\rho$ is the density-symbol at that point; bearing in mind that $d\text{v}$ is the symbol expressing the number of times which the unit-volume is taken, and that the quantity of matter contained in an unit-volume of the substance, whatever that substance is, is $\rho$ times the quantity of matter contained in an unit-volume of the substance of the unit-density, we have

$$\text{Quantity of matter in } d\text{v} = \rho\, d\text{v}; \tag{1}$$

and therefore

$$\text{Mass-element of body} = \rho\, d\text{v}. \tag{2}$$

Now mass does not depend on the place of the body with respect to the earth: it is the same, as far as we know, for the same body everywhere, whether it is at the moon or at the most distant planets. Let us however suppose the body to be at a certain place on the earth, and let the earth's attraction at that place on an unit-mass be $g$; or, in other words, let $g$ be the weight of the unit-mass at a given place; then since the volume-element of the body contains $\rho\, d\text{v}$ units of mass, and the weight of each of these units is $g$; therefore

$$\text{The weight of the element of the body contained in } d\text{v} = \rho g\, d\text{v}, \tag{3}$$

and therefore

$$\text{Weight-element of body} = \rho g\, d\text{v} \tag{4}$$

$$= \text{mass-element} \times g; \tag{5}$$

$$\therefore \quad \text{Weight of body} = \int \rho g\, d\text{v}. \tag{6}$$

It appears from the preceding that the mass of an unit-volume, or the density of a given substance, is expressed by the symbol $\rho$, meaning that the density of the substance is $\rho$ times some assumed unit-density; $\rho$ is commonly called *the specific density* of the substance; and the unit-density is assumed to be that of distilled water at a certain temperature, and under a certain atmospheric pressure. From the assumptions which we have made it also follows that the weight of an unit-volume is $\rho g$; this weight is commonly called "specific gravity;" and sometimes and more correctly "specific weight." Ordinarily of homogeneous bodies the unit-volume is a cubic inch; and the weight of a cubic inch of any given substance is compared with that of a cubic inch of distilled water, at a certain temperature and under certain conditions. I must however refer the reader desirous of further information on these subjects to physical treatises wherein they are fully discussed; though I may remark that it is by these means that we avoid the difficulty of determining $\rho$ for every different substance.

In (6) $g$ is plainly constant on account of the volume of the body being small in comparison of that of the earth: and if $\rho$ is also constant, then

$$\text{Weight of the body} = \rho g \int d\text{v} = \rho g \text{v}, \qquad (7)$$

that is, the weight of a body is equal to the product of its volume and its specific gravity.

Suppose however $\rho$ to be variable; and let us calculate the weight of some bodies for which this is the case.

Ex. 1. To find the weight of a circular plate of uniform thickness, the density of which varies as the distance from the centre.

Let $\tau$ be the thickness of the plate and $a$ its radius: let the centre of the plate be the origin, and let it be referred to polar coordinates; so that $d\text{v} = \tau r\, dr\, d\theta$: let $\rho = kr$; then

$$\text{Weight of the plate} = \int_0^{2\pi}\int_0^a gk\tau r^2\, dr\, d\theta$$

$$= \frac{2\pi g k \tau a^3}{3}.$$

Suppose the density to be constant, and the thickness to vary directly as the distance from the centre; then $\tau = kr$, and we have

$$\text{Weight of the plate} = \int_0^{2\pi}\int_0^a g\rho k r^2\, dr\, d\theta$$

$$= \frac{2\pi g \rho k a^3}{3}.$$

Ex. 2. To find the weight of a straight wire or rod, the density of which varies directly as the distance from one end.

Let the end of the rod be taken as the origin, and let $a$ be the length of it; and let the distance of any point of it from that end $= x$; let $\omega =$ the area of a transverse section of it; then $d\text{v} = \omega\, dx$; and $\rho = kx$; therefore

$$\text{Weight of the rod} = \int_0^a gk\omega x\, dx$$

$$= \frac{gk\omega a^2}{2}.$$

Ex. 3. To determine the bounding curve of a thin ribbon of uniform thickness and density, such that the breadth of it corresponding to each ordinate may be proportional to the weight of the ribbon beyond it.

Let the curve be that delineated in fig. 63. Let the axis of $x$ be vertical, and that of $y$ horizontal. OM $= x$, MP $= y$, OA $= a$. Let $\tau$ be the constant thickness of the ribbon, $\rho$ its density; then taking the part of the ribbon on the positive side of the axis of $x$, the weight of it below MP

$$= \int_{y=0}^{y=y} \tau\rho g y\, dx;$$

therefore by the data

$$\int_{y=0}^{y=y} \tau\rho g y\, dx = \mu y,$$

$$\therefore \quad y\,dx = k\,dy,$$

$$dx = k\frac{dy}{y},$$

$$y = a^{\frac{x}{k}};$$

the equation to the logarithmic curve. Similarly, if OA$' = a'$, for the curve on the other side we shall have

$$y' = a'^{\frac{x}{k}}.$$

If therefore the ribbon with its edges cut into two logarithmic curves according to the figure is suspended from a horizontal bar A$'$OA, the tension on every horizontal section will be proportional to the length of that section.

108.] From equation (4) then it appears that each mass-element of the body corresponding to the volume-element $d$V is acted on by a force which is equal to $g\rho\, d$V; and the lines of action of all these are parallel and are vertical; their resultant therefore is equal to their sum, as appears by equation (6), and may be inferred from Art. 59; and has a line of action parallel to those of the several components; the resultant is called the weight of the body: it has also a definite point of application; and by virtue of Art. 59, the position of this point is independent of the direction of the line of action of the forces, and therefore is the same, whatever is the position of the body. It is called the *centre of gravity;* and of course, if it is fixed, the body will rest in all positions about it. The position of it is determined in the following manner:

First let us take the case of many material particles separate from each other; let their masses be $m_1, m_2, \dots m_n$; and the coordinates of their positions $x_1 y_1 z_1$, $x_2 y_2 z_2, \dots x_n y_n z_n$; then their weights are $m_1 g$, $m_2 g, \dots m_n g$: let the coordinates to their centre of gravity be $\bar{x}, \bar{y}, \bar{z}$; then referring to Art. 59, equations (116), $\text{P}_1 = m_1 g$, $\text{P}_2 = m_2 g, \ldots \text{P}_n = m_n g$; therefore

$$\text{R} = \Sigma.mg = g\Sigma.m, \tag{8}$$

$$\left.\begin{array}{ll} \bar{x}\Sigma.mg = \Sigma.mgx, & \therefore\ \bar{x}\Sigma.m = \Sigma.mx \\ \bar{y}\Sigma.mg = \Sigma.mgy, & \phantom{\therefore\ }\bar{y}\Sigma.m = \Sigma.my \\ \bar{z}\Sigma.mg = \Sigma.mgz, & \phantom{\therefore\ }\bar{z}\Sigma.m = \Sigma.mz \end{array}\right\}; \tag{9}$$

whereby both the resultant and the position of its point of application are known. And from the form of these equations it follows that, in the investigation of the centre of gravity of a system of material particles or bodies, we may, if it is convenient, divide the system into groups, and calculate separately the centre of gravity of each group; and by a similar process deduce from them the centre of gravity of the whole system.

Secondly let us take the case of many material particles aggregated into a continuous body, so that the symbol of summation becomes that of integration; and let the coordinates to the type volume-element of the body be $x, y, z$: then the type-force is $\rho g\, d\text{v}$; let $\bar{x}, \bar{y}, \bar{z}$ be the coordinates to the centre of gravity; then by (116) Article 59,

$$\left.\begin{array}{l} \bar{x}\int \rho g\, d\text{v} = \int \rho g x\, d\text{v} \\ \bar{y}\int \rho g\, d\text{v} = \int \rho g y\, d\text{v} \\ \bar{z}\int \rho g\, d\text{v} = \int \rho g z\, d\text{v} \end{array}\right\}; \tag{10}$$

$\int$ is used on both sides of the equations as a general symbol of summation; and is to be replaced by the symbols of single, double, or triple integration according to the different values of $d\text{v}$; and the integrals are definite; the limits being assigned by the particular circumstances of the problem. $g$ will of course be divided out on both sides of the equations; and so will $\rho$, if the density is constant: but I have left these quantities that the student may see the true meaning of the equations.

If the body whose centre of gravity is to be found is a plane plate or wire and of thickness infinitesimal in comparison of the surface or of the length, we may take its plane to be that of $xy$, so that $z$ will vanish for all the particles of the body, and we shall have from (9) and (10) respectively,

$$\left.\begin{array}{l} \bar{x}\Sigma.m = \Sigma.mx \\ \bar{y}\Sigma.m = \Sigma.my \end{array}\right\}, \tag{11}$$

$$\left.\begin{aligned}\bar{x}\int\rho g\,d\text{v} &= \int\rho g x\,d\text{v}\\ \bar{y}\int\rho g\,d\text{v} &= \int\rho g y\,d\text{v}\end{aligned}\right\}. \qquad (12)$$

109.] The general values (9) and (10) require closer consideration. The centre of gravity is the point of application of the resultant of all the weights of the several component particles of a body, which resultant is equal to the sum of the separate weights; it is therefore that point at which, if the weight of the whole body acts, an effect is produced the same as that of all the particles of the body taken in combination; or, in other and equivalent words, *the centre of gravity is that point at which, if the body is collected into a material particle, the circumstances of pressure are the same as those of the body occupying the space which it does.* This is immediately apparent from (12): for if we imagine the whole mass of the body to be condensed into a particle at the centre of gravity, then $\int\rho g\,d\text{v}$ is the weight of the mass therein condensed, and the left-hand members of the equations are the moments of the mass thus condensed about the axes of $y$ and $x$ respectively: and the right-hand members are the sums of the moments of the weights of the several component elements of the body with reference to the same axes. And this is, be it observed, the mechanical interpretation of the equations.

Also with reference to the equations (10) it is to be observed that the product of a body's weight, and the distance of its centre of gravity from any plane, is equal to the sum of the products of the weight of every element of it, and the distance of that element from the same plane.

Another point also requires some remarks. In Article 107 different concrete units are involved: now the symbols $\rho$, $d\text{v}$, $g$ are symbols of numbers; and therefore their product is a number; but the quantity which we commence with is volume-element, and that which we end with is weight-element: it remains therefore to seek the source whence this change arises; it is true, as it is convenient, that $d\text{v}$ expresses the *number* of the volume-units, $\rho$ the *number* of mass-units in a volume-unit, and $g$ the *number* of earth's attraction-units in a mass-unit: but how does the result of all this imply weight? In the first place, the process "multiplication" must be used in a sense wider than its numerical one, so as to include within its subjects of operation quantities of different kinds; and so that the product may be of a kind different to that of either of the multiplicands: and thus the product of two concrete units is a concrete unit of a different kind; the product of the volume-unit and of the density-unit is mass-unit; and the product of the mass-unit and of the earth's attraction-unit is weight-unit; the change of concrete unit therefore arises from the product of the different concrete units; and weight-unit is the product of three different concrete units. The units are of course arbitrary, and therefore we choose those which are most convenient; and thus we take a cubic inch to be the volume-unit; the density of distilled water, at a certain temperature and under certain atmospheric pressure, to be the density unit; and the earth's attraction at Greenwich on a mass-unit to be the gravity-unit; and by means of these we obtain the weight of a cubic inch of distilled water at a certain place, and compare all other weights with it.

Or we may arrive at the same result by the laws of variation; the quantity of matter in a given volume varies as the volume, and as the density or quantity of matter contained in a volume-unit; so that if $k$ is the coefficient of variation,

$$\text{Mass} = k\text{v} \times \rho; \tag{13}$$

but as $k$ is arbitrary, let us assume the unit of mass to be that which corresponds to the volume-unit and the density-unit; therefore $k = 1$, and

$$\text{Mass} = \text{v} \times \rho. \tag{14}$$

Again, the weight of a given body varies as the mass of the body, and as the earth's attraction; therefore if $m$ represents the mass of a body, and $g$ the earth's attraction on an unit of mass at a given place, and if $k$ is the coefficient of variation,

$$\text{Weight} = kmg; \tag{15}$$

and if we assume the unit of weight to be that which corresponds to an unit of mass, and to an unit of earth's attraction at a given place, $k = 1$; therefore

$$\text{Weight} = mg = \text{v}\rho g. \tag{16}$$

There are of course many cases where the centre of gravity is known at once, by reason of the symmetry of the figure; thus the centre of gravity of a straight wire or rod, of the same density and thickness throughout, is at the middle point of the rod; the centre of gravity of a circular wire of the same density and thickness throughout is at the centre of the circle: that of a circular, or of an elliptical plate of constant thickness and density is at the centre: that of a homogeneous sphere and of a homogeneous ellipsoid is at the centre: and in a similar manner we shall frequently conclude that the centre of gravity of a body is in a particular line which can be at once determined by means of the symmetry of the figure. We proceed now to the determination of the centres of gravity of various figures, lines, plates, and bodies.

### SECTION 2.—*The centre of gravity of material lines, or wires, straight and curved.*

110.] Let us first consider the centre of gravity of a curved material line or wire, of which the thickness is infinitesimal in comparison of the length, so that all the particles of it may be considered to be in one plane; let it be referred to rectangular coordinates, and let the plane of the wire be that of $xy$; then the equations (12) are applicable.

Let $\omega =$ the area of a transverse section of the wire, and $ds =$ a length-element, so that $d\text{v} = \omega\, ds$; let $\rho$ be the density at the point $(x, y)$, and $g =$ the earth's attraction; and let $\bar{x}$, $\bar{y}$ be the coordinates of the centre of gravity; then

$$\left.\begin{aligned} \bar{x}\int \rho g \omega\, ds &= \int \rho g \omega x\, ds \\ \bar{y}\int \rho g \omega\, ds &= \int \rho g \omega y\, ds \end{aligned}\right\}. \tag{17}$$

The integrals are of course definite, and the limits are fixed by the conditions of the problem. If the equation to the curved wire is $y = f(x)$, then

$$ds^2 = dx^2 + dy^2,$$

$$ds = \{1 + (f'(x))^2\}^{\frac{1}{2}}\, dx. \tag{18}$$

Also if the curve of the wire is referred to polar coordinates, and if the equation to it is $r = f(\theta)$, then $x = r\cos\theta$, $y = r\sin\theta$, and

$$ds = \{r^2 + (f'(\theta))^2\}^{\frac{1}{2}}\, d\theta, \tag{19}$$

and therefore

$$\left.\begin{aligned} \bar{x}\int \rho g \omega \{r^2 + (f'(\theta))^2\}^{\frac{1}{2}}\, d\theta &= \int \rho g \omega r \cos\theta \{r^2 + (f'(\theta))^2\}^{\frac{1}{2}}\, d\theta \\ \bar{y}\int \rho g \omega \{r^2 + (f'(\theta))^2\}^{\frac{1}{2}}\, d\theta &= \int \rho g \omega r \sin\theta \{r^2 + (f'(\theta))^2\}^{\frac{1}{2}}\, d\theta \end{aligned}\right\}. \tag{20}$$

It will be found that in many cases the centre of gravity of a material line is outside of the line; and therefore it is necessary that it should be rigidly connected with it if the wire or rod is to have physical support.

111.] Ex. 1. To find the centre of gravity of a wire of uniform thickness and density bent into the form of a quadrant of a circle.

Let the radius of the circle be $a$; fig. 43; then as $\rho\ \omega$ and $g$ are constant, they may be divided out, and (17) become

$$\bar{x}\int ds = \int x\,ds, \qquad \bar{y}\int ds = \int y\,ds; \tag{21}$$

$$\text{also} \qquad x^2+y^2 = a^2,$$

$$\therefore \quad \frac{dx}{y} = \frac{dy}{-x} = \frac{ds}{a},$$

$$\therefore \quad \bar{x}\int_0^a \frac{a\,dx}{(a^2-x^2)^{\frac{1}{2}}} = \int_0^a \frac{ax\,dx}{(a^2-x^2)^{\frac{1}{2}}},$$

$$\bar{x}\left[\sin^{-1}\frac{x}{a}\right]_0^a = \left[-(a^2-x^2)^{\frac{1}{2}}\right]_0^a,$$

$$\bar{x}\frac{\pi}{2} = a, \qquad \bar{x} = \frac{2a}{\pi};$$

$$\bar{y}\int_0^a \frac{a\,dx}{(a^2-x^2)^{\frac{1}{2}}} = \int_0^a a\,dx, \qquad \bar{y}\left[\sin^{-1}\frac{x}{a}\right]_0^a = \left[x\right]_0^a,$$

$$\therefore \quad \bar{y} = \frac{2a}{\pi}.$$

Or thus by means of polar coordinates; $r = a$,

$$\therefore \quad \bar{x}\int_0^{\frac{\pi}{2}} d\theta = \int_0^{\frac{\pi}{2}} a\cos\theta\,d\theta, \qquad \therefore \quad \bar{x} = \frac{2a}{\pi},$$

$$\bar{y}\int_0^{\frac{\pi}{2}} d\theta = \int_0^{\frac{\pi}{2}} a\sin\theta\,d\theta, \qquad \therefore \quad \bar{y} = \frac{2a}{\pi}.$$

Ex. 2. To find the centre of gravity of a wire of constant thickness and density, and bent into the form of a complete cycloid.

Let the starting point of the cycloid be the origin, and let the equation to the curve be

$$x = a\,\text{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}},$$

$$\therefore \quad \frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a-y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}};$$

it is evident that the centre of gravity will be in the line perpendicular to the base at its point of bisection; therefore $\bar{x} = \pi a$; and as $\rho$, $g$, $\omega$ are constant,

$$\bar{y}\int_0^{2a} \frac{dy}{(2a-y)^{\frac{1}{2}}} = \int_0^{2a} \frac{y\,dy}{(2a-y)^{\frac{1}{2}}}; \qquad \therefore \quad \bar{y} = \frac{4a}{3}.$$

Ex. 3. To find the centre of gravity of a wire of constant thickness and density, bent into the form of an arc of a circle.

Let the radius of the circle be $a$; and let the line passing through the middle point (the vertex) of the circular arc and the centre of the circle be the axis of $x$; then as the arc, fig. 44, is symmetrical with respect to this line, $\bar{y} = 0$. Let the arc BOB$' = 2s$, and let the chord BB$' = 2c$, OD $= b$; then

$$y^2 = 2ax - x^2,$$

$$\frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a},$$

$$\text{and} \qquad \bar{x}\int_0^{b} \frac{dx}{(2ax-x^2)^{\frac{1}{2}}} = \int_0^{b} \frac{x\,dx}{(2ax-x^2)^{\frac{1}{2}}},$$

$$\therefore \quad \bar{x} = a - \frac{ac}{s}.$$

Ex. 4. To find the centre of gravity of a straight rod, the thickness of which varies directly as the distance from one end.

Let the end of the rod whence the variation of the thickness is reckoned be taken as the origin, and the line as the axis of $x$: then $\omega = kx$; let $a =$ the length of the rod; and we have

$$\bar{x}\int_0^{a} \rho g \kappa x\,dx = \int_0^{a} \rho g \kappa x^2\,dx, \qquad \bar{x} = \frac{2a}{3}.$$

Ex. 5. To find the centre of gravity of a straight rod, the density of which varies as the $n$th power of the distance of each point from a given point in the line of the rod produced.

Let O be the point from which the variation of the density takes place; fig. 45; OA $= a$, OB $= b$, OP $= x$, PQ $= dx$; $\rho = \kappa x^n$; then

$$\bar{x}\int_a^{b} \kappa\omega g x^n\,dx = \int_a^{b} \kappa\omega g x^{n+1}\,dx,$$

$$\therefore \quad \bar{x} = \frac{n+1}{n+2}\,\frac{b^{n+2}-a^{n+2}}{b^{n+1}-a^{n+1}}.$$

If $n = -2$, then

$$\bar{x}\int_a^{b} \kappa\omega g\,\frac{dx}{x^2} = \int_a^{b} \kappa\omega g\,\frac{dx}{x}, \qquad \bar{x} = \frac{ab}{b-a}\log\frac{b}{a}.$$

Ex. 6. To find the centre of gravity of a wire bent into the form of a cycloid, the thickness of which varies directly as the distance from the middle point of the wire.

The middle point of the wire is the highest point of the cycloid; let it be taken as the origin; and let the axis of the cycloid be the axis of $x$; then $\bar{y} = 0$; let the length of the wire be $8a$; then Vol. II, Art. 103, Ex. 3, the radius of the base-circle is $a$; and the equation to the cycloid is

$$y = a \operatorname{versin}^{-1} \frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

$$\frac{dy}{(2a-x)^{\frac{1}{2}}} = \frac{dx}{x^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}},$$

$$\therefore \quad s = 2(2ax)^{\frac{1}{2}}, \qquad x = \frac{s^2}{8a};$$

therefore $\rho = \kappa s$; and we have

$$\bar{x} \int g \omega \kappa s \, ds = \int g \omega \kappa s x \, ds, \qquad \bar{x} \int_0^{4a} s \, ds = \frac{1}{8a} \int_0^{4a} s^3 \, ds,$$

$$\bar{x} = a.$$

112.] If the curve, into which the wire is bent, is in space, whether plane or of double curvature, then

$$\left.\begin{aligned} \bar{x} \int \rho g \omega \, ds &= \int \rho g \omega x \, ds \\ \bar{y} \int \rho g \omega \, ds &= \int \rho g \omega y \, ds \\ \bar{z} \int \rho g \omega \, ds &= \int \rho g \omega z \, ds \end{aligned}\right\}. \tag{22}$$

Ex. 1. A wire of constant thickness and density is bent into the form of a helix; find its centre of gravity.

Let $a =$ the radius of the base-cylinder; and let the wire commence at the axis of $x$, that is, at the point $(a, 0, 0)$, see fig. 125, Vol. I; and let it end at $(x, y, z)$; then

$$x = a \cos \phi, \qquad y = a \sin \phi, \qquad z = ka\phi,$$

$$ds = (1 + k^2)^{\frac{1}{2}} a \, d\phi,$$

$$\bar{x} \int_0^{\phi} \rho g \omega (1 + k^2)^{\frac{1}{2}} a \, d\phi = \int_0^{\phi} \rho g \omega (1 + k^2)^{\frac{1}{2}} a^2 \cos \phi \, d\phi,$$

$$\bar{x}\phi = a \sin \phi, \qquad \therefore \quad \bar{x} = ka \frac{y}{z},$$

$$\bar{y}\phi = a(1 - \cos \phi), \qquad \bar{y} = ka \frac{a - x}{z},$$

$$\bar{z}\phi = \frac{ka\phi^2}{2}, \qquad \bar{z} = \frac{z}{2}.$$

Ex. 2. To find the centre of gravity of the perimeter of a triangle in space, the three sides of which are thin rods of constant thickness and density.

Let the lengths of the sides be $l_1, l_2, l_3$; and the angular points be $(x_1, y_1, z_1) \dots (x_3, y_3, z_3)$: $\rho =$ the constant density, $\omega =$ the area of a transverse section of the rod: then the centres of gravity of $l_1, l_2, l_3$ are manifestly at the points

$$\frac{x_2+x_3}{2}, \quad \frac{y_2+y_3}{2}, \quad \frac{z_2+z_3}{2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{x_1+x_2}{2}, \quad \frac{y_1+y_2}{2}, \quad \frac{z_1+z_2}{2},$$

and therefore by the formulæ (9)

$$\bar{x}(l_1+l_2+l_3) = \frac{1}{2}\{l_1(x_2+x_3)+l_2(x_3+x_1)+l_3(x_1+x_2)\},$$

$$\bar{y}(l_1+l_2+l_3) = \frac{1}{2}\{l_1(y_2+y_3)+l_2(y_3+y_1)+l_3(y_1+y_2)\},$$

$$\bar{z}(l_1+l_2+l_3) = \frac{1}{2}\{l_1(z_2+z_3)+l_2(z_3+z_1)+l_3(z_1+z_2)\}.$$

By a similar process the centre of gravity of the perimeter of a polygon formed by heavy rods in space may be determined.

113.] The determination of the centres of gravity of material lines or wires also suggests the following problem, which is solved by the calculus of variations:

To find the equation to the curve into which a thin heavy rod of uniform thickness and density and of given length is to be bent, so that its ends being fixed at two given points, the centre of gravity may be in the lowest possible position.

Let the axis of $z$ be parallel to the direction of gravity; and let $2c$ be the length of the rod; and $(x_1, y_1, z_1)$ and $(x_0, y_0, z_0)$ the ends of the line; then

$$\int_0^1 ds = 2c, \tag{23}$$

$$\bar{z}\,2c = \int_0^1 z\,ds, \tag{24}$$

and $\bar{z}$ will be a maximum or a minimum according as the plane of $xy$ is above or below the centre of gravity of the suspended wire; in either case, $\delta\bar{z} = 0$; therefore from (24)

$$2c\delta\bar{z} = 0 = \int_0^1 \delta . z\,ds$$

$$2c\delta\bar{z} = 0 = \int_0^1 (z\,\delta ds + ds\,\delta z)$$

$$= \int_0^1 \left\{ z\left(\frac{dx}{ds}\,d.\delta x + \frac{dy}{ds}\,d.\delta y + \frac{dz}{ds}\,d.\delta z\right) + ds\,dz \right\}$$

$$= \left[ z\left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right)\right]_0^1$$

$$-\int_0^1 \left\{ d.z\frac{dx}{ds}\delta x + d.z\frac{dy}{ds}\delta y + \left(d.z\frac{dz}{ds} - ds\right)\delta z \right\}; \quad (25)$$

Of this quantity the first part vanishes by reason of the limits being fixed; also from (23),

$$\delta.2c = 0 = \delta\int_0^1 ds$$

$$= \int_0^1 \left\{ \frac{dx}{ds}\,d.\delta x + \frac{dy}{ds}\,d.\delta y + \frac{dz}{ds}\,d.\delta z \right\}$$

$$= \left[\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right]_0^1$$

$$-\int_0^1 \left\{ d\frac{dx}{ds}\delta x + d.\frac{dy}{ds}\delta y + d.\frac{dz}{ds}\delta z \right\}; \quad (26)$$

and of this quantity the first part vanishes by reason of the limits being fixed; and as the second part is to consist with the second part of (25), we have

$$\frac{d.z\dfrac{dx}{ds}}{d.\dfrac{dx}{ds}} = \frac{d.z\dfrac{dy}{ds}}{d.\dfrac{dy}{ds}} = \frac{d.z\dfrac{dz}{ds} - ds}{d.\dfrac{dz}{ds}} = \lambda \text{ (say)}; \qquad (27)$$

from the first two members of which equality we have

$$\frac{d.\dfrac{dx}{ds}}{\dfrac{dx}{ds}} = \frac{d.\dfrac{dy}{ds}}{\dfrac{dy}{ds}}, \qquad \therefore \quad \frac{dx}{x_1 - x_0} = \frac{dy}{y_1 - y_0};$$

the constants being introduced consistently with the curve passing through $(x_1, y_1, z_1)$ and $(x_0, y_0, z_0)$;

$$\therefore \quad \frac{x - x_0}{x_1 - x_0} = \frac{y - y_0}{y_1 - y_0},$$

whence it follows that the curve is a plane curve, and is in a plane perpendicular to that of $xy$. Let the plane of $xz$ be taken so as to contain the curve; then $y = 0$; and from the first of (27) we have

$$d.z\frac{dx}{ds} = \lambda d.\frac{dx}{ds},$$

$$\therefore \quad z\,d.\frac{dx}{ds}+dz\frac{dx}{ds}=\lambda d.\frac{dx}{ds}, \qquad \frac{dz}{z-\lambda}+\frac{d.\dfrac{dx}{ds}}{\dfrac{dx}{ds}}=0,$$

$$\therefore \quad \log(z-\lambda)+\log\frac{dx}{ds}=\log a, \qquad \text{or,} \quad (z-\lambda)\frac{dx}{ds}=a,$$

where $a$ is an arbitrary constant of integration; and since $ds^2=dx^2+dz^2$,

$$\frac{dx}{a}=\frac{dz}{\{(z-\lambda)^2-a^2\}^{\frac{1}{2}}},$$

$$\therefore \quad z-\lambda=\frac{b}{2}\left\{e^{\frac{x-b}{a}}+e^{-\frac{x-b}{a}}\right\}; \tag{28}$$

where $b$ is another arbitrary constant of integration; and the three undetermined constants $a, b, \lambda$ may be determined by the conditions of the curve passing through two given points, and of the length of the curve between those points being given. The equation (28) is that of the catenary, the properties of which will be investigated hereafter; and the result is important, inasmuch as it shews that the curve in which a perfectly flexible and inextensible string will hang when suspended from two fixed points is also that of which the centre of gravity has the lowest possible position.

114.] And there is another property of the centre of gravity of a curved material line, which it is necessary to explain. Suppose the line to be of constant thickness and density, and to be infinitesimally thin; then

$$\bar{y}\int ds=\int y\,ds,$$

$$\therefore \quad 2\pi\bar{y}\times s=\int 2\pi y\,ds. \tag{29}$$

Now if the plane curve whose length is $s$ revolves about the axis of $x$, and generate thereby a thin shell (or surface) of revolution, the right-hand member of (29) is the area of the surface generated; see Art. 129, Vol. II; and the left-hand member of (29) is the product of the length of the generating line and of the path described during an entire revolution by the centre of gravity of it; hence we conclude that,

If a plane curve lies wholly on one side of a line in its own plane, and revolving about that line generates thereby a surface of revolution, the area of the surface is equal to the (geometrical) product of the length of the revolving line, and of the path described by its centre of gravity.

This theorem is one of those known by the name of the Theorems of Pappus or of Guldinus; it is a geometrical relation existing between a curve, the surface which it generates by revolving about a line in its own plane, and the distance of its centre of gravity from that line; the curve must not intersect the axis of $x$; for if it does, $y$ will change its sign; and (29) may be an inexact expression; the generating curve may also be a closed figure. Also as (29) expresses the equality of the two sides of the equation for a whole revolution, so will a similar theorem be true for any part of a revolution. Two or three examples are subjoined.

Ex. 1. A circle of radius $a$ revolves about an axis in its own plane at a distance $c$ from its centre; it is required to find the area of the surface of the ring thereby generated.

The circumference of the generating curve is $2\pi a$; and as the centre of gravity of it is at its centre, the path described by the centre of gravity during a complete revolution is $2\pi c$;

$$\therefore \quad \text{the area of the surface of the ring} = 4\pi^2 ac.$$

Ex. 2. A right-angled triangle revolves about its hypothenuse, and its sides thereby describe a surface; it is required to find the area of the surface described.

Let $a$, $b$ be the sides of the triangle, and $h$ the length of the perpendicular from the right-angle to the hypothenuse, so that

$$\frac{1}{h^2} = \frac{1}{a^2} + \frac{1}{b^2};$$

then the area of the surface $= \pi(a+b)h$

$$= \frac{\pi(a+b)ab}{(a^2+b^2)^{\frac{1}{2}}},$$

Also if the area of a surface is known, and the length of the generating line is known, the distance of the centre of gravity of the line from the axis of revolution may be determined. Thus, the surface of a sphere of radius $a = 4\pi a^2$, the length of a semicircle $= \pi a$; therefore from (29),

$$2\pi\bar{y} \times \pi a = 4\pi a^2,$$

$$\therefore \quad \bar{y} = \frac{2a}{\pi}.$$

## SECTION 3.—*Centre of gravity of thin plates and curved shells, bounded by lines straight or curved.*

115.] In the next place let us consider a plane plate of infinitesimal thickness, bounded by curved or straight lines, and refer it to rectangular coordinates. Let the plane of the plate be that of $xy$, and let the coordinates of any element in the plane surface of the plate be $x, y$; so that the area of an element E is $dx\,dy$; see fig. 46. Let the thickness of the plate at E $= \tau$; then

$$dV = \tau\, dx\, dy. \tag{30}$$

Let $\rho =$ density, $g =$ earth's attraction; equations (12) become

$$\left.\begin{aligned}\bar{x}\iint \rho g \tau\, dy\, dx &= \iint \rho g \tau x\, dy\, dx\\ \bar{y}\iint \rho g \tau\, dy\, dx &= \iint \rho g \tau y\, dy\, dx\end{aligned}\right\}, \tag{31}$$

the limits being assigned by the geometrical conditions of the problem; and the conditions of the definite integrals being the same as those explained in Vol. II. Chap. VII.

Ex. 1. It is required to find the centre of gravity of a thin plate of uniform thickness and density, bounded by a parabola, its axis, and an ordinate; fig. 46.

Let OA $= a$, AB $= b$; $\tau =$ thickness of plate, $\rho =$ density: then

$$y^2 = \frac{b^2}{a}x.$$

Let $\frac{b}{a^{\frac{1}{2}}}x^{\frac{1}{2}} = Y$, so that from (31), introducing the limits, we have

$$\bar{x}\int_0^a\int_0^Y dy\, dx = \int_0^a\int_0^Y x\, dy\, dx,$$

$$\bar{x}\int_0^a x^{\frac{1}{2}}\, dx = \int_0^a x^{\frac{3}{2}}\, dx,$$

$$x = \frac{3}{5}a;$$

$$\bar{y}\int_0^a\int_0^Y dy\, dx = \int_0^a\int_0^Y y\, dy\, dx,$$

$$\therefore\quad \bar{y} = \frac{3b}{8}.$$

Ex. 2. To find the centre of gravity of a thin plate of uniform thickness and density in the form of an elliptic quadrant.

The equation to the bounding curve is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let $$\text{Y}=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}},$$

then from (31) in this case

$$\bar{x}\int_0^a\int_0^{\text{Y}}dy\,dx=\int_0^a\int_0^{\text{Y}}x\,dy\,dx,$$

$$\bar{x}\int_0^a(a^2-x^2)^{\frac{1}{2}}dx=\int_0^a x(a^2-x^2)^{\frac{1}{2}}dx,$$

$$\therefore\quad \bar{x}=\frac{4a}{3\pi}.$$

$$\bar{y}\int_0^a\int_0^{\text{Y}}dy\,dx=\int_0^a\int_0^{\text{Y}}y\,dy\,dx,$$

$$\therefore\quad \bar{y}=\frac{4b}{3\pi}.$$

Ex. 3. To find the centre of gravity of a thin triangular plate of constant thickness and density.

Let $\tau$ be the thickness of the plate, and $\rho=$ the density. Take the angle O, fig. 47, for the origin, and the sides OA, OB for the coordinate axes; OA $=a$, OB $=b$, so that the equation to AB is

$$\frac{x}{a}+\frac{y}{b}=1.$$

Let the angle at O $=\omega$; then the area of the surface at E $=dx\,dy\sin\omega$; $d\text{V}=\tau\,dx\,dy\sin\omega$. Then if

$$\text{Y}=\frac{b}{a}(a-x),$$

the equations of moments about the axes are

$$\bar{x}\sin\omega\int_0^a\int_0^{\text{Y}}dy\,dx\sin\omega=\int_0^a\int_0^{\text{Y}}x(\sin\omega)^2dy\,dx,$$

$$\bar{y}\sin\omega\int_0^a\int_0^{\text{Y}}dy\,dx\sin\omega=\int_0^a\int_0^{\text{Y}}y(\sin\omega)^2dy\,dx,$$

$$\therefore\quad \bar{x}=\frac{a}{3},\qquad \bar{y}=\frac{b}{3};$$

the centre of gravity therefore is situated on the line passing through O and bisecting AB, at a distance from O equal to two-thirds of the bisecting line; and as the result is independent of the particular angle, it is equally true for all the angles; and therefore the centre of gravity of a triangular thin plate is at the point of intersection of the three bisectors of the sides drawn from the opposite angles. This is also manifest from the following reason: let OAB be a triangular plate, fig. 48; and let OC be drawn from O to C, the middle point of the opposite side AB; and imagine the plate to be divided into a series of thin slices by lines parallel to AB; then the centre of gravity of each of these slices will be at its middle point, that is, at its intersection with OC. Imagine therefore each slice to be condensed into its centre of gravity; there is then a series of particles of increasing weight arranged along the line OC, the law of increase being that of the distance directly, because PP′ varies as OM; if therefore OM $= x$, and OC $= h$, we have from (17)

$$\bar{x}\int_0^h x\,dx = \int_0^h x^2\,dx,$$

$$\therefore \quad \bar{x} = \frac{2}{3}h. \tag{32}$$

Hereby also we conclude that if the coordinates to the angles of a triangular plate in space are $x_1 y_1 z_1$, $x_2 y_2 z_2$, $x_3 y_3 z_3$, then

$$\left.\begin{aligned} \bar{x} &= \frac{x_1+x_2+x_3}{3} \\ \bar{y} &= \frac{y_1+y_2+y_3}{3} \\ \bar{z} &= \frac{z_1+z_2+z_3}{3} \end{aligned}\right\}. \tag{33}$$

Ex. 4. To find the centre of gravity of a cycloidal plate, the thickness of which varies as the $n$th power of the distance from the base, and of which the density is constant.

In this case taking the starting point as the origin, and the base as the axis of $x$,

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}},$$

$$dx = \frac{y\,dy}{(2ay-y^2)^{\frac{1}{2}}}.$$

Let $\tau = \kappa y^n =$ thickness, $\rho =$ density; it is plain that $\bar{x} = \pi a$; and from (31)

$$\bar{y}\int_0^{2\pi a}\int_0^y y^n\,dy\,dx = \int_0^{2\pi a}\int_0^y y^{n+1}\,dy\,dx;$$

$$\therefore \quad \bar{y}\int_0^{2\pi a}\frac{y^{n+1}}{n+1}\,dx = \int_0^{2\pi a}\frac{y^{n+2}}{n+2}\,dx,$$

$$\bar{y}\int_0^{2a}\frac{y^{n+2}\,dy}{(2ay-y^2)^{\frac{1}{2}}} = \frac{n+1}{n+2}\int_0^{2a}\frac{y^{n+3}\,dy}{(2ay-y^2)^{\frac{1}{2}}}$$

$$= \frac{n+1}{n+2}\,\frac{2n+5}{n+3}\,a\int_0^{2a}\frac{y^{n+2}\,dy}{(2ay-y^2)^{\frac{1}{2}}};$$

$$\therefore\quad \bar{y} = \frac{n+1}{n+2}\,\frac{2n+5}{n+3}\,a.$$

116.] If the plane surface of the plate is referred to polar coordinates, and rectangular coordinates are retained for the centre of gravity, then the area of the surface-element of the plate is $r\,dr\,d\theta$, and $x = r\cos\theta$, $y = r\sin\theta$, so that the equations (31) become

$$\left.\begin{aligned}\bar{x}\iint\rho g\tau r\,dr\,d\theta &= \iint\rho g\tau r^2\cos\theta\,dr\,d\theta\\ \bar{y}\iint\rho g\tau r\,dr\,d\theta &= \iint\rho g\tau r^2\sin\theta\,dr\,d\theta\end{aligned}\right\}. \qquad (34)$$

Ex. 1. To find the centre of gravity of a plate in the form of a sector of a circle, the thickness of which varies directly as the distance from the centre of the circle.

Let $a$ = radius of circle, $2\alpha$ = the angle which the sector subtends at the centre; and let the axis of $x$ be the line bisecting the angle $2\alpha$, so that $\bar{y} = 0$; then $\tau = kr$, and we have

$$\bar{x}\int_{-\alpha}^{\alpha}\int_0^{a} r^2\,dr\,d\theta = \int_{-\alpha}^{\alpha}\int_0^{a} r^3\cos\theta\,dr\,d\theta,$$

$$\bar{x} = \frac{3a}{4}\,\frac{\sin\alpha}{\alpha}.$$

Ex. 2. To find the centre of gravity of a thin plate of uniform thickness and density in the form of the loop of the lemniscata.

The equation to the bounding curve is

$$r^2 = a^2\cos 2\theta;$$

and as the loop is symmetrical with respect to the axis of $x$, $\bar{y} = 0$. Let $\mathrm{r} = a(\cos 2\theta)^{\frac{1}{2}}$, then from (34),

$$\bar{x}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\mathrm{r}} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\mathrm{r}} r^2\cos\theta\,dr\,d\theta,$$

$$\bar{x}\,\frac{a^2}{2} = \frac{a^3}{3}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\cos 2\theta)^{\frac{3}{2}}\cos\theta\,d\theta$$

$$= \frac{a^3}{3}\,2^{\frac{3}{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left\{\frac{1}{2}-(\sin\theta)^2\right\}^{\frac{3}{2}}d.\sin\theta$$

$$\bar{x} = \frac{2^{\frac{5}{2}}a}{3}\int_{-k}^{k}(k^2-x^2)^{\frac{3}{2}}\,dx,$$

if $k^2 = \frac{1}{2}$, and $\sin\theta = x$;

$$\therefore \quad \bar{x} = \frac{2^{\frac{5}{2}}a}{3}\,\frac{3}{8}\,\frac{\pi}{4}$$

$$= \frac{\pi a}{2^{\frac{5}{2}}}.$$

117.] Centre of gravity of a thin shell of revolution.

Let the axis of revolution be the axis of $x$; and let the equation to the curve, by the revolution of which the exterior surface of the shell is generated, be $y = f(x)$: let $\tau =$ the thickness of the shell; $\rho =$ the density; $g =$ the earth's attraction; and imagine the shell, see fig. 49, to be divided into a series of circular rings or annuli of breadth $dx$ by means of planes perpendicular to the axis of revolution, and at an infinitesimal distance apart: then, if $ds$ is a length-element of the generating curve, the volume of any one of these rings corresponding to a point $(x, y)$ on the generating curve is $2\pi y\tau\,ds$; and therefore the weight of it is $2\pi\rho g\tau y\,ds$: now imagine this weight to be condensed into a point at the centre of gravity of the ring, which is at M on the axis of $x$: the circumstances of pressure are not hereby changed: and let us imagine the weight of each ring to be similarly collected at its centre of gravity; then we have a series of weights arranged along the line O$x$, of variable magnitude, the law of variation depending on the equation of the generating curve: but such that the weight at the distance $x$ is equal to $2\pi y\rho g\tau\,ds$: hence we have to find the centre of gravity of this rod of variable density; and therefore, by virtue of equations (17),

$$\bar{x}\int 2\pi\rho g\tau y\,ds = \int 2\pi\rho g\tau xy\,ds, \tag{35}$$

and cancelling $2\pi g$,

$$\bar{x}\int \rho\tau y\,ds = \int \rho\tau xy\,ds. \tag{36}$$

Ex. 1. To find the centre of gravity of a thin shell of uniform thickness and density, the exterior surface of which is generated by the revolution of a quadrant of a circle about one of its bounding radii.

Let $\tau =$ thickness of shell; $\rho =$ density; then, fig. 50, the equation to the generating curve is

$$x^2+y^2 = a^2,$$

$$\therefore \quad \frac{dx}{y} = \frac{-dy}{x} = \frac{ds}{a};$$

$$\therefore \quad \bar{x}\int_0^a a\,dx = \int_0^a a\,x\,dx,$$

$$\bar{x} = \frac{a}{2}.$$

This result is also manifest by the method of infinitesimals: in Vol. I, Art. 24, Ex. 7, it is shewn that each zone of the shell is equal to the corresponding zone of the cylinder of the same thickness circumscribing the spherical shell; and therefore as these zones are equal and equivalent as to the position of their centres of gravity, the latter may replace the former, and the centre of gravity of the hemispherical shell is the same as that of the cylindrical shell; and this latter is evidently on OA in the middle point of OA.

Ex. 2. To find the centre of gravity of a thin right conical shell, of uniform thickness and density.

Let $\tau =$ the thickness of the shell; $\rho =$ the density; and let the equation to the generating straight line be

$$y = a\,x,$$

let the altitude of the shell $= a$: then $ds^2 = (1+a^2)\,dx^2$; and from (36) we have

$$\bar{x}\int_0^a x\,dx = \int_0^a x^2 dx,$$

$$\bar{x} = \frac{2a}{3}.$$

This is also manifest by the following reasoning: the conical shell may be imagined to be resolved into a series of triangular plates all the vertices of which meet at the vertex of the cone, and the bases of which form the circular base of the conical shell: now the centre of gravity of a triangular plate is on the line which is drawn from the vertex to the middle point of the base, and is at a distance from the vertex equal to two-thirds of that line; and therefore the centre of gravity of the shell is on the axis at a distance from the vertex equal to two-thirds of the axis.

And suppose the thickness of the conical shell to vary as the distance from the vertex: then $\rho = k(1+a^2)^{\frac{1}{2}}x$,

$$\therefore \quad \bar{x}\int_0^a x^2\,dx = \int_0^a x^3\,dx,$$

$$\bar{x} = \frac{3a}{4}.$$

Ex. 3. To find the centre of gravity of a thin shell of uniform thickness and density formed by the revolution about its base of a wire bent into a semi-cycloid.

The equation to the generating curve is

$$x = a \operatorname{versin}^{-1}\frac{y}{a} - (2ay-y^2)^{\frac{1}{2}},$$

$$\frac{dx}{y} = \frac{dy}{(2ay-y^2)^{\frac{1}{2}}} = \frac{ds}{(2ay)^{\frac{1}{2}}},$$

$$\bar{x}\int_0^{2a}\frac{y\,dy}{(2a-y)^{\frac{1}{2}}} = \int_0^{2a}\frac{xy\,dy}{(2a-y)^{\frac{1}{2}}};$$

$$\therefore \quad \bar{x} = \frac{26a}{15}.$$

118.] Centre of gravity of a thin curved shell.

Lastly, let us investigate the coordinates of the centre of gravity of a thin curved shell; of which let the thickness $= \tau$, the density $= \rho$; and let the equation to the bounding surface of the shell be

$$\mathrm{F}(x, y, z) = 0. \tag{37}$$

Then using the symbols of Articles 133 and 134, Vol. II, let $x, y, z$ be the coordinates to a point on the surface; and let $d\mathrm{A}$ be the surface-element abutting at it; then

$$d\mathrm{V} = \tau\, d\mathrm{A}, \tag{38}$$

where

$$d\mathrm{A} = \frac{\mathrm{Q}}{\mathrm{U}}\,dy\,dz = \frac{\mathrm{Q}}{\mathrm{V}}\,dz\,dx = \frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy, \tag{39}$$

$$= \{1+p^2+q^2\}^{\frac{1}{2}}\,dx\,dy; \tag{40}$$

so that (10) become, taking for $d\mathrm{A}$ the last value of (39),

$$\left.\begin{aligned}
\bar{x}\iint \rho g\tau\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy &= \iint \rho g\tau x\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy\\
\bar{y}\iint \rho g\tau\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy &= \iint \rho g\tau y\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy\\
\bar{z}\iint \rho g\tau\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy &= \iint \rho g\tau z\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy
\end{aligned}\right\}. \tag{41}$$

Ex. 1. To find the centre of gravity of the octant of a thin spherical shell of uniform thickness and density

$$x^2+y^2+z^2 = a^2,$$

$$\text{U} = 2x, \qquad \text{V} = 2y, \qquad \text{W} = 2z,$$

$$\therefore \quad \text{Q}^2 = 4(x^2+y^2+z^2),$$

$$= 4a^2.$$

Let $\tau$ = the thickness, $\rho$ = density; and as these are constant, they may be divided out of the equations (41). Also let $(a^2-x^2)^{\frac{1}{2}} = \text{Y}$; then from (41)

$$\bar{x}\int_0^a\int_0^{\text{Y}}\frac{a\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}} = \int_0^a\int_0^{\text{Y}}\frac{a\,x\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}},$$

$$\bar{x}\int_0^a \frac{\pi}{2}\,dx = \int_0^a \frac{\pi}{2}\,x\,dx,$$

$$\therefore \quad \bar{x} = \frac{a}{2};$$

$$\bar{y}\int_0^a\int_0^{\text{Y}}\frac{a\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}} = \int_0^a\int_0^{\text{Y}}\frac{a\,y\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}},$$

$$\bar{y} = \frac{a}{2};$$

$$\bar{z}\int_0^a\int_0^{\text{Y}}\frac{a\,dy\,dx}{(a^2-x^2-y^2)^{\frac{1}{2}}} = \int_0^a\int_0^{\text{Y}} a\,dy\,dx,$$

$$\bar{z} = \frac{a}{2}.$$

Suppose the thickness of the shell to vary as the $z$-ordinate to any point of it; then $\tau = kz$, and

$$\bar{x}\int_0^a\int_0^{\text{Y}} a\,k\,dy\,dx = \int_0^a\int_0^{\text{Y}} a\,k\,x\,dy\,dx,$$

$$\bar{x} = \frac{4a}{3\pi};$$

$$\bar{y}\int_0^a\int_0^{\text{Y}} a\,k\,dy\,dx = \int_0^a\int_0^{\text{Y}} a\,k\,y\,dy\,dx,$$

$$\bar{y} = \frac{4a}{3\pi};$$

$$\bar{z}\int_0^a\int_0^{\text{Y}} a\,k\,dy\,dx = \int_0^a\int_0^{\text{Y}} a\,k\,(a^2-x^2-y^2)^{\frac{1}{2}}\,dy\,dx,$$

$$\bar{z} = \frac{2a}{3}.$$

119.] The following theorem, due to Pappus, expresses a relation between a plane area, the volume of the solid generated by it as it revolves about a line on its own plane, and the distance of the centre of gravity of the area from the axis, whereby, when any two of these quantities are given, we are able to discover the third.

Let the revolving area be of constant density and thickness, and be so thin as to be conceived to be a geometrical surface, then, if $\bar{y}$ is the distance of the centre of gravity of this area from the axis of $x$, we have, by the second of (31),

$$\bar{y}\iint dy\,dx = \iint y\,dy\,dx,$$

$$\therefore \quad 2\pi\bar{y} \times \iint dy\,dx = \iint 2\pi y\,dy\,dx. \tag{42}$$

Now these integrals being definite, the second factor of the left-hand member of the equation expresses the area in the plane $xy$, and the first factor is the length of the path described by the centre of gravity of that area, as it revolves through four right angles about the axis of $x$: and because $dy\,dx$ is the area-element, and $2\pi y$ is the path described by the area-element during a complete revolution of the area about the axis of $x$, the right-hand member is the product of all the area-elements of the given area, and of their paths, and is therefore the volume described by the area during a complete revolution: if therefore the curve lies wholly on the same side of the axis of $x$, so that $y$ does not change sign, the above equation authorizes us to infer that,

If a plane area, lying wholly on the same side of a line in its own plane, revolves about that line, and thereby generates a solid of revolution, the volume of the solid thus generated is equal to the (geometrical) product of the revolving area, and of the path described by its centre of gravity during the revolution.

As (42) is true for the whole revolution, a similar theorem is also true for any part of the revolution: and if the generating area is such as that described in fig. 46, where the axis of $x$ is one of the bounding lines, then the limits of the $y$-integration in (42) are the ordinate to the curve and zero: therefore

$$2\pi\bar{y}\int y\,dx = \int \pi y^2\,dx, \tag{43}$$

and the right-hand member is the ordinary expression for the volume of a solid of revolution, which is investigated in Art. 137, Vol. II. In other cases the limits of $y$ are given by the geometrical conditions of the problem.

Ex. 1. An ellipse revolves about a line in its own plane, the perpendicular distance of which from the centre is equal to $c$;

it is required to find the volume of the ring generated during a complete revolution.

Let $a$ and $b$ be the semi-axes of the generating ellipse; then the generating area $= \pi ab$; and as $2\pi c$ is the path described by the centre of gravity,

$$\text{the volume} = 2\pi^2 abc.$$

It will be observed that the volume is the same, whatever direction the axis of revolution has with respect to the axes of the ellipse, provided that the perpendicular distance from the centre to the axis of revolution is the same.

Ex. 2. The volume of a sphere of radius $a$ is $\frac{4\pi a^3}{3}$; and the area of a semicircle is $\frac{\pi a^2}{2}$: it is required to deduce from these data the position of the centre of gravity of the semicircle.

Let $\bar{y}$ be the distance of the centre of gravity of the semicircle from the diameter; then considering it as the generating area of the sphere, we have

$$2\pi\bar{y} \times \frac{\pi a^2}{2} = \frac{4\pi a^3}{3}, \qquad \therefore \quad \bar{y} = \frac{4a}{3\pi};$$

and by reason of the symmetry, the centre of gravity is on the line which is perpendicular to the diameter through the centre of the circle.

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## SECTION 4.—*Centre of gravity of heavy bodies bounded by plane and curved surfaces.*

120.] Before I proceed to the general case, I will consider that of a solid bounded by a surface of revolution, and refer the body to the axis of revolution as the axis of $x$: let the equation to the generating curve of the bounding surface be $y = f(x)$. Imagine the solid, (see fig. 51,) to be divided into thin circular slices by planes at an infinitesimal distance apart, and perpendicular to the axis of revolution: of these let the circular slice PP′Q′Q be the type, and let $\text{OM} = x$, $\text{MN} = dx$, so that $dx$ is the thickness of it. Of this slice take a particle at a distance $r$ from the axis, and so that the plane passing through O$x$ and that particle may be inclined at an angle $\theta$ to the plane passing through O$x$ and O$y$; then the volume of the element is equal to $r\,d\theta\,dr\,dx$. Let $\rho =$ the density of the body at the particle, then

$$\text{mass-element} = \rho r\,dr\,d\theta\,dx, \tag{44}$$

$$\text{and the weight-element} = \rho g r\,dr\,d\theta\,dx. \tag{45}$$

Now if the constitution of the body as to density is symmetrical with respect to the axis of revolution, the centre of gravity is plainly on the axis of $x$, and therefore we have to find only $\bar{x}$; and we have from (10)

$$\bar{x}\iiint \rho g r\, d\theta\, dr\, dx = \iiint \rho g x r\, d\theta\, dr\, dx, \tag{46}$$

and performing the $\theta$-integration through a whole revolution, so as to obtain the required result for a ring of radius $r$, and observing that the symmetry of the body renders $\rho$ independent of $\theta$, we have, dividing out $2\pi g$,

$$\bar{x}\iint \rho r\, dr\, dx = \iint \rho x r\, dr\, dx. \tag{47}$$

And if the density is uniform throughout a complete slice, we may perform the $r$-integration between $r = 0$, and $r = y$, where $y$ is the ordinate to the generating curve: and (47) becomes

$$\bar{x}\int \rho y^2\, dx = \int \rho y^2 x\, dx, \tag{48}$$

and the limits of $x$ will depend on the circumstances of the problem.

Ex. 1. To find the centre of gravity of a paraboloid of revolution of uniform density, the length of whose axis is $c$.

Let the equation to the generating parabola be $y^2 = 4ax$; therefore from (48), as $\rho$ is constant,

$$\bar{x}\int_0^c 4ax\, dx = \int_0^c 4ax^2\, dx, \qquad \bar{x} = \frac{2}{3}c.$$

Ex. 2. To find the centre of gravity of a portion of a prolate spheroid of uniform density, the length of whose axis measured from the vertex is $c$.

Let the equation to the generating curve of the bounding surface be

$$y^2 = \frac{b^2}{a^2}(2ax - x^2);$$

then, as $\rho$ is constant, (48) becomes

$$\bar{x}\int_0^c (2ax - x^2)\, dx = \int_0^c (2ax - x^2)\, x\, dx,$$

$$\bar{x} = \frac{c}{4}\,\frac{8a - 3c}{3a - c}.$$

Thus for a hemi-spheroid, $c = a$, and we have

$$\bar{x} = \frac{5a}{8}.$$

As $b$ does not enter into either of the last two values, they are the same for a spherical segment and for a hemisphere.

Ex. 3. To find the centre of gravity of a double convex lens of uniform density.

Let the equations to the generating circles of the two intersecting spheres be, fig. 52,

$$x^2+y^2 = a^2, \qquad (x-c)^2+y^2 = b^2,$$

where OA $= a$, BC $= b$, OC $= c$; then the equation to the plane of intersection of the spheres is

$$x = \frac{a^2+c^2-b^2}{2c} = k \text{ (say)};$$

then from (48)

$$\bar{x}\left\{\int_k^a (a^2-x^2)\,dx + \int_{c-b}^k \{b^2-(x-c)^2\}\,dx\right\}$$
$$= \int_k^a (a^2-x^2)\,x\,dx + \int_{c-b}^k \{b^2-(x-c)^2\}\,x\,dx,$$

whence may $\bar{x}$ be determined.

Ex. 4. To find the centre of gravity of a cone, the density of each circular slice of which varies as the $n$th power of its distance from a parallel plane through the vertex.

Let the vertex be the origin, and the equation to the generating line of the cone be $y = ax$; and let $a$ be the altitude; then $\rho = kx^n$: and (48) becomes

$$\bar{x}\int_0^a x^{n+2}\,dx = \int_0^a x^{n+3}\,dx, \qquad \bar{x} = \frac{n+3}{n+4}\,a.$$

Ex. 5. To find the centre of gravity of a cone, the density of every particle of which increases as its distance from the axis.

Let the vertex be the origin, $a =$ the altitude, and let the equation of the generating line of the bounding surface be $y = ax$; then in equation (47) $\rho = kr$, so that

$$\bar{x}\int_0^a\int_0^{ax} r^2\,dr\,dx = \int_0^a\int_0^{ax} r^2 x\,dr\,dx, \qquad \bar{x} = \frac{4}{5}\,a.$$

Ex. 6. To find the centre of gravity of the volume of uniform density contained between a hemisphere and a cone whose vertex is the vertex of the hemisphere and base is the base of the hemisphere.

Let the common vertex, see fig. 53, be the origin; and let the equations to the bounding surfaces be

$$y^2 = 2ax-x^2 = \mathrm{Y}^2, \qquad y^2 = x^2,$$

so that Y and $x$ are the limits of the $r$-integration in equation (47): then, as $\rho$ is constant,

$$\bar{x}\int_0^a\int_x^{\mathrm{Y}} r\,dr\,dx = \int_0^a\int_x^{\mathrm{Y}} r\,x\,dr\,dx,$$

$$\bar{x}\int_0^a (2ax-x^2-x^2)\,dx = \int_0^a (2ax-x^2-x^2)\,x\,dx,$$

$$\bar{x} = \frac{a}{2}.$$

121.] Now let us take the most general case of a body in space; and first let it be referred to three rectangular axes originating at O: let $x, y, z$ be the coordinates to any particle of it, so that the volume-element abutting at it is $dx\,dy\,dz$; then

$$d\mathrm{V} = dx\,dy\,dz:$$

let the density $=\rho$, so that equations (10) become

$$\left.\begin{aligned}\bar{x}\iiint\rho\,dx\,dy\,dz &= \iiint\rho\,x\,dx\,dy\,dz\\ \bar{y}\iiint\rho\,dx\,dy\,dz &= \iiint\rho\,y\,dx\,dy\,dz\\ \bar{z}\iiint\rho\,dx\,dy\,dz &= \iiint\rho\,z\,dx\,dy\,dz\end{aligned}\right\}. \qquad (49)$$

The integrals are of course definite and the limits are assigned by the conditions of the problem.

Ex. 1. To find the centre of gravity of a homogeneous body in the form of the octant of an ellipsoid.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1;$$

and let
$$\mathrm{Z} = c\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)^{\frac{1}{2}},$$

$$\mathrm{Y} = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}},$$

then
$$\bar{x}\int_0^a\int_0^{\mathrm{Y}}\int_0^{\mathrm{Z}} dz\,dy\,dx = \int_0^a\int_0^{\mathrm{Y}}\int_0^{\mathrm{Z}} x\,dz\,dy\,dx,$$

$$\therefore\quad \bar{x} = \frac{3a}{8};$$

similarly,
$$\bar{y} = \frac{3b}{8}, \qquad \bar{z} = \frac{3c}{8}.$$

Ex. 2. To find the centre of gravity of a body of uniform density bounded by the Cono-Cuneus of Wallis and the planes $z=0$, $y=c$.

The equation to the Cono-Cuneus is, equation (89), Art. 315, Vol. I,

$$c^2 z^2 = y^2 (a^2 - x^2),$$

and performing the $z$-, $y$-, $x$-integrations in order, the limits are $\frac{y}{c}(a^2-x^2)^{\frac{1}{2}}$ and 0, $c$ and 0, $a$ and 0; so that if

$$z = \frac{y}{c}(a^2-x^2)^{\frac{1}{2}},$$

$$\bar{x}\int_0^a\int_0^c\int_0^z dz\,dy\,dx = \int_0^a\int_0^c\int_0^z x\,dx\,dy\,dz$$

$$\bar{x}\int_0^a\int_0^c y(a^2-x^2)^{\frac{1}{2}}\,dy\,dx = \int_0^a\int_0^c xy(a^2-x^2)^{\frac{1}{2}}\,dy\,dx,$$

$$\bar{x}\int_0^a (a^2-x^2)^{\frac{1}{2}}\,dx = \int_0^a x(a^2-x^2)^{\frac{1}{2}}\,dx,$$

$$\bar{x} = \frac{4a}{3\pi};$$

$$\bar{y}\int_0^a\int_0^c\int_0^z dz\,dy\,dx = \int_0^a\int_0^c\int_0^z y\,dz\,dy\,dx,$$

$$\bar{y}\int_0^a\int_0^c y(a^2-x^2)^{\frac{1}{2}}\,dy\,dx = \int_0^a\int_0^c y^2(a^2-x^2)^{\frac{1}{2}}\,dx,$$

$$\bar{y} = \frac{2}{3}c;$$

$$\bar{z}\int_0^a\int_0^c\int_0^z dz\,dy\,dx = \int_0^a\int_0^c\int_0^z z\,dz\,dy\,dx,$$

$$\bar{z}\int_0^a\int_0^c \frac{y}{c}(a^2-x^2)^{\frac{1}{2}}\,dy\,dx = \int_0^a\int_0^c \frac{y^2(a^2-x^2)}{2c^2}\,dy\,dx,$$

$$\bar{z} = \frac{8a}{9\pi}.$$

122.] Again, let the curved bounding surface be referred to a system of polar coordinates of the construction of Art. 146, Vol. II, so that

$$\left.\begin{aligned}
\bar{x}\iiint\rho r^2\sin\theta\,dr\,d\theta\,d\phi &= \iiint\rho r^3(\sin\theta)^2\cos\phi\,dr\,d\theta\,d\phi\\
\bar{y}\iiint\rho r^2\sin\theta\,dr\,d\theta\,d\phi &= \iiint\rho r^3(\sin\theta)^2\sin\phi\,dr\,d\theta\,d\phi\\
\bar{z}\iiint\rho r^2\sin\theta\,dr\,d\theta\,d\phi &= \iiint\rho r^3\sin\theta\cos\theta\,dr\,d\theta\,d\phi
\end{aligned}\right\}; \quad (50)$$

the integrals of course being definite, and the limits being assigned by the geometrical conditions of the problem.

Ex. 1. To find the centre of gravity of an octant of a sphere, the density of which varies as the $n$th power of the distance of any particle from the centre.

Let $a =$ the radius of the sphere; and let $\rho = kr^n$; then equations (50) become

$$\bar{x}\int_0^a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^{n+2}\sin\theta\,d\theta\,d\phi\,dr = \int_0^a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} r^{n+3}(\sin\theta)^2\cos\phi\,d\theta\,d\phi\,dr,$$

$$\bar{x} = \frac{n+3}{n+4}\frac{a}{2} = \bar{y} = \bar{z},$$

the last two being inferred from the symmetry of the body.

Ex. 2. The vertex of a right circular cone is at the centre of a sphere; it is required to find the centre of gravity of a body of uniform density contained within the cone and the sphere.

Let the axis of $z$ be the axis of the cone: and let $\alpha$ be the semi-vertical angle of the cone; $a =$ the radius of the sphere; $\rho =$ the constant density: then $\bar{x}$ and $\bar{y}$ are evidently equal to zero; and we have

$$\bar{z}\int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^2\sin\theta\,dr\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\alpha}\int_0^{a} r^3\sin\theta\cos\theta\,dr\,d\theta\,d\phi,$$

$$\bar{z}\frac{a^3}{3}(1-\cos\alpha)\,2\pi = \frac{a^4}{4}\frac{(\sin\alpha)^2}{2}\,2\pi,$$

$$\bar{z} = \frac{3a}{8}(1+\cos\alpha).$$

123.] I shall conclude this section with a few examples of determining the centres of gravity of bodies which do not come under any of the former methods, but to which the principles are equally applicable.

Ex. 1. To find the centre of gravity of a right pyramid of uniform density, whose base is any regular plane figure.

Let the vertex of the pyramid be the origin, and the axis of the pyramid the axis of $x$; divide the pyramid into slices of the thickness $dx$ by planes perpendicular to the axis: then as the areas of the sections thus formed will vary as the squares of their homologous sides, and as these sides will vary as the distances from the vertex, so will the areas of the sections vary as the squares of the distances from the vertex; and therefore if the axis of the pyramid is divided into equal infinitesimal elements, the masses of the several slices will vary as the squares of the distance from the vertex. Now imagine each slice to be condensed into its centre of gravity, which point will be on the axis of $x$; then if $a =$ the altitude of the pyramid, we shall have

$$\bar{x}\int_0^a x^2\,dx = \int_0^a x^3\,dx, \qquad \therefore\quad \bar{x} = \frac{3}{4}a.$$

Ex. 2. On the base of a hemisphere a right circular cone is constructed, the whole body being of uniform density; determine the altitude of the cone, so that the centre of gravity of the whole may be at the centre of the circular base of the hemisphere.

Let $a =$ the radius of the hemisphere,
$c =$ the altitude of the cone:

then if we imagine the hemisphere and the cone to be condensed into their centres of gravity, the moments of these weights must be equal about the centre of the circular base of the hemisphere: that is,

$$\int_0^a (a^2-x^2)\,x\,dx = \int_0^c \frac{a^2}{c^2}(c-x)^2 x\,dx,$$

$$\therefore \quad c^2 = 3a^2;$$

and therefore the vertical angle of the cone is 60°.

Ex. 3. When a heavy body with a convex surface rests on a horizontal plane, the vertical line through the centre of gravity also passes through the point of contact: because as the body is acted on by only two forces, viz., the weight acting downwards at the centre of gravity, and the reaction of the plane upwards at the point of contact, these forces cannot be in equilibrium unless they are equal, and act along the same line in opposite directions.

Hence it appears that the compound body of the last example will rest in any position on its convex spherical surface.

Hence also it follows that if a body is suspended from any point, the point of suspension and the centre of gravity are in the same vertical line.

A body in the form of a paraboloid of revolution of given altitude and uniform density is suspended from a point in the edge of its circular base; it is required to find the inclination of its axis to the vertical.

Let $a =$ the altitude of the paraboloid; $b =$ the radius of its circular base; $\theta =$ the angle between the axis of the paraboloid and the vertical: then, since the distance of the centre of gravity from the centre of the circular base $= \frac{a}{3}$, see Ex. 1. Art. 120,

$$\tan\theta = \frac{3b}{a}.$$

Ex. 4. If a heavy body is placed on a rough inclined plane, the friction of which is sufficient to prevent sliding, the body will be at rest so long as the vertical line through the centre of gravity passes within the part of the body which is in contact with the inclined plane; and if it falls beyond that part, the body will fall over; and if it passes through the edge of it, the body is just in its limiting position of rest.

A given cone rests with its base on an inclined plane: it is required to determine the inclination of the plane, when the cone is just on the point of falling over.

Let $a =$ the altitude of the cone, and $b$ the radius of the base: then $\mathrm{CG} = \frac{a}{4}$, see fig. 54: let $\mathrm{CO}x = \alpha$:

$$\begin{aligned} \therefore \quad \tan\alpha &= \tan \mathrm{CO}x, \\ &= \tan \mathrm{CGB}, \\ &= \frac{4b}{a}; \end{aligned}$$

and when the angle of inclination of the plane exceeds this angle, the cone will fall over.

## Section 5.—*Stability and instability of equilibrium of heavy bodies.*

124.] And we have now means at our disposal for determining the stability or instability of bodies under the action of gravity, in accordance with the principles already explained in Section 5 of the preceding Chapter.

If a heavy body is in equilibrium with one point of it resting on a plane or on a surface, the centre of gravity of the body is in the same vertical line with the point of contact, and the line of action of the common pressures is vertical. Now imagine a body thus at rest on a plane or on a surface to receive a slight displacement by means of a new force, and to be kept at rest in its displaced position; then either (1) this new force may act in a direction to hinder the body from returning to its former position; or (2) the body may be at rest in its new position without the continued action of the new force; or (3) the force may act in a direction to hinder the body from receding further from its former position; thus, when the body is slightly displaced, the tendency is in the first case to return to its original state; in the second, to remain at rest in its new position; in the third, to recede further from it: in the first state we have stable equilibrium, in the second, neutral or continuous, and in the third, unstable: and the mechanical circumstances are these: in the first case the centre of gravity ascends in consequence of the displacement; in the second it remains in the same horizontal line; in the third it descends: thus if a heavy ellipsoid is balanced at the extremity of its greatest axis the equilibrium is unstable; if it is placed on the end of its mean axis the equilibrium will be stable for oscillations about the least axis, and unstable for those about the greatest axis, and will be neutral for rotation about some axes intermediate to these; and if it is placed on the end of its least axis, the position is that of stable equilibrium: also evidently the equilibrium of a heavy sphere resting on a smooth plane is continuous; and thus the analytical problem is, to determine whether the centre of gravity of a body resting on a surface, ascends or moves horizontally or descends, when the body receives a slight displacement: we therefore calculate its distance from a horizontal plane, and determine whether that distance is for a small displacement a minimum, a constant, or a maximum. Thus if $h =$ the distance above a horizontal plane, in all these cases $dh = 0$, and the change of sign will determine the nature of equilibrium. And this is in accordance with the results of Art. 103: let the axes of coordinates be so chosen that the axis of $z$ is vertical: then we have

$$\bar{z}\,\Sigma.\mathrm{P} = \Sigma.\mathrm{P}z,$$

and therefore if $\bar{z}$ is a maximum or a minimum

$$d.\Sigma.\mathrm{P}z = 0;$$

$$\therefore\quad \Sigma.\mathrm{P}\,dz = 0.$$

To take the most general case of this kind, let us consider that of a heavy body bounded by a convex surface resting on another body also with a convex surface. And let fig. 55 represent the bodies: the continuous lines indicating the position of the bodies when they are at rest at first, and the dotted lines the position of displacement. Let CAO be the vertical line passing through A the point of contact of the two surfaces when they are at rest, and through the centre of gravity of the upper body: let C be the centre of curvature of the lower body corresponding to the point A, and O that of the upper body; let G be the centre of gravity of the upper body: now suppose a small displacement of the upper body to take place by means of *rolling* on the lower one, so that P being the new point of contact, and A′ being the point which was originally in contact with A, $A'P = AP$, the axis about which the rolling takes place being perpendicular to the plane of the paper. Let the curvature of the two surfaces be continuous about the points A and P; and by reason of the small displacement let O and G respectively be moved to O′ and G′; let $CA = CP = \rho_1$; $OA = O'A' = O'P = \rho_2$; $ACP = \theta$; $OG = O'G' = c$; therefore since the arc $AP =$ the arc $A'P$;

$$\therefore \quad \rho_1 \theta = \rho_2 A'O'P, \qquad \therefore \quad A'O'P = \frac{\rho_1}{\rho_2} \theta.$$

Let $h = G'K =$ vertical height of G′ above the horizontal line through C; therefore

$$h = (\rho_1 + \rho_2) \cos \theta - c \cos \left(1 + \frac{\rho_1}{\rho_2}\right) \theta,$$

and replacing the cosines by the first two terms of their equivalent series, because $\theta$ is small, we have

$$h = \rho_1 + \rho_2 - c - (\rho_1 + \rho_2) \left(1 - c \frac{\rho_1 + \rho_2}{\rho_2^2}\right) \frac{\theta^2}{1.2},$$

$$\frac{dh}{d\theta} = -(\rho_1 + \rho_2) \left(1 - c \frac{\rho_1 + \rho_2}{\rho_2^2}\right) \theta,$$

$$= 0, \text{ if } \theta = 0,$$

and changes sign from + to −, if $c$ is less than $\dfrac{\rho_2^2}{\rho_1 + \rho_2}$,

- - - - - - - - - - — to +, if $c$ is greater than $\dfrac{\rho_2^2}{\rho_1 + \rho_2}$;

and therefore $h$ is a maximum or a minimum according as $AG = \rho_2 - c$ is greater or less than $\dfrac{\rho_1 \rho_2}{\rho_1 + \rho_2}$; that is, as

$$\frac{1}{AG} \text{ is less than or greater than } \frac{1}{\rho_1} + \frac{1}{\rho_2}; \tag{51}$$

and therefore the equilibrium is stable or is unstable according as

$$\frac{1}{AG} \text{ is greater than or less than } \frac{1}{\rho_1} + \frac{1}{\rho_2}. \tag{52}$$

If the equilibrium is neutral

$$\frac{1}{AG} = \frac{1}{\rho_1} + \frac{1}{\rho_2}; \tag{53}$$

and in this case, for a small displacement, the centre of gravity of the upper body neither ascends nor descends.

If the lower surface is plane, $\rho_1 = \infty$, and the equilibrium is stable or unstable, according as AG is less or greater than $\rho_2$; that is, according as the centre of gravity is below or above the centre of curvature corresponding to the point A.

If the lower surface is concave, $\rho_1$ is negative, and the equilibrium is stable or unstable according as

$$\frac{1}{\text{AG}} \text{ is greater or less than } \frac{1}{\rho_2} - \frac{1}{\rho_1}. \tag{54}$$

125.] The values of $\rho_1$ and $\rho_2$ will of course depend on the position of the normal planes of the greatest and least curvature of the two surfaces, and therefore the stability will be different for the different normal planes through A; the stability therefore will be greatest or least according as

$$\frac{1}{\rho_1} + \frac{1}{\rho_2}$$

is a minimum or a maximum.

If therefore in this latter case, which is the most unfavourable, the equilibrium is stable, it is also stable for every normal section passing through A, and therefore the position of the body is one of complete stability.

Suppose however that the upper and lower surfaces are so arranged, that $a$ is the angle between the normal section of greatest curvature in the lowest, and that of the greatest curvature in the upper; and suppose that it is required to find the nature of the stability of any particular normal plane.

Let $\theta$ be the angle between the normal plane of displacement, and that of maximum curvature in the lowest surface: then if $\text{R}_1$ and $r_1$ are the principal radii of curvature of the lower surface, by Euler's theorem, Art. 351, Vol. I,

$$\frac{1}{\rho_1} = \frac{(\cos\theta)^2}{r_1} + \frac{(\sin\theta)^2}{\text{R}_1};$$

and if $\text{R}_2$ and $r_2$ are the principal radii of curvature of the upper surface

$$\frac{1}{\rho_2} = \frac{\{\cos(\theta+a)\}^2}{r_2} + \frac{\{\sin(\theta+a)\}^2}{\text{R}_2};$$

$$\therefore \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{(\cos\theta)^2}{r_1} + \frac{\{\cos(\theta+a)\}^2}{r_2} + \frac{(\sin\theta)^2}{\text{R}_1} + \frac{\{\sin(\theta+a)\}^2}{\text{R}_2}:$$

whereby the normal plane of least stability may be determined.

126.] One or two problems involving conditions of stability or instability are subjoined.

Ex. 1. A heavy uniform beam rests against a smooth curve, and against a vertical wall, all of which are in the same vertical plane; it is required to find the nature of the curve so that the beam may be at rest in all positions.

Let the beam be QP, fig. 56, of which let G be the middle point and the centre of gravity; and let the horizontal line, in which the centre of gravity is in all positions of the beam, be the axis of $x$, and let it meet the vertical wall in the point O; let O be the origin, let the length of the beam be $2a$, so that the curve required meets the wall at a distance OA $(= a)$ below O; let OA be the axis of $y$; $\text{OM} = x$, $\text{MP} = y$, $\text{QGO} = \theta$,

$$\therefore \quad \frac{x}{2a} = \cos\theta, \qquad \frac{y}{a} = \sin\theta;$$

therefore squaring and adding,

$$\frac{x^2}{4a^2} + \frac{y^2}{a^2} = 1,$$

the equation to an ellipse, whose centre is O, horizontal semi-axis is $2a$, and vertical semi-axis is $a$.

The property of the curve required in the problem is evidently the same as that of the elliptic compasses.

Ex. 2. A heavy uniform beam rests against a smooth vertical wall, and on a smooth curve; determine the nature of the curve so that the beam may rest in all positions.

Let RQ be the beam of length $2a$, whose centre of gravity is G, fig. 57; P the point in the curve at which the beam touches it; let the horizontal line OMG, in which in all positions of the beam its centre of gravity is, be the axis of $x$; and let it meet the wall at O, and let O be the origin, $\text{OM} = x$, $\text{MP} = y$, $\text{QG} = \text{GR} = a$. Then, as the line RQ is a tangent to the required curve at P,

$$\tan \text{OGQ} = -\frac{dy}{dx}.$$

Therefore

$$a = \text{QP} + \text{PG},$$

$$= \frac{x\,ds}{dx} - \frac{y\,ds}{dy};$$

$$\therefore \quad y = x\frac{dy}{dx} - \frac{a\,dy}{(dx^2 + dy^2)^{\frac{1}{2}}};$$

which is a differential equation of Clairault's form: and of which the singular solution is, see Art. 314, Vol. II,

$$y^{\frac{2}{3}} + x^{\frac{2}{3}} = a^{\frac{2}{3}};$$

the equation of a hypocycloid, see Art. 179, Vol. I.

Ex. 3. To determine whether the position of the beam resting on two planes, as investigated in Ex. 2, Art. 50, is of stable or of unstable equilibrium.

In fig. 29 let GK $= h$; therefore

$$\begin{aligned} h &= \text{AC} \sin a - a \sin \theta, \\ &= 2a \frac{\sin a \sin(\beta+\theta)}{\sin(a+\beta)} - a \sin\theta, \\ &= \frac{a}{\sin(a+\beta)} \{\sin(a-\beta)\sin\theta + 2\sin a \sin\beta\cos\theta\}; \end{aligned}$$

$$\therefore \frac{dh}{d\theta} = 0 = \frac{a}{\sin(a+\beta)}\{\sin(a-\beta)\cos\theta - 2\sin a\sin\beta\sin\theta\},$$

$$\therefore \tan\theta = \frac{\sin(a-\beta)}{2\sin a \sin\beta}, \text{ (see Ex. 2, Art. 50)};$$

and $\frac{dh}{d\theta}$ changes sign from + to −; therefore $h$ is a maximum, and the equilibrium is unstable.

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## SECTION 6.—*General properties of the centre of gravity.*

127.] In the first place let me remark that by reason of equations (9), if $\bar{x}, \bar{y}, \bar{z}$ are the coordinates to the centre of gravity of a series of heavy particles of the masses $m_1, m_2, \dots m_n$, and if the origin of coordinates is placed at the centre of gravity, so that $\bar{x} = \bar{y} = \bar{z} = 0$, then

$$\left.\begin{aligned} \Sigma.mx &= m_1x_1 + m_2x_2 + \dots + m_nx_n = 0 \\ \Sigma.my &= m_1y_1 + m_2y_2 + \dots + m_ny_n = 0 \\ \Sigma.mz &= m_1z_1 + m_2z_2 + \dots + m_nz_n = 0 \end{aligned}\right\}. \qquad (55)$$

And thus, apart from all mechanical considerations, such as weight, or any of the properties which have been investigated in the preceding Sections of this Chapter, it is often convenient to consider the centre of gravity to be geometrically the point whereat the conditions (55) are satisfied.

THEOREM I. Of all points in space the centre of gravity is, with reference to a system of material particles, such that the sum of the products of each particle and the square of its distance from the point is a minimum.

Let $(x, y, z)$ be the required point; $m_1 m_2 \dots m_n$ the particles; $(x_1 y_1 z_1)$, $(x_2 y_2 z_2) \dots (x_n y_n z_n)$ their positions; then if

$$\left.\begin{aligned} u^2 = m_1&\{(x-x_1)^2 + (y-y_1)^2 + (z-z_1)^2\} \\ +m_2&\{(x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2\} \\ +&\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ +m_n&\{(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2\} \end{aligned}\right\}, \qquad (56)$$

and if $u^2$ is to be a minimum,

$$\left.\begin{aligned} u\,\mathrm{D}u = {} & m_1\{(x-x_1)dx+(y-y_1)dy+(z-z_1)dz\} \\ & +m_2\{(x-x_2)dx+(y-y_2)dy+(z-z_2)dz\} \\ & +\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\ & +m_n\{(x-x_n)dx+(y-y_n)dy+(z-z_n)dz\} \end{aligned}\right\}; \qquad (57)$$

and equating to zero the coefficients of $dx$, $dy$, $dz$, we have

$$x = \frac{\Sigma.mx}{\Sigma.m}, \qquad y = \frac{\Sigma.my}{\Sigma.m}, \qquad z = \frac{\Sigma.mz}{\Sigma.m}; \qquad (58)$$

and as (56) by the form of the expression admits of infinite increase, $u$ evidently cannot be a maximum; (58) therefore render $u$ a minimum; and these are the coordinates of the centre of gravity.

128.] **Theorem** II. If a system of material particles is invariable in form, and its centre of gravity is at a constant distance from a fixed point, the sum of the products of each particle and the square of its distance from the fixed point is constant.

Let the fixed point be the origin, and let $\bar{x}\,\bar{y}\,\bar{z}$ be the coordinates of the centre of gravity, and $(x_1\,y_1\,z_1)$, $(x_2\,y_2\,z_2)\ldots(x_n\,y_n\,z_n)$ the positions of the particles in a given position of the system, these coordinates being measured from the centre of gravity; also let

$$\bar{x}^2+\bar{y}^2+\bar{z}^2 = a^2;$$

and let $r_1\,r_2\ldots r_n$ be the distances of the particles from the fixed point: then

$$\begin{aligned} \Sigma.mr^2 = {} & m_1\{(\bar{x}+x_1)^2+(\bar{y}+y_1)^2+(\bar{z}+z_1)^2\} \\ & +\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\ & +m_n\{(\bar{x}+x_n)^2+(\bar{y}+y_n)^2+(\bar{z}+z_n)^2\}, \\ = {} & a^2(m_1+m_2+\ldots+m_n) \\ & +2\bar{x}\Sigma.mx+2\bar{y}\Sigma.my+2\bar{z}\Sigma.mz \\ & +\Sigma.m\rho^2, \end{aligned}$$

if $\rho_1\,\rho_2\ldots\rho_n$ are the distances of $m_1\,m_2\ldots m_n$ from the centre of gravity. But $\Sigma.mx = 0$, $\Sigma.my = 0$, $\Sigma.mz = 0$, by reason of (55); therefore

$$\Sigma.mr^2 = a^2\Sigma.m+\Sigma.m\rho^2; \qquad (59)$$

and as the right-hand member is constant, so is the left-hand member, and the proposition is proved.

129.] **Theorem** III. If there is a system of heavy material particles, the product of the sum of the masses and of the sum of the products of each mass and the square of its distance from the centre of gravity is equal to the sum of the product of every two masses and of the square of the distance between them.

Let the centre of gravity be the origin: then by (55)

$$\left.\begin{aligned} m_1 x_1 + m_2 x_2 + \dots + m_n x_n &= 0 \\ m_1 y_1 + m_2 y_2 + \dots + m_n y_n &= 0 \\ m_1 z_1 + m_2 z_2 + \dots + m_n z_n &= 0 \end{aligned}\right\}. \tag{60}$$

Let $\rho_1\, \rho_2 \dots \rho_n$ be the distances of $m_1\, m_2 \dots m_n$ from the origin; then squaring and adding the above, we have

$$\begin{aligned} &m_1^2 \rho_1^2 + m_2^2 \rho_2^2 + \dots + m_n^2 \rho_n^2 \\ &\qquad + 2 m_1 m_2 (x_1 x_2 + y_1 y_2 + z_1 z_2) \\ &\qquad + \dots \dots \dots \dots \\ &\qquad + 2 m_{n-1} m_n (x_{n-1} x_n + y_{n-1} y_n + z_{n-1} z_n) = 0; \end{aligned} \tag{61}$$

$$\therefore\quad \Sigma . m^2 \rho^2 + 2 \Sigma . m m' \rho \rho' \cos(\rho, \rho') = 0, \tag{62}$$

if $m, m'$ are the symbols for every two of the material particles, and $(\rho, \rho')$ is the angle contained between $\rho$ and $\rho'$. Now suppose $u$ to be the distance between the positions of the two particles $m$ and $m'$, then

$$u^2 = \rho^2 + \rho'^2 - 2 \rho \rho' \cos(\rho, \rho');$$

$$\therefore\quad 2 \rho \rho' \cos(\rho, \rho') = \rho^2 + \rho'^2 - u^2.$$

Therefore (62) becomes

$$\Sigma . m^2 \rho^2 + \Sigma . m m' (\rho^2 + \rho'^2 - u^2) = 0:$$

and when written at length

$$\begin{aligned} &m_1 \{ m_1 \rho_1^2 + m_2 \rho_2^2 + \dots + m_n \rho_n^2 \} \\ &+ m_2 \{ m_1 \rho_1^2 + m_2 \rho_2^2 + \dots + m_n \rho_n^2 \} \\ &+ \dots \dots \dots \dots \\ &+ m_n \{ m_1 \rho_1^2 + m_2 \rho_2^2 + \dots + m_n \rho_n^2 \} - \Sigma . m m' u^2 = 0; \end{aligned} \tag{63}$$

and if $\text{M} = \Sigma . m = m_1 + m_2 + \dots + m_n$; we have

$$\text{M}\, \Sigma . m \rho^2 = \Sigma . m m' u^2, \tag{64}$$

which is the proposition required*.

130.] THEOREM IV. If a material particle is in equilibrium under the action of many pressures which are represented as to intensity and line of action by straight lines meeting at the particle; and if at the extremities of each of these lines heavy particles equal in weight are placed, the centre of gravity of these is at the point which is at rest under the action of the impressed pressures.

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* In the "Mécanique Analytique" of Lagrange, Premiere partie, Section III, Art. 20, an extension of this Theorem is given.

By reason of equations (69), Art. 28, we have

$$\Sigma.\mathrm{P}\cos\alpha = 0, \qquad \Sigma.\mathrm{P}\cos\beta = 0, \qquad \Sigma.\mathrm{P}\cos\gamma = 0: \qquad (65)$$

let $s_1, s_2, \dots s_n$ be the line-representatives of the impressed forces acting on the material particle, the place of which we will take to be the origin: so that the equations (65) become

$$\Sigma.s\cos\alpha = 0, \qquad \Sigma.s\cos\beta = 0, \qquad \Sigma.s\cos\gamma = 0. \qquad (66)$$

Let $(x_1\, y_1\, z_1), (x_2\, y_2\, z_2), \dots (x_n\, y_n\, z_n)$ be the extremities of $s_1, s_2, \dots s_n$; so that

$$\begin{array}{lll} x_1 = s_1\cos\alpha_1, & y_1 = s_1\cos\beta_1, & z_1 = s_1\cos\gamma_1, \\ x_2 = s_2\cos\alpha_2, & y_2 = s_2\cos\beta_2, & z_2 = s_2\cos\gamma_2, \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ x_n = s_n\cos\alpha_n, & y_n = s_n\cos\beta_n, & z_n = s_n\cos\gamma_n, \end{array}$$

whereby (66) become

$$\Sigma.x = 0, \qquad \Sigma.y = 0, \qquad \Sigma.z = 0; \qquad (67)$$

and if the mass of the particle at the extremity of every line-representative is $m$, we have

$$\Sigma.mx = 0, \qquad \Sigma.my = 0, \qquad \Sigma.mz = 0, \qquad (68)$$

and therefore by (55), the origin is the centre of gravity of all the particles.

# CHAPTER V.

## THE ACTION OF FORCES ON BODIES OF VARIABLE FORM.

### SECTION 1.—*The action of forces on flexible and inextensible strings or cords.*

131.] Thus far the bodies or systems of material particles, on which the statical forces act, have been assumed to be rigid, and their forms, or the relative position of the particles, have been supposed not to change on account of the acting forces. We shall now extend the inquiry to the case of bodies whose form varies by the action of the pressures, but becomes permanent, and may be considered rigid, under the action of the impressed forces. I shall first shortly investigate the case of the Funicular Polygon.

Suppose a string or cord, fig. 58, AB to be fastened at the two points A, B; the cord being without weight, perfectly flexible, and perfectly inextensible; and suppose at $Q_1, Q_2, Q_3, Q_4$, definite points of it, pressures $P_1, P_2, P_3, P_4$ to act with definite intensities and along definite lines of action, so that the cord assumes the permanent position indicated in the figure; the object is the determination of the form of the polygonal figure which the cord of given length assumes under the action of these forces, and of the tensions of each of its component straight elements.

It is manifest that the tension is the same throughout each element; and that as each point $Q_1, Q_2, \ldots Q_4$ is at rest, the forces acting at each are in equilibrium. Let the tensions along $AQ_1, Q_1Q_2, \ldots Q_4B$, be respectively $T_1, T_2, \ldots T_5$, so that the pressures at the fixed points A and B are respectively $T_1$ and $T_5$; and let the angles between the successive straight parts of the cord be $\alpha_1, \alpha_2, \ldots \alpha_4$; then as the point $Q_1$ is kept at rest by the three forces $T_1, P_1$, and $T_2$, the lines of action of all are in the same plane, and we have

$$\frac{T_1}{\sin P_1Q_1Q_2} = \frac{P_1}{\sin \alpha_1} = \frac{T_2}{\sin P_1Q_1A}. \tag{1}$$

In the same way for the point $Q_2$ we have

$$\frac{T_2}{\sin P_2Q_2Q_3} = \frac{P_2}{\sin \alpha_2} = \frac{T_3}{\sin P_2Q_2Q_1}; \tag{2}$$

and so on for the other points; and therefore when the form of the polygon and the magnitudes and lines of action of the forces $P_1, P_2, \ldots\ldots$ are given, the tensions of the several connecting strings may be determined.

132.] Suppose that the lines of action of the forces $P_1, P_2, \ldots P_4$ bisect the angles $a_1, a_2, \ldots a_4$; then $T_1, = T_2, = \ldots = T_5$; and

$$\frac{P_1}{\cos\frac{a_1}{2}} = \frac{P_2}{\cos\frac{a_2}{2}} = \ldots = \frac{P_4}{\cos\frac{a_4}{2}}; \tag{3}$$

and this condition may be secured in two ways; (1) by fixing smooth pins at the points $Q_1 \ldots Q_4$, and passing the strings round them, so that the tension of the string is the same on both sides of the pin, and the pressure on the pin is the resultant of these two equal forces, and therefore its line of action bisects the angle between their lines of action: and (2) by making the impressed forces act on the cord at the points $Q_1 \ldots$ by means of smooth rings which slide on the cord, and are at rest at these points; and the line of action of $P_1$ will manifestly under this arrangement bisect the angle $AQ_1Q_2$, because considering $A$ and $Q_2$ to be fixed, and the cord to be also fastened at them, if the ring $Q_1$ slides, it can move only on the surface of a prolate spheroid, of the generating ellipse of which $A$ and $Q_2$ are the foci, and the length $AQ_1Q_2$ of the cord is the major axis, and therefore the normal at $Q_1$ which is the line of action of $P_1$ bisects the focal angle between the distances.

And also besides suppose that the two sides of the polygon abutting at (say) $Q_1$ are equal; then if $AQ_1 = Q_1Q_2 = s_1$, and the radius of the circle passing through $A_1Q_1Q_2$ is $\rho_1$, we have

$$\cos\frac{a_1}{2} = \frac{s_1}{2\rho_1}; \tag{4}$$

and therefore if all the sides are equal, from (3) it follows that each impressed force is inversely as the radius of the circle passing through its point of application and the two angular points of the polygon adjacent on each side.

Now of such a polygon a curve is a particular case, if the length of the curve is the equicrescent variable; and the circles just mentioned are the circles lying in the osculating plane at the point, and whose radius is the radius of absolute curvature of the curve at the point; and therefore when a funicular curve fastened at its two ends is acted on in all its equal elements by normal forces, the tension is the same throughout, and each normal force varies as the absolute curvature of the curve at the point where it is applied.

Thus suppose a cord to be stretched by given forces at its ends on a curved surface, then the pressure caused by the surface is at every point in the direction of the normal of the surface, and is therefore proportional to the absolute curvature of the curve which the cord assumes on the surface; and as the normal-line of the reaction is in the same plane with two consecutive elements of the funicular curve, the osculating plane of the curve is a normal plane to the surface at the common point; and therefore, see Art. 227, Vol. II, the curve is the geodesic line on the surface joining the two points: and this geodesic line may evidently be either the maximum or the minimum; thus, a cord stretched between two given points on a sphere will arrange itself along the geodesic line, which is a great circle; and as one great circle-arc abutting at the points will be a minimum, so will the remainder of the same great circle be the maximum.

133.] If the lines of action of all the forces acting on the funicular polygon are parallel, the cord is evidently in one plane. Let the lines of action of the forces be vertical; then $\sin \mathrm{P}_1\mathrm{Q}_1\mathrm{Q}_2 = \sin \mathrm{P}_2\mathrm{Q}_2\mathrm{Q}_1$, $\sin \mathrm{P}_2\mathrm{Q}_2\mathrm{Q}_3 = \sin \mathrm{P}_3\mathrm{Q}_3\mathrm{Q}_2, \ldots\ldots$; so that if $\beta_1, \beta_2, \ldots$ are the angles of inclination of the successive lengths to the horizontal line,

$$\mathrm{T}_1 \cos \beta_1 = \mathrm{T}_2 \cos \beta_2 = \mathrm{T}_3 \cos \beta_3 = \ldots; \tag{5}$$

and therefore the successive tensions are inversely as the cosines of the angles of inclination to the horizon of the sides along which they act; and therefore if $k$ is the tension of a side which is horizontal, and if $\mathrm{T}$ is the tension along any side whose inclination to the horizontal line is $\beta$,

$$\mathrm{T} = k \sec \beta. \tag{6}$$

Suppose however that the vertical forces are the weights of the several parts of the cord, so that $\mathrm{P}_1, \mathrm{P}_2, \ldots$ are proportional to the lengths $\mathrm{A}\mathrm{Q}_1, \mathrm{Q}_1\mathrm{Q}_2, \ldots$; and moreover suppose that the lengths of the elements are infinitesimal, so that the polygon becomes a plane curve, then if the density and thickness of the cord are constant throughout, and if $\rho =$ the density, and $\omega =$ the area of a transverse section,

$$\mathrm{P} = \rho\omega g\, ds,$$

$$\sin\beta = \frac{dy}{ds}, \qquad \cos\beta = \frac{dx}{ds};$$

and if $\mathrm{T}$ and $\mathrm{T}'$ are the tensions at the beginning and end of an element respectively,

$$\left.\begin{aligned} \mathrm{T}'\cos\beta' &= \mathrm{T}\cos\beta + d.\mathrm{T}\cos\beta \\ \mathrm{T}'\sin\beta' &= \mathrm{T}\sin\beta + d.\mathrm{T}\sin\beta \end{aligned}\right\}; \tag{7}$$

therefore taking vertical forces

$$\mathrm{P} = d.\mathrm{T}\sin\beta;$$

and if $k = \rho\omega c g$,

$$\rho\omega g\, ds = d.\rho\omega c g \frac{dy}{dx},$$

and placing the origin at the lowest point of the curve,

$$s = c\frac{dy}{dx}; \tag{8}$$

from which equation all the properties of the curve, which is called the *Catenary*, may be deduced. I propose however to consider the problem of the equilibrium-form of a flexible and inextensible string under the action of given forces in a more general way.

134.] Suppose a perfectly flexible and inextensible string to be in space, and to be at all its parts subject to the action of certain given forces; let it be referred to a system of coordinate axes, and at the point $(x, y, z)$, let $\rho$ be the density, $\omega$ the area of a transverse section of the cord, and $ds$ the length-element; so that $\rho\omega ds$ is the mass-element of the cord. Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the impressed forces acting on an unit of mass at that point; so that the pressures acting on the mass-element at the point are

$$\rho\omega\mathrm{X}\, ds, \qquad \rho\omega\mathrm{Y}\, ds, \qquad \rho\omega\mathrm{Z}\, ds. \tag{9}$$

Let $\mathrm{T}$ be the tension of the cord at the point $(x, y, z)$; then as it acts along the curve, its resolved parts are

$$\mathrm{T}\frac{dx}{ds}, \qquad \mathrm{T}\frac{dy}{ds}, \qquad \mathrm{T}\frac{dz}{ds}; \tag{10}$$

and therefore at the point $(x+dx, y+dy, z+dz)$ the resolved parts of the tension are

$$\mathrm{T}\frac{dx}{ds} + d.\mathrm{T}\frac{dx}{ds}, \qquad \mathrm{T}\frac{dy}{ds} + d.\mathrm{T}\frac{dy}{ds}, \qquad \mathrm{T}\frac{dz}{ds} + d.\mathrm{T}\frac{dz}{ds}; \tag{11}$$

the tension varying continuously as we pass along the curve; now let us suppose $x, y, z$, and $s$ to increase simultaneously; then the element of the curve being in equilibrium under the action of the forces (9) (10) and (11), the conditions of equilibrium, equations (94), Art. 57, become

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds}+\rho\omega\text{X}\,ds &= 0\\ d.\text{T}\frac{dy}{ds}+\rho\omega\text{Y}\,ds &= 0\\ d.\text{T}\frac{dz}{ds}+\rho\omega\text{Z}\,ds &= 0\end{aligned}\right\}; \tag{12}$$

and from these equations all the properties of the curve are to be deduced.

First, integrating the equations, we have

$$\frac{\int\rho\omega\text{X}\,ds}{dx}=\frac{\int\rho\omega\text{Y}\,ds}{dy}=\frac{\int\rho\omega\text{Z}\,ds}{dz},$$

and therefore the numerators are proportional to the direction-cosines of the arc-element on which the forces act.

Also expressing at length the first terms of these equations, and taking $s$ to be equicrescent, we have

$$\left.\begin{aligned} \text{T}\,d.\frac{dx}{ds}+\frac{dx}{ds}d\text{T}+\rho\omega\text{X}\,ds &= 0\\ \text{T}\,d.\frac{dy}{ds}+\frac{dy}{ds}d\text{T}+\rho\omega\text{Y}\,ds &= 0\\ \text{T}\,d.\frac{dz}{ds}+\frac{dz}{ds}d\text{T}+\rho\omega\text{Z}\,ds &= 0\end{aligned}\right\}. \tag{13}$$

Multiply these equations severally by $dx, dy, dz$ and add, and we have

$$d\text{T}+\rho\omega\{\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz\}=0; \tag{14}$$

where $d\text{T}$ is the total differential of T. Let the integral of this be taken between the limits which carry the subscripts $n$ and 0; and we have

$$\text{T}_n-\text{T}_0+\int_0^n\rho\omega\{\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz\}=0. \tag{15}$$

If therefore $\rho, \omega, \text{X}, \text{Y}, \text{Z}$ are functions of the coordinates of the point to which they apply, and are such that the quantity under the sign of integration is a complete differential, then the difference between the tensions at the limits is a function of the coordinates of those points only, and is independent of the form of the curve which joins them.

135.] When the force, of which the components are X, Y, Z, is normal to the curve at all its points,

$$\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz=0, \tag{16}$$

then

$$\text{T}_n-\text{T}_0=0; \tag{17}$$

so that the tension of the string is the same at all its points.

The condition (16) holds good when the string is stretched on a smooth surface, and the pressure of the surface is the only acting force.

Also if the tension T is constant, (12) become

$$\left.\begin{aligned} \mathrm{T}\, d.\frac{dx}{ds} + \rho\omega \mathrm{X}\, ds &= 0 \\ \mathrm{T}\, d.\frac{dy}{ds} + \rho\omega \mathrm{Y}\, ds &= 0 \\ \mathrm{T}\, d.\frac{dz}{ds} + \rho\omega \mathrm{Z}\, ds &= 0 \end{aligned}\right\}; \tag{18}$$

$$\therefore \quad \mathrm{T}^2\left\{\left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2\right\} = \rho^2\omega^2(\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2)\,ds^2. \tag{19}$$

Let $\mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2 = \mathrm{P}^2$; then, by equation (23), Art. 325, Vol. I, if $\rho'$ is the radius of absolute curvature of the curve at the point $(x, y, z)$,

$$\frac{\mathrm{T}^2\, ds^2}{\rho'^2} = \rho^2\omega^2\mathrm{P}^2\, ds^2; \tag{20}$$

$$\therefore \quad \mathrm{T}^2 = \rho^2\omega^2\mathrm{P}^2\rho'^2, \qquad \mathrm{P} = \frac{\mathrm{T}}{\rho\omega\rho'}; \tag{21}$$

and therefore, whatever is the normal force, the form of the curve will be such, that the force varies inversely as the radius of absolute curvature, (see Art. 132). If therefore the force is constant and the curve is plane, it will be also the arc of a circle.

Again if we transfer to the right-hand members of the equations, the last term in each of (13), and then square them, and add them, we have, if $s$ is equicrescent,

$$\mathrm{T}^2\frac{ds^2}{\rho'^2} + d\mathrm{T}^2 = \rho^2\omega^2\mathrm{P}^2\, ds^2;$$

$$\therefore \quad \frac{\mathrm{T}_2}{\rho'^2} + \left(\frac{d\mathrm{T}}{ds}\right)^2 = \rho^2\omega^2\mathrm{P}^2. \tag{22}$$

Also from (13) eliminating T and $d$T, we have

$$\left(dz\, d.\frac{dy}{ds} - dy\, d.\frac{dz}{ds}\right)\mathrm{X} + \left(dx\, d.\frac{dz}{ds} - dz\, d.\frac{dx}{ds}\right)\mathrm{Y} + \left(dy\, d.\frac{dx}{ds} - dx\, d.\frac{dy}{ds}\right)\mathrm{Z} = 0; \tag{23}$$

$$\therefore \quad (dz\, d^2y - dy\, d^2z)\mathrm{X} + (dx\, d^2z - dz\, d^2x)\mathrm{Y} + (dy\, d^2x - dx\, d^2y)\mathrm{Z} = 0; \tag{24}$$

and as the former factors of each term are proportional to the direction-cosines of the binormal, we conclude that the impressed force lies in the osculating plane of the string-curve.

Also if $\phi$ is the angle between the line of action of P and the arc-element,

$$\mathrm{X}\, dx + \mathrm{Y}\, dy + \mathrm{Z}\, dz = \mathrm{P}\, ds \cos\phi;$$

therefore from (14)

$$d\mathrm{T} + \rho\omega\mathrm{P}\, ds \cos\phi = 0,$$

and substituting this value for $d\text{T}$ in (22) we have

$$\text{T} = \rho\rho'\omega\text{P}\sin\phi,$$

which are indeed the tangential and normal components: and hence, as before, if $\phi = 90°$, that is, if the line of action of P is normal to the curve

$$d\text{T} = 0, \qquad \therefore \quad \text{T} = \text{a constant},$$

$$\text{and} \qquad \text{T} = \rho\rho'\omega\text{P},$$

and therefore P varies inversely as the radius of absolute curvature.

136.] Next let us investigate the equations of equilibrium of a string stretched on a given curved surface, and under the action of certain forces by which it is kept on the surface; and let us suppose the surface to be smooth.

Let the equation to the surface be $\text{F}(x, y, z) = 0$; and let its partial derived functions be U, V, W; and let $\text{Q}^2 = \text{U}^2 + \text{V}^2 + \text{W}^2$: let $\text{R}\,ds$ be the pressure of the surface against the mass-element whose length is $ds$, so that the equations of equilibrium are

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds} + \rho\omega\text{X}\,ds + \text{R}\frac{\text{U}}{\text{Q}}\,ds &= 0 \\ d.\text{T}\frac{dy}{ds} + \rho\omega\text{Y}\,ds + \text{R}\frac{\text{V}}{\text{Q}}\,ds &= 0 \\ d.\text{T}\frac{dz}{ds} + \rho\omega\text{Z}\,ds + \text{R}\frac{\text{W}}{\text{Q}}\,ds &= 0 \end{aligned}\right\}. \qquad (25)$$

Multiply these equations severally by $dx, dy, dz$, and add, and let $s$ be equicrescent; then because

$$\text{U}\,dx + \text{V}\,dy + \text{W}\,dz = 0,$$

we have

$$d\text{T} + \rho\omega\{\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz\} = 0; \qquad (26)$$

and therefore if $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz = 0$, T is constant throughout the length of the string, whatever is the form of the surface.

Again, differentiating the first terms of (25), and multiplying the equations severally by $\frac{\text{U}}{\text{Q}}, \frac{\text{V}}{\text{Q}}, \frac{\text{W}}{\text{Q}}$, and adding, we have

$$\text{T}\left\{\frac{\text{U}}{\text{Q}}\,d.\frac{dx}{ds} + \frac{\text{V}}{\text{Q}}\,d.\frac{dy}{ds} + \frac{\text{W}}{\text{Q}}\,d.\frac{dz}{ds}\right\} + \rho\omega\,ds\left\{\frac{\text{XU} + \text{YV} + \text{ZW}}{\text{Q}}\right\} + \text{R}\,ds = 0; \quad (27)$$

and therefore if $\theta =$ the angle between the normal to the surface and the principal normal to the curve at a common point, and if $\phi =$ the angle between the normal to the surface and the line of action of the resultant of the impressed forces, viz., P, and if $\rho' =$ the radius of absolute curvature of the curve, we have

$$\frac{\text{T}\cos\theta}{\rho'} + \rho\omega\text{P}\cos\phi + \text{R} = 0. \qquad (28)$$

Again, suppose that $\text{X} = \text{Y} = \text{Z} = 0$, and that we differentiate the first terms of each of the equations (25), and eliminate $\text{T}$ and $d\text{T}$ by cross-multiplication, then

$$(dz\,d^2y - dy\,d^2z)\,\text{U} + (dx\,d^2z - dz\,d^2x)\,\text{V} + (dy\,d^2x - dx\,d^2y)\,\text{W} = 0; \quad (29)$$

and therefore the binormal of the curve is perpendicular to the normal of the surface; the curve therefore along which the string is laid is a geodesic line on the surface.

137.] To pass however from the general investigations of the preceding Articles to the particular cases: first let us suppose, in the case of the free catenary, the impressed forces to act in one plane, that of $xy$; then the equations (12) become

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds} + \rho\omega\text{X}\,ds &= 0 \\ d.\text{T}\frac{dy}{ds} + \rho\omega\text{Y}\,ds &= 0 \end{aligned}\right\}; \quad (30)$$

and therefore, as before,

$$d\text{T} + \rho\omega\,(\text{X}\,dx + \text{Y}\,dy) = 0: \quad (31)$$

whence the tension at any point may be determined for any given forces.

And if $\text{T}$ is constant, and $\text{P}$ is the impressed force of which $\text{X}$ and $\text{Y}$ are the components, and $\rho'$ is the radius of curvature at $(x, y)$,

$$\text{T}^2 = \rho^2\omega^2\text{P}^2\rho'^2. \quad (32)$$

Let us suppose gravity, or the earth's attraction, to be the only acting force, in which case the curve is the catenary, and let the axis of $x$ be horizontal, and that of $y$ vertical; then $\text{X} = 0$, $\text{Y} = -g$; so that the equations (30) become

$$d.\text{T}\frac{dx}{ds} = 0, \qquad d.\text{T}\frac{dy}{ds} - g\omega\rho\,ds = 0; \quad (33)$$

$$\therefore \quad \text{T}\frac{dx}{ds} = c; \quad (34)$$

where $c$ is the constant of integration, and is the value of $\text{T}$ when $\frac{dx}{ds} = 1$, that is, is the horizontal tension of the catenary. Also from (33) and (34)

$$c\frac{dy}{dx} = \int g\omega\rho\,ds: \quad (35)$$

and if $\omega$ and $\rho$ are constant throughout the curve

$$c\frac{dy}{dx} = g\omega\rho s, \quad (36)$$

if $s=0$, when $\frac{dy}{dx}=0$, that is, if $s$ is measured from the lowest point of the curve; and from (34), we infer that the horizontal tension of the catenary is constant throughout.

Let us however deduce the properties of the curve from first principles; and suppose the curve, see fig. 59, to be suspended from two fixed points, A and B, in the plane of the paper, which is supposed to be vertical; let C be the lowest point of the catenary, and let a vertical line through it be taken for the axis of $y$, and let the horizontal line, which will also touch the curve at C, be the axis of $x$. Let $\text{CM}=x$, $\text{MP}=y$, the arc $\text{CP}=s$, $\rho=$ density at P, $\omega=$ the area of the transverse section of the cord. Then the arc CP, after it has assumed its permanent form of equilibrium, may be considered as a rigid body, and kept at rest by three forces, (1) T the tension acting at P in the direction of the tangent, (2) the weight of the cord CP acting vertically downwards and which is equal to $\int_0^s g\rho\omega\, ds$, and (3) the horizontal tension at the lowest point C; as to the last force, let us suppose $\sigma$ to be the density of the cord at C, $a$ to be the area of a transverse section at the same point, and $c$ to be the length of cord such that $ga\sigma c$ is equal to the tension at C; then by the triangle of forces, these forces are proportional to the three lines PT′, T′N, NP, which their lines of action are respectively parallel to; and therefore we have

$$\frac{\text{T}}{\text{PT}'}=\frac{\int_0^s g\rho\omega\, ds}{\text{NT}'}=\frac{ga\sigma c}{\text{NP}}; \tag{37}$$

but $$\frac{\text{PT}'}{ds}=\frac{\text{T}'\text{N}}{dy}=\frac{\text{NP}}{dx};$$

$$\therefore\quad \frac{\text{T}}{ds}=\frac{\int_0^s g\rho\omega\, ds}{dy}=\frac{ga\sigma c}{dx}; \tag{38}$$

so that the equation to the curve is given by

$$a\sigma c\frac{dy}{dx}=\int_0^s \rho\omega\, ds; \tag{39}$$

and the tension at any point by the equation

$$\text{T}=ga\sigma c\frac{ds}{dx}. \tag{40}$$

138.] Let $\omega$ and $\rho$ be constant throughout the cord; so that the curve becomes that of a cord of constant thickness and density, suspended from two given points A and B: therefore $\rho=\sigma$, $\omega=a$; and from (39),

$$\frac{dy}{dx}=\frac{s}{c}; \tag{41}$$

therefore differentiating, and making $x$ equicrescent,

$$c\frac{d^2y}{dx^2}=\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}; \tag{42}$$

$$\frac{d.\frac{dy}{dx}}{\left\{1+\frac{dy^2}{dx^2}\right\}^{\frac{1}{2}}}=\frac{dx}{c}; \tag{43}$$

and integrating, and taking the limits such that $\frac{dy}{dx}=0$, when $x=0$, we have

$$\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}=\frac{x}{c};$$

$$\left.\begin{aligned}\therefore\quad \frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}&=e^{\frac{x}{c}}\\ \therefore\quad -\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}&=e^{-\frac{x}{c}}\end{aligned}\right\}; \tag{44}$$

$$\therefore\quad 2\frac{dy}{dx}=e^{\frac{x}{c}}-e^{-\frac{x}{c}}; \tag{45}$$

and integrating again, and observing that $y=0$, when $x=0$, we have

$$2y=c\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}-2c;$$

$$\therefore\quad y+c=\frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}. \tag{46}$$

Also equating the values of $\frac{dy}{dx}$ in (41) and (45) we have

$$s=\frac{c}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\}; \tag{47}$$

and either (46) or (47) is the equation to the catenary of constant thickness and density, when the lowest point of the curve is the origin, and the horizontal tangent at it is the axis of $x$.

To simplify the equation, let the origin be moved to a point O, see fig. 60, at a distance $c$ below C and on the axis of $y$, so that (46) becomes

$$y=\frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}, \tag{48}$$

and (47) is unaltered. The horizontal line through O is called the Directrix of the Catenary.

Now $c=$ OC is the length of a cord of the same thickness and density as the cord of the curve, the weight of which is equal to the tension of the curve at its lowest point: if therefore a smooth small pulley were placed at C, and if over it a cord of length $c$, and of the same thickness and density as the cord of the curve, and joined to the arc CP, were suspended, the curve would be in equilibrium.

From (47)
$$\frac{ds}{dx} = \frac{1}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\},$$
$$= \frac{y}{c}; \tag{49}$$
therefore from (40),
$$\text{T} = ga\sigma y; \tag{50}$$
that is, the tension at every point of the curve is equal to the weight of a cord of the same thickness and density, and equal in length to the ordinate at the point. The tension therefore is the least at the lowest point of the catenary, and increases directly as the ordinate: it is the same also for the two points in the same horizontal line. And therefore if, see fig. 61, a cord of constant thickness and density is suspended over two small pulleys A and B, and is at rest by means of certain lengths hanging over the pulleys, the two ends H and K are in the same horizontal line, and the tension at the lowest point C is equal to the weight of a cord similar in all respects, and whose length is CO.

139.] Let us investigate some of the more prominent geometrical properties of the catenary. From (47) and (48) we have
$$y = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}, \qquad s = \frac{c}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\};$$
$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\} = \frac{s}{c}, \tag{51}$$
$$\frac{d^2y}{dx^2} = \frac{1}{2c}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\} = \frac{1}{c}\frac{ds}{dx} \tag{52}$$
$$= \frac{y}{c^2}, \tag{53}$$
$$\therefore \quad y = c\frac{ds}{dx}, \tag{54}$$
Now as (48) is unaltered when $x$ is replaced by $-x$, it follows that the catenary is symmetrical with respect to the axis of $y$.

Also squaring (47) and (48) and subtracting, we have
$$y^2 - s^2 = c^2. \tag{55}$$
From the preceding equation it will be found that the radius

of curvature of the catenary is equal to the normal, and that the lines are drawn from the curve in opposite directions; hence the radius of curvature at C is equal to $c$. Also from (50)

$$\begin{aligned}
\text{T}^2 &= g^2 a^2 \sigma^2 y^2 \\
&= g^2 a^2 \sigma^2 (c^2 + s^2) \\
&= g^2 a^2 \sigma^2 c^2 + g^2 a^2 \sigma^2 s^2 \\
&= (\text{tension of curve at lowest point})^2 \\
&\qquad + (\text{weight of cord of length} = s)^2.
\end{aligned}$$

Also let a tangent PΠ, fig. 60, be drawn to the catenary at the point P, and from M, the foot of the ordinate, let a perpendicular to PΠ be drawn; then since $\frac{dx}{ds}$ is the sine of ΠPM,

$$\begin{aligned}
\Pi\text{M} &= y\frac{dx}{ds} \\
&= c;
\end{aligned}$$

and therefore from (53) or (41) PΠ $= s =$ the arc CP. Therefore the point Π is on the involute of the catenary which originates from the curve at C, and ΠM is a tangent to this involute; and as ΠM is the tangent to this last curve, and is equal to the constant quantity $c$, the involute is the equitangential curve or tractrix, the asymptote of which is the axis of $x$. Let therefore $\eta$ and $\xi$ be the current coordinates to this curve; ON $= \xi$, NΠ $= \eta$; then

$$\begin{aligned}
\frac{d\eta}{d\xi} &= \tan \Pi\text{M}x \\
&= -\tan \Pi\text{MN} \\
&= -\frac{\Pi\text{N}}{\text{NM}} = -\frac{\eta}{\{c^2-\eta^2\}^{\frac{1}{2}}},
\end{aligned}$$

which is the differential equation to the equitangential curve: see Ex. 1, Art. 101, Vol. II. And producing PΠ, so that it cuts the axis of $x$ in T, PΠ is the radius of curvature of the tractrix at the point Π, and ΠT is the normal; and therefore as ΠMT is a right-angle, PΠ × ΠT $=$ ΠM$^2$; therefore in the tractrix

$$\text{the radius of curvature} \times \text{the normal} = c^2.$$

Also the catenary at its lowest point approximately coincides with a conical parabola. For taking the equation (46), the origin of which is at the lowest point,

$$y + c = \frac{c}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right\}$$

$$y+c = \frac{c}{2}\left\{1+\frac{x}{c}+\frac{x^2}{1.2.c^2}+\frac{x^3}{1.2.3.c^3}+\dots\right.$$

$$\left.+1-\frac{x}{c}+\frac{x^2}{1.2.c^2}-\frac{x^3}{1.2.3.c^3}+\dots\right\}$$

$$= c\left\{1+\frac{x^2}{1.2.c^2}+\frac{x^4}{1.2.3.4.c^4}+\dots\right\};$$

and omitting terms which involve powers of $x$ higher than the second,

$$y = \frac{x^2}{2c};$$

$$\therefore\quad x^2 = 2cy;$$

the equation to a parabola, whose vertex is C, whose principal axis is C$y$, and whose latus rectum is $2c$.

140.] The constant $c$ which enters into the equations (47) and (48) is to be determined experimentally by means of the tension at the lowest point C. Suppose however that the data of the problem are different to those which we have taken. Suppose, for instance, that a homogeneous heavy cord of the length $2l$ is suspended from two points in the same horizontal line at a distance $2b$ apart, and that it is required to determine the equation of the catenary, the position of the lowest point, and the tension at every point.

Let the origin be taken at the point of bisection of the horizontal line which joins the two given points; see fig. 62; the horizontal line being the axis of $x$, and the vertical line measured downwards being the axis of $y$; OB = OB′ = $b$; let OC = $h$; so that the equations become

$$h+c-y = \frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right\}, \tag{56}$$

$$s = \frac{c}{2}\left\{e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right\}; \tag{57}$$

and in these we have to determine $h$ and $c$ in terms of $l$ and $b$.

Let $a$ be the angle at which the curve is inclined to OB at the point B; then since

$$\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\frac{dy}{dx} = e^{\frac{x}{c}},$$

$$\sec a+\tan a = e^{\frac{b}{c}}.$$

Also from (41) $\qquad \tan a = \dfrac{l}{c},$ (58)

$$\therefore \quad \frac{b}{l} = \cot a \log(\sec a + \tan a)$$

$$= \cot a \log \tan\left(45^\circ + \frac{a}{2}\right);$$

whence by successive approximation $a$ may be determined. And from (58)

$$c = l \cot a;$$

and from (56) and (57) we have

$$h + c = \frac{c}{2}\left\{e^{\frac{b}{c}} + e^{-\frac{b}{c}}\right\}, \qquad l = \frac{c}{2}\left\{e^{\frac{b}{c}} - e^{-\frac{b}{c}}\right\},$$

$$\therefore \quad h + l + c = c e^{\frac{b}{c}},$$

$$h = l\{\operatorname{cosec} a - \cot a\}$$

$$= l \tan \frac{a}{2}.$$

therefore the tension at the lowest point $= \rho g \omega l \cot a$,

and the tension at B, and at B$' = \rho g \omega l \operatorname{cosec} a$;

thus all the circumstances of the curve are determined.

Another problem of the same kind is, To determine the form and circumstances of the catenary when a heavy homogeneous string of given length is suspended over two smooth pulleys in the same horizontal line, and the ends of the string hang freely so that the string supports itself.

141.] To determine the position of the centre of gravity of the cord of the catenary of uniform thickness and density, beginning at the lowest point C; fig. 60.

$$\bar{x}\int g\rho\omega\, ds = \int g\rho\omega x\, ds, \qquad \therefore \quad \bar{x}\int ds = \int x\, ds;$$

$$\bar{x}s = xs - \int_0^x \frac{c}{2}\left\{e^{\frac{x}{c}} - e^{-\frac{x}{c}}\right\} dx$$

$$= xs - \frac{c^2}{2}\left\{e^{\frac{x}{c}} + e^{-\frac{x}{c}} - 2\right\}$$

$$= xs - cy + c^2,$$

$$\therefore \quad \bar{x} = x - \frac{c(y-c)}{s}; \qquad (59)$$

$$\bar{y}\int ds = \int y\, ds,$$

$$\bar{y}s = \int_0^x \frac{c}{4}\left(e^{\frac{x}{c}} + e^{-\frac{x}{c}}\right)^2 dx$$

$$= \frac{c^2}{8}\left\{e^{\frac{2x}{c}} - e^{-\frac{2x}{c}}\right\} + \frac{cx}{2}$$

$$\bar{y}s = \frac{sy}{2} + \frac{cx}{2};$$

$$\bar{y} = \frac{y}{2} + \frac{cx}{2s}. \qquad (60)$$

And by geometrical construction in fig. 60,

$$\bar{x} = \text{OT} + \text{T}\Pi, \qquad \bar{y} = \frac{\text{MP} + \text{OT}'}{2}.$$

In Art. 113 it has been proved that of all curves which a wire or a flexible string of uniform thickness and density and of given length with its ends at fixed points can assume, the catenary is that of which the centre of gravity has the lowest position. The form therefore which a heavy flexible cord of uniform thickness and density assumes when suspended from two fixed points is that of stable equilibrium.

142.] Next let us consider the circumstances of a heavy string of varying thickness and density, under the action of gravity only.

In this case, of course, the equations (38) hold good, and are sufficient, viz.

$$\frac{\text{T}}{ds} = \frac{\int_0^s g\rho\omega\, ds}{dy} = \frac{ga\sigma c}{dx}, \qquad (61)$$

$$\therefore \quad ga\sigma c\frac{dy}{dx} = \int_0^s g\rho\omega\, ds,$$

$$ga\sigma c\frac{d^2y}{dx^2} = g\rho\omega\frac{ds}{dx}; \qquad (62)$$

from which the variation of the density or of the thickness may be determined, when the catenarian curve is given; and the curve may be found, when the law of the thickness or of the density is given: also

$$\text{T} = ga\sigma c\frac{ds}{dx}; \qquad (63)$$

whereby the tension at any point of the curve may be found. Some examples are subjoined.

Ex. 1. It is required to determine the law of variation of the thickness of a heavy homogeneous string, that it may be in equilibrium in the form of a parabola with its vertex downwards and its axis vertical.

Let the equation be $x^2 = 4ay$;

$$\therefore \quad \frac{dx}{2a} = \frac{dy}{x} = \frac{ds}{(4a^2+x^2)^{\frac{1}{2}}}, \qquad \frac{d^2y}{dx^2} = \frac{1}{2a};$$

and therefore from (62), as $\rho$ is constant and equal to $\sigma$,

$$\omega = \frac{ac}{(4a^2+x^2)^{\frac{1}{2}}}, \qquad \text{T} = \frac{ga\sigma c}{2a}(4a^2+x^2)^{\frac{1}{2}};$$

therefore when $x$ is small, $\omega$ is constant, which fact is accordant with the proposition of Article 139.

Ex. 2. It is required to find the law of variation of the density of a heavy string of uniform thickness that it may hang in the form of a semicircle with its diameter horizontal under the action of gravity.

$$x^2 = 2ay-y^2; \qquad \therefore \quad \frac{dx}{a-y} = \frac{dy}{x} = \frac{ds}{a}.$$

Therefore from (62) $\quad \rho = \dfrac{a\sigma ca}{(a-y)^2},$

that is, the density varies inversely as the square of the depth below the horizontal diameter of the semicircle.

Also $\qquad \text{T} = \dfrac{ga\sigma ca}{a-y}.$

If therefore $y=a$, $\rho=\infty$, $\text{T}=\infty$: that is, the density and the tension are both infinite; and rightly so, because the string is vertical at the points of its support at the extremities of the horizontal diameter of the circle, and there is at them no counteracting horizontal force to balance the horizontal tension at the lowest point.

Ex. 3. To find the form of a heavy string, the thickness of which varies inversely as the square root of its length from the lowest point, when it is acted on by gravity.

In this case $\qquad \omega = \mu s^{-\frac{1}{2}};$

Therefore from (61)

$$\begin{aligned} ga\sigma c\frac{dy}{dx} &= \int_0^s g\rho\mu s^{-\frac{1}{2}}\,ds \\ &= 2g\rho\mu s^{\frac{1}{2}}; \end{aligned}$$

$$\frac{d\left(\dfrac{dy}{dx}\right)^2}{\left\{1+\dfrac{dy^2}{dx^2}\right\}^{\frac{1}{2}}} = \frac{4\rho^2\mu^2}{a^2\sigma^2c^2}\,dx; \qquad \therefore \quad \left\{1+\frac{dy^2}{dx^2}\right\}^{\frac{1}{2}}-1 = \frac{2\rho^2\mu^2}{a^2\sigma^2c^2}x,$$

because the origin is at the lowest point, where the curve is horizontal; and making obvious substitutions,

$$a\frac{dy}{dx} = (x^2+2ax)^{\frac{1}{2}},$$

whence the equation to the curve will be found without difficulty. Also

$$\text{T} = ga\sigma c \frac{a+x}{a}.$$

Ex. 4. To find the equation to the catenarian curve, when the weight of each element of the curve varies as the horizontal projection of it.

This case is approximately that of suspension bridges, in which each element of the chain has to bear that part of the roadway which corresponds to the horizontal projection of it.

In this case $\rho\omega g\, ds = \mu\rho\omega g\, dx$;

therefore from (61)

$$ga\sigma c \frac{dy}{dx} = \int_0^x \mu\rho\omega g\, dx$$

$$= \mu\rho\omega g x$$

$$\therefore \quad dy = \frac{1}{2k} x\, dx,$$

$$4ky = x^2;$$

the equation of a parabola with its axis vertical and vertex downwards.

Ex. 5. To determine the equation to the catenarian curve of uniform density, and the law of variation of the thickness, so that the thickness may be at all points porportional to the tension.

In this case $\omega = \mu \text{T};$ (64)

therefore (61) becomes

$$\frac{\text{T}}{ds} = \frac{\int_0^s g\rho\mu\text{T}\, ds}{dy} = \frac{ga\sigma c}{dx},$$

$$\therefore \quad ga\sigma c d.\frac{dy}{dx} = g\rho\mu\text{T}\, ds, \qquad d.\frac{dy}{dx} = g\rho\mu\frac{ds^2}{dx},$$

$$\frac{d.\frac{dy}{dx}}{1+\frac{dy^2}{dx^2}} = g\rho\mu\, dx,$$

$$\therefore \quad \log\sec \rho g\mu x = g\rho\mu y, \tag{65}$$

$$\sec g\rho\mu x = e^{g\rho\mu y}, \tag{66}$$

which is the equation to the required curve. And if we substitute $\frac{1}{a}$ for $g\rho\mu$ we have

$$e^{\frac{y}{a}} = \sec\frac{x}{a};$$

and if $y = \infty$, $x = \frac{\pi a}{2}$, which is therefore the greatest value of $x$. Also

$$\text{T} = ga\sigma c \sec g\rho\mu x, \tag{67}$$

$$\omega = \mu g a\sigma c \sec g\rho\mu x. \tag{68}$$

143.] In Article 113 it is shewn that of all uniform and heavy curved lines of given length joining two given points in the same vertical plane, the catenary is that of which the centre of gravity has the lowest position; I propose therefore to extend the problem to the case of heavy flexible strings of given varying density and thickness, and to find the form of the curve that the centre of gravity of it may be the lowest possible.

Let the axis of $z$ be vertical, and let the coordinates to any point on the curve be $x, y, z$, and let the element $ds$ begin at this point; let $\mu\, ds =$ the mass-element of the string-curve, where $\mu$ is a function of $x, y, z$; then $\bar{z}$ is to be a minimum, where

$$\bar{z}\int_0^1 \mu\, ds = \int_0^1 \mu z\, ds. \tag{69}$$

Now $\int_0^1 \mu\, ds$ is the mass of the string, and this evidently is constant, so that the variation of the right-hand member of (69) is to vanish consistently with this condition,

$$\therefore \quad \delta.\int_0^1 \mu z ds = 0, \qquad \text{and} \quad \delta.\int_0^1 \mu\, ds = 0; \tag{70}$$

from the former of which we have

$$\begin{aligned} 0 &= \int_0^1 \delta.\mu z\, ds \\ &= \int_0^1 \Big\{\mu z\Big(\frac{dx}{ds}\, d.\delta x + \frac{dy}{ds}\, d.\delta y + \frac{dz}{ds}\, d.\delta z\Big) + ds\, \delta.\mu z\Big\} \\ &= \Big[\mu z\Big(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\Big)\Big]_0^1 \\ &\quad + \int_0^1 \Big\{\Big(z\, ds\Big(\frac{d\mu}{dx}\Big) - d.\mu z\frac{dx}{ds}\Big)\delta x + \Big(z\, ds\Big(\frac{d\mu}{dy}\Big) - d.\mu z\frac{dy}{ds}\Big)\delta y \\ &\qquad \Big(\mu\, ds + z\, ds\Big(\frac{d\mu}{dz}\Big) - d.\mu z\frac{dz}{ds}\Big)\delta z\Big\}; \end{aligned} \tag{71}$$

and from the latter of (70),

$$
\begin{aligned}
0 &= \int_0^1 \delta.\mu\, ds \\
&= \left[\mu\left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right)\right]_0^1 \\
&\quad + \int_0^1 \left\{\left(ds\left(\frac{d\mu}{dx}\right) - \delta.\mu\frac{dx}{ds}\right)\delta x + \left(ds\left(\frac{d\mu}{dy}\right) - \delta.\mu\frac{dy}{ds}\right)\delta y\right. \\
&\qquad\qquad \left.\left(ds\left(\frac{d\mu}{dz}\right) - \delta.\mu\frac{dz}{ds}\right)\delta z\right\}. \qquad (72)
\end{aligned}
$$

Now for (71) and (72) to consist, it is necessary that

$$
\begin{aligned}
\frac{z\,ds\left(\frac{d\mu}{dx}\right) - d.\mu z\frac{dx}{ds}}{ds\left(\frac{d\mu}{dx}\right) - d.\mu\frac{dx}{ds}} &= \frac{z\,ds\left(\frac{d\mu}{dy}\right) - d.\mu z\frac{dy}{ds}}{ds\left(\frac{d\mu}{dy}\right) - d.\mu\frac{dy}{ds}} \\
&= \frac{\mu\,ds + z\,ds\left(\frac{d\mu}{dz}\right) - d.\mu z\frac{dx}{ds}}{ds\left(\frac{d\mu}{dz}\right) - d.\mu\frac{dx}{ds}} = \lambda, \qquad (73)
\end{aligned}
$$

where $\lambda$ is an undetermined constant; and from these equations, when $\mu$ is given, the equation to the catenary is to be deduced. If $\mu = 1$, the equations (73) become (27), Art. 113.

144.] Let us also consider some general properties of a string-curve which is under the action of certain forces of attraction or repulsion originating in fixed centres.

Let the central force acting on an unit of mass of the string be P, and let the centre of it be the origin; let the force be repulsive so that its tendency is to remove the molecules of the string further from the origin, and therefore the string will be concave towards it; if the force is attractive P will be affected with a negative sign and the string-curve will be convex towards the origin. The components of P along the coordinate-axes acting on an element of the curve which begins at the point $(x, y, z)$ and at a distance $r$ from the centre, are

$$
\mathrm{X} = \frac{\mathrm{P}x}{r}, \qquad \mathrm{Y} = \frac{\mathrm{P}y}{r}, \qquad \mathrm{Z} = \frac{\mathrm{P}z}{r};
$$

so that the equations (12) become

$$
\left.
\begin{aligned}
d.\mathrm{T}\frac{dx}{ds} + \rho\omega\, ds\,\frac{\mathrm{P}x}{r} &= 0 \\
d.\mathrm{T}\frac{dy}{ds} + \rho\omega\, ds\,\frac{\mathrm{P}y}{r} &= 0 \\
d.\mathrm{T}\frac{dz}{ds} + \rho\omega\, ds\,\frac{\mathrm{P}z}{r} &= 0
\end{aligned}
\right\}; \qquad (74)
$$

multiplying the second of these equations by $z$, and the third by $y$, and subtracting,

$$z\,d.\mathrm{T}\frac{dy}{ds} - y\,d.\mathrm{T}\frac{dz}{ds} = 0;$$

$$\left.\begin{aligned} \therefore\ \text{integrating}\quad & z\mathrm{T}\frac{dy}{ds} - y\mathrm{T}\frac{dz}{ds} = h_1 \\ \text{similarly}\quad & x\mathrm{T}\frac{dz}{ds} - z\mathrm{T}\frac{dx}{ds} = h_2 \\ & y\mathrm{T}\frac{dx}{ds} - x\mathrm{T}\frac{dy}{ds} = h_3 \end{aligned}\right\}; \tag{75}$$

and therefore multiplying these last equations severally by $x, y, z$, and adding,

$$h_1 x + h_2 y + h_3 z = 0; \tag{76}$$

which is the equation to a plane passing through the origin, which is the centre of force: whence we infer that the curve and the centre of force are in one and the same plane.

Let this plane be the plane of $xy$, so that $z = 0$; equations (74) become

$$\left.\begin{aligned} d.\mathrm{T}\frac{dx}{ds} + \rho\omega\,ds\frac{\mathrm{P}x}{r} = 0 \\ d.\mathrm{T}\frac{dy}{ds} + \rho\omega\,ds\frac{\mathrm{P}y}{r} = 0 \end{aligned}\right\}. \tag{77}$$

Differentiating the first terms of these equations, and eliminating $d\mathrm{T}$, we have

$$\mathrm{T}\frac{dy\,d^2x - dx\,d^2y}{ds} + \rho\omega\,ds\frac{\mathrm{P}}{r}(x\,dy - y\,dx) = 0.$$

If therefore $\rho'$ is the radius of curvature of the string-curve at the point $(x, y, z)$, and if $p$ is the perpendicular from the origin on the tangent at the same point,

$$\frac{\mathrm{T}}{\rho'} - \rho\omega\frac{\mathrm{P}p}{r} = 0;$$

$$\therefore\quad \mathrm{T}r - \rho\rho'\omega p\mathrm{P} = 0. \tag{78}$$

Again, differentiating the first terms of (77), and multiplying the first by $\frac{dx}{ds}$, and the second by $\frac{dy}{ds}$, and adding, we have

$$d\mathrm{T} + \mathrm{T}\left\{\frac{dx}{ds}\,d.\frac{dx}{ds} + \frac{dy}{ds}\,d.\frac{dy}{ds}\right\} + \rho\omega\mathrm{P}\frac{x\,dx + y\,dy}{r} = 0,$$

and if $s$ is equicrescent

$$\frac{dx}{ds}\,d.\frac{dx}{ds} + \frac{dy}{ds}\,d.\frac{dy}{ds} = 0;$$

$$\text{and}\qquad x^2 + y^2 = r^2, \qquad \therefore\quad x\,dx + y\,dy = r\,dr;$$

$$\therefore \quad d\mathrm{T} + \rho\omega\mathrm{P}\,dr = 0; \tag{79}$$

and therefore from (78), $d\mathrm{T} + \dfrac{\mathrm{T}}{\rho'}\,\dfrac{r\,dr}{p} = 0,$

but $\quad \rho' = \dfrac{r\,dr}{dp}; \qquad \therefore \quad \dfrac{d\mathrm{T}}{\mathrm{T}} + \dfrac{dp}{p} = 0;$

and integrating, if $\mathrm{T}_0$ and $p_0$ are simultaneous values of $\mathrm{T}$ and $p$, we have

$$\mathrm{T}p = \mathrm{T}_0 p_0 = \text{a constant}; \tag{80}$$

hence we conclude that the tension varies inversely as the perpendicular on the tangent from the centre of force; equation (79) is the equation of statical moments with respect to the centre of force, and might therefore have been deduced directly from equation (57), Art. 48.

Hence eliminating $\mathrm{T}$ between (78) and (80), and replacing $\rho'$ by its value $\dfrac{r\,dr}{dp}$, we have

$$\frac{dp}{p^2} - \frac{\omega\rho\mathrm{P}\,dr}{p_0\mathrm{T}_0} = 0;$$

$$\therefore \quad \frac{1}{p} = -\int \frac{\omega\rho\mathrm{P}\,dr}{p_0\mathrm{T}_0}; \tag{81}$$

the limits of the integral being given by the conditions of the problem. From (81), when $\mathrm{P}$ is given, the equation to the catenarian curve is to be found; and if the curve is given, $\mathrm{P}$ may be found; also from (79) or (80) the tension at any point of the curve may be found.

145.] As an example of the preceding expressions, let us suppose the force to vary inversely as the square of the distance, and to be attractive; and let us also suppose the cord or chain on which the force acts to be uniform in thickness and density. Then

$$\mathrm{P} = -\frac{\mu}{r^2};$$

therefore from (79)

$$\mathrm{T} - \mathrm{T}_0 + \rho\omega\mu\left(\frac{1}{r} - \frac{1}{r_0}\right) = 0;$$

where $\mathrm{T}_0$ and $r_0$ are initial given values of $\mathrm{T}$ and $r$.

Also from (81) $\quad \dfrac{1}{p} - \dfrac{1}{p_0} = \dfrac{\omega\rho\mu}{p_0\mathrm{T}_0}\left(\dfrac{1}{r_0} - \dfrac{1}{r}\right),$

and if $\dfrac{\omega\rho\mu}{p_0\mathrm{T}_0} = -c$, $\dfrac{1}{p_0} + \dfrac{\omega\rho\mu}{p_0 r_0\mathrm{T}_0} = -ck$, we have

$$\begin{aligned} \frac{1}{p} &= \frac{c}{r} - ck, \\ &= cu - ck, \end{aligned}$$

if $u = \frac{1}{r}$; and therefore by equation (19), Art. 221, Vol. I,

$$c^2u^2 - 2c^2ku + c^2k^2 = \frac{1}{p^2},$$
$$= u^2 + \frac{du^2}{d\theta^2};$$
$$\therefore \quad \frac{du^2}{d\theta^2} = (c^2-1)u^2 - 2c^2ku + c^2k^2; \tag{82}$$

and the integral of this equation will be of three different forms, according as $c$ is greater than, equal to, or less than, unity.

(1) Let $c^2$ be greater than unity; then if $c^2-1 = n^2$, the integral of (82) is of the form

$$u - a = \frac{b}{2}\{e^{n\theta} + e^{-n\theta}\}.$$

(2) Let $c^2 = 1$, then the integral is of the form

$$r = \frac{c}{1-\theta^2}.$$

(3) Let $c^2$ be less than unity; then if $1-c^2 = n^2$,

$$u - a = b\cos n\theta.$$

146.] There is one other class of problems connected with the present subject, and which must not be omitted. Suppose a perfectly flexible and inextensible string to be in contact with a *rough* surface under the action of given forces, and to be on the point of motion in the direction of its length, so that the friction of the surface is a force which acts in a direction contrary to that of the motion: then if $\text{R}\,ds$ is the pressure on the surface of a length-element of the string, and $\text{F}\,ds$ is the friction corresponding to $ds$, and $\mu$ is the coefficient of friction, see Art. 68,

$$\text{F}\,ds = \mu\text{R}\,ds; \tag{83}$$

and as F acts in the direction of the line along which motion is about to take place, the components of $\text{F}\,ds$ are

$$\text{F}\,dx, \qquad \text{F}\,dy, \qquad \text{F}\,dz,$$
$$\text{or,} \quad \mu\text{R}\,dx, \qquad \mu\text{R}\,dy, \qquad \mu\text{R}\,dz;$$

so that the equations of pressure become, see Art. 136,

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds} + \rho\omega\text{X}\,ds - \mu\text{R}\,dx + \text{R}\frac{\text{U}}{\text{Q}}\,ds &= 0 \\ d.\text{T}\frac{dy}{ds} + \rho\omega\text{Y}\,ds - \mu\text{R}\,dy + \text{R}\frac{\text{V}}{\text{Q}}\,ds &= 0 \\ d.\text{T}\frac{dz}{ds} + \rho\omega\text{Z}\,ds - \mu\text{R}\,dz + \text{R}\frac{\text{Z}}{\text{Q}}\,ds &= 0 \end{aligned}\right\}; \tag{84}$$

and from these equations general properties may be deduced.

Let us however consider a special case: that, viz., of a flexible cord, which I shall imagine to be without weight, in contact with a rough circular cylinder, in a plane perpendicular to the axis of the cylinder, and with given forces acting at the ends of the string.

Let fig. 64 represent the string resting on the cylinder, of which the plane of the paper is a section perpendicular to the axis of the cylinder: let the string be in contact with the cylinder over an arc which subtends at the centre the angle $\text{ACB} = a$; and let the tension at the ends of the string be F and F′; and these are also the tensions at A and B. Let $\text{AC} = a$, $\text{ACP} = \theta$, $\text{PCQ} = d\theta$; and let the line AC be the axis of $x$, and C be the origin. Then at the point $\text{P}(x, y)$, since $\text{X} = \text{Y} = 0$,

$$\left.\begin{aligned} d.\text{T}\frac{dx}{ds} - \mu\text{R}\,dx + \text{R}\,ds\frac{x}{a} &= 0 \\ d.\text{T}\frac{dy}{ds} - \mu\text{R}\,dy + \text{R}\,ds\frac{y}{a} &= 0 \end{aligned}\right\}; \tag{85}$$

therefore differentiating the first terms in each, and multiplying them respectively by $dx$ and $dy$, and observing that $x\,dx + y\,dy = 0$, because $x^2 + y^2 = a^2$,

$$d\text{T} - \mu\text{R}\,ds = 0. \tag{86}$$

And again, differentiating the first terms of (85), and multiplying the first by $dy$, and the second by $dx$, and subtracting,

$$\text{T}\left\{dy\,d.\frac{dx}{ds} - dx\,d.\frac{dy}{ds}\right\} + \frac{\text{R}\,ds}{a}(x\,dy - y\,dx) = 0; \tag{87}$$

$$\therefore \quad \text{T}\,\frac{dy\,d^2x - dx\,d^2y}{ds^2} + \frac{\text{R}}{a}(x\,dy - y\,dx) = 0;$$

$$\text{T} = a\text{R}. \tag{88}$$

And since $ds = a\,d\theta$, from (86) and (88) we have

$$\frac{d\text{T}}{\text{T}} = \mu\,d\theta, \qquad \log\frac{\text{T}}{\text{F}} = \mu\theta;$$

$$\therefore \quad \text{T} = \text{F}e^{\mu\theta}; \tag{89}$$

and therefore if F′ is the tension of the string at B, just as motion begins to take place,

$$\text{F}' = \text{F}e^{\mu a}; \tag{90}$$

therefore also $\qquad \text{T} = \text{F}'e^{\mu(\theta - a)};$

$$\therefore \quad \text{R} = \frac{\text{T}}{a} = \frac{\text{F}}{a}e^{\mu\theta}; \tag{91}$$

therefore the whole pressure on the cylinder

$$= \int_0^{a} \mathrm{R}\, a\, d\theta,$$

$$= \int_0^{a} \mathrm{F} e^{\mu\theta} d\theta,$$

$$= \frac{\mathrm{F}}{\mu}(e^{\mu a}-1). \tag{92}$$

Now from (90), $\mathrm{F}'-\mathrm{F} = \mathrm{F}(e^{\mu a}-1)$: if therefore a string, with the forces $\mathrm{F}$ and $\mathrm{F}'$ acting at its ends, passes over a rough cylinder, and if the part of the string in contact with the cylinder subtends an angle $a$ at the centre, motion will not take place so long as the difference between the two forces does not exceed

$$\mathrm{F}(e^{\mu a}-1). \tag{93}$$

147.] Ex. 1. A string passes over three rough cylindrical horizontal bars which are at equal distances apart, and the lower two of which are in the same horizontal plane; and at the ends of the string weights are suspended: find the difference between them just as motion begins to take place.

Let $\mathrm{F}$ be the less, and $\mathrm{F}'$ the greater weight, $\mathrm{T}_1$ the tension of the string between the first and the upper bars; $\mathrm{T}_2$ the tension between the upper and the third bars: therefore

$$\mathrm{T}_1 = \mathrm{F} e^{\mu\frac{\pi}{6}}, \qquad \mathrm{T}_2 = \mathrm{T}_1 e^{\mu\frac{2\pi}{3}}, \qquad \mathrm{F}' = \mathrm{T}_2 e^{\mu\frac{\pi}{6}};$$

$$\therefore \quad \mathrm{F}' = \mathrm{F} e^{\mu\pi}; \qquad \therefore \quad \mathrm{F}'-\mathrm{F} = \mathrm{F}(e^{\mu\pi}-1).$$

Ex. 2. A string passes over a rough horizontal cylinder; and two weights $\mathrm{P}$ and $\mathrm{Q}$ are suspended at its ends so that $\mathrm{P}$ is just beginning to descend; what weight must be added to $\mathrm{Q}$, so that $\mathrm{Q}$ may be beginning to descend?

Let $\mathrm{Q}'$ be the additional weight required; then we have

$$\mathrm{P} = \mathrm{Q} e^{\mu\pi}, \qquad \mathrm{Q}+\mathrm{Q}' = \mathrm{P} e^{\mu\pi};$$

$$\therefore \quad \mathrm{Q}' = \frac{\mathrm{P}^2-\mathrm{Q}^2}{\mathrm{Q}}.$$

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## SECTION 2.—*The action of forces on elastic bodies.*

148.] Our knowledge of the internal constitution of bodies is doubtless very imperfect; but so far as it goes there is no material substance in nature, the relative positions of the particles of which are not changed when the matter is acted on by external pressures: if a force acts on a body at a certain point, and in the way of pressure against it, the particles of the body at, or about, the point of application, approach to each other; and if the force is a pulling force, the distances between the constituent molecules of the body, at and about the point of application, are increased. It seems indeed that a body is made up of a system of molecules, infinitesimal in volume, and at an infinitesimal distance apart, and that these are held in a state of relative rest by forces acting reciprocally from one to another; and that these forces are functions of the distances between the molecules; and that when an external force acts on the system, the molecules are either separated farther from, or are brought nearer to, each other, by reason of the action of the force; so that either a compression or a dilatation of the system takes place; all bodies, that is, are compressible and extensible to a certain degree: the relative position of the molecules is not the same when the body is free from, and when it is subject to, external pressures. Into the particular mode of action of such forces on the constitution of a body, or the change of molecular action of the internal forces under the influence of such external force, I shall enter only briefly, and generally; the special study of such a subject belongs to other branches of physical mathematics; and now we have not data sufficient for the full solution of the problem. But I would observe, that our previous results of forces acting on rigid bodies, that is, on bodies the constituent molecules of which are in a state of relative rest, are not hereby falsified, because the molecules of the body though disturbed at first are ultimately in relative rest. It is the amount of this disturbance which we shall generally calculate: and upon the hypothesis of the truth of certain laws, which laws will not be deduced from more remote principles.

The disturbances or displacements which the molecules undergo are of three kinds: there may be (1) a longitudinal compression or dilatation; I shall calculate the effects of this on a bar or a string: (2) a flexure, as of a shell or a lamina: (3) a twisting or a torsion, as of a twisted bar. Now in all these, as in all similar displacements, there is the same result; no disturbance or disarrangement, at least within certain limits, takes place, unless there is also called into action a force of restitution, whereby the body tends to recover its former state; the molecular forces are such that, so long as temperature, &c., remain the same, they tend to bring the body back again into that state which it had before the disturbance due to the external force: this energy of restitution is called *Elasticity*; "La force elastique," says D'Alembert in the Encyclopédie, art. Élasticité, "est une propriété ou puissance des corps, au moyen de laquelle ils se rétablissent dans la figure et l'étendue, qu'une cause extérieure leur avait fait perdre." Thus elasticity in the first of the three cases mentioned above, is the tendency which a stretched string has to return to its former and unstretched length: in the second case it is the force of a spring, as that of a coil which is the motive power of a watch: in the third case it is the force of return which a twisted wire exhibits, as in Coulomb's Torsion Balance, or in Cavendish's experiment with leaden balls. Let this term then be plainly distinguished from expansibility, extensibility, compressibility, and so on: it is consequent upon these last, but expresses a property quite distinct from them; and the greater or less perfectness of elasticity of a given substance depends on the degree with which it recovers the state, as to the arrangement of its molecules, whence it has been displaced: if the state is altogether recovered, elasticity is perfect: if the body remains in the state into which it has been put by the disturbing force, it is said to be wholly inelastic: neither of these conditions is ever fully satisfied in nature. Thus much as to elasticity is sufficient for our present purpose.

149.] In the first place let us investigate the most simple case: that, viz., of an extensible string, which is stretched by the action of certain forces.

The law to which the extension is subject, and which is commonly called Hooke's law, is, The extension is as the tension: that is, the length added to an extensible string by means of a stretching force varies as the force. Also the same law may be supposed to be applicable to compression, that is, the compression varies as the compressing force. Suppose the length of an extensible string of an unit-length, and the area of whose transverse section is an unit, to be by the action of an unit-force increased by a quantity $e$, so that 1 becomes $1+e$; then, by reason of the preceding law, under the action of a force $\text{T}$, the length is increased by $e\text{T}$, so that 1 becomes $1+e\text{T}$; and therefore, the circumstances as to thickness, density, &c., of the string being the same throughout, the length of a string of length $a$ becomes $a(1+e\text{T})$; $e$ is called the coefficient of elasticity. If the stretching force is not the same throughout the length of the string, this formula is inapplicable as it stands; but we may resolve the string into infinitesimal parts, and apply it to each of these.

It is sometimes convenient to express $e$ in another form. Let $a'$ be the length of $a$ when stretched by the constant force T throughout; so that
$$a' = a(1+e\mathrm{T}), \tag{94}$$
and let E be the value of T, when $a$ is stretched so that its length is doubled:

then
$$2a = a(1+e\mathrm{E});$$
$$\therefore \quad e = \frac{1}{\mathrm{E}}; \tag{95}$$
and (94) becomes
$$a' = a\left(1+\frac{\mathrm{T}}{\mathrm{E}}\right): \tag{96}$$
E is called the Modulus of Elasticity.

Ex. 1. A heavy extensible string of constant thickness and density is suspended by one end, and hangs vertically; it is required to find the length of it thus stretched.

Let O, fig. 68, be the end by which it is suspended: $a =$ the length of it when unstretched: $\mathrm{OA} = a' =$ the length when stretched: $\rho =$ the density: $\omega =$ area of transverse section: $g =$ earth's attraction on an unit-mass: $\mathrm{OP} = x'$, $\mathrm{PQ} = dx'$: and suppose $x$ to be the distance of P from O, when the string is not stretched: then the weight of $\mathrm{PA} = \rho g\omega(a-x)$: and this is the stretching force on PQ: therefore
$$dx' = dx\,\{1+e\rho g\omega\,(a-x)\};$$
$$\therefore \quad \Big[x'\Big]_0^{a'} = \int_0^a \{1+e\rho g\omega\,(a-x)\}\,dx;$$
$$a' = a + \frac{e\rho g\omega a^2}{2},$$
$$= a\left\{1+\frac{e\rho g\omega a}{2}\right\}. \tag{97}$$

If the whole weight of the string had been collected at the lowest point, then
$$a_1 = a\{1+e\rho g\omega a\}; \tag{98}$$
and therefore by its own weight the string is stretched only half as much as it would be, if it were collected at its lowest point.

If $\rho$ or $\omega$ varies, the corresponding alteration must be made in the preceding integral.

Ex. 2. A heavy extensible string of constant thickness and density is suspended by one end, and hangs vertically; at a given point in it a weight is fixed; it is required to find the length of the string thus stretched.

Let O be the end by which it is suspended: let A be the point at which the weight, say W, is placed, $\text{OA} = a$, $\text{AB} = b$, $a$ and $b$ referring to the string unstretched: then, using the same symbols as in the preceding Article, we have

$$a_1 + b_1 = \int_0^a \{1 + \rho g \omega e(a+b-x) + e\text{W}\}\, dx + \int_0^b \{1 + \rho g \omega e(b-x)\}\, dx$$

$$= a + b + e\text{W}a + \frac{\rho g \omega e}{2}(a+b)^2.$$

Ex. 3. Two weights P and Q resting on two inclined planes, fig. 69, are connected by an elastic string PQ; it is required to find the position of equilibrium.

Let $\text{CP} = x$, $\text{CQ} = y$; let the inclinations to the horizon of CA, CB, PQ be $\alpha, \beta, \theta$; let the tension of $\text{PQ} = \text{T}$, and the unstretched length $= a$:

$$\therefore \quad \text{PQ} = a\{1 + e\text{T}\}.$$

Then resolving along the planes, and eliminating T, we have

$$\tan\theta = \frac{\text{Q}\sin\beta\cos\alpha - \text{P}\sin\alpha\cos\beta}{(\text{P}+\text{Q})\sin\alpha\sin\beta},$$

$$\text{PQ} = a\left\{1 + \frac{e\text{P}\sin\alpha}{\cos(\theta+\alpha)}\right\}. \tag{99}$$

Ex. 4. A heavy elastic ring is placed round a smooth vertical cone, and descends by its own weight; it is required to find the position of equilibrium.

Consider the cone to be the limiting form of a regular pyramid of $n$ sides, of which two adjacent ones are the triangles APQ, AQR in fig. 70: and let PQ and QR be two adjoining elements of the string which rest on these sides: let the triangles APQ and AQR be bisected by the lines A$p$ and A$q$ drawn to the middle points of their bases; and so that the string contained between $p$ and $q$ is the $n$th part of the whole ring. Let W = the weight of the ring, $a =$ the radius of it unstretched; $r =$ the radius of it stretched; $2\alpha =$ the vertical angle of the cone; then the weight of $p\text{Q}q = \frac{\text{W}}{n}$, and this resolved along $\text{AQ} = \frac{\text{W}}{n}\cos\alpha$: now the other forces acting on $p\text{Q}q$ are the two tensions along $p$Q and Q$q$, and these are equal to each other and to T (say); let $\text{AQP} = \text{AQR} = \beta$; and resolving along AQ, we have

$$\frac{\text{W}}{n}\cos\alpha = 2\text{T}\cos\beta;$$

$$\therefore\quad 2\pi r = 2\pi a\{1+e\text{T}\},$$

$$r = a\left\{1+\frac{e\text{W}}{2n}\frac{\cos a}{\cos\beta}\right\};$$

but $$\cos\beta = \frac{p\text{Q}}{\text{AQ}} = \frac{\pi\,\text{OQ}}{n\,\text{OQ}}\sin a;$$

$$\therefore\quad r = a\left\{1+\frac{e\text{W}\cot a}{2\pi}\right\}; \tag{100}$$

and this determines the position of the ring.

150.] The last example of the preceding Article differs from the former in that the string, by reason of its increased length, also undergoes a change of curvature; and this change of form is doubtless to a certain extent resisted, or favoured, as the case may be, by the elastic forces of the matter of the string: that is, by those forces of elasticity which affect the curvature of the string. And of these forces no account has been taken; the ring is supposed to be perfectly flexible, and yet extensible.

We may however consider, in a more general form, the curve which is taken by a string, perfectly flexible, and extensible according to Hooke's Law, under the action of given forces.

Let $ds'$ be a length-element of the curve before it is stretched, and $ds$ the corresponding length-element in its stretched state: then, if T is the tension,

$$ds = ds'(1+e\text{T}); \tag{101}$$

also let the thickness and density of the curve when stretched be the same throughout the length; this supposition is of course generally only approximately true in applications; and let X, Y, Z be the impressed forces acting on an unit-mass of the string before it is stretched, then the equations (12), Art. 134, become

$$\left.\begin{aligned} \rho\omega\text{X}\,ds + (1+e\text{T})\,d.\text{T}\frac{dx}{ds} &= 0 \\ \rho\omega\text{Y}\,ds + (1+e\text{T})\,d.\text{T}\frac{dy}{ds} &= 0 \\ \rho\omega\text{Z}\,ds + (1+e\text{T})\,d.\text{T}\frac{dz}{ds} &= 0 \end{aligned}\right\}; \tag{102}$$

from which the general properties of the curve are to be deduced, and the properties of any particular curve when the impressed forces are given.

Let $s$ be equicrescent; then, expanding the last terms of each of the equations (102), and multiplying the equations severally by $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, and adding, we have

$$\rho\omega\{\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz\}+(1+e\text{T})\,d\text{T} = 0. \tag{103}$$

And from (101), $e\,d\text{T} = d.\frac{ds}{ds'}$;

$$\therefore\quad \rho\omega\{\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz\}+\frac{1}{e}\frac{ds}{ds'}\,d.\frac{ds}{ds'} = 0,$$

$$\therefore\quad \rho\omega\{\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz\}+\frac{1}{2e}\,d.\left(\frac{ds}{ds'}\right)^2 = 0; \tag{104}$$

whence by integration $\frac{ds}{ds'}$, and thence the extension of the string, may be determined.

151.] Suppose however the string to be heavy, and gravity to be the sole acting force: the string-curve will manifestly be wholly in one vertical plane. Let the plane be that of $xy$, and let the axes of $x$ and $y$ be respectively horizontal and vertical: and let the curve be above the axis of $x$: then

$$\text{X} = \text{Z} = 0, \qquad \text{Y} = -g;$$

therefore from the first of (102)

$$d.\text{T}\frac{dx}{ds} = 0; \tag{105}$$

and therefore the horizontal tension is constant throughout the curve: let it be equal to the weight of a string of length $c$, the thickness and density of which are the same as those of the string-curve: then integrating (105) we have

$$\text{T}\frac{dx}{ds} = \rho\omega c g. \tag{106}$$

Again, from the second of (102),

$$\rho\omega g\,ds = (1+e\text{T})\,d.\text{T}\frac{dy}{ds},$$

$$\therefore\quad ds = c\left(1+e\rho\omega c g\frac{ds}{dx}\right)d.\frac{dy}{dx}. \tag{107}$$

To integrate these, let $\frac{dy}{dx} = \tan\tau$; therefore from (107),

$$ds = c\,(1+e\rho\omega c g\sec\tau)\,d.\tan\tau, \tag{108}$$

$$\therefore\quad \left.\begin{aligned} dx &= ds\cos\tau = c\,(\cos\tau+e\rho\omega c g)\,d.\tan\tau, \\ dy &= ds\sin\tau = c\,(\sin\tau+e\rho\omega c g\tan\tau)\,d.\tan\tau \end{aligned}\right\}; \tag{109}$$

and integrating

$$x = c\left\{\log\tan\left(\frac{\pi}{4}+\frac{\tau}{2}\right)+e\rho\omega cg\tan\tau\right\}, \tag{110}$$

$$y = c\left\{\sec\tau+\frac{e\rho\omega cg}{2}(\tan\tau)^2\right\}, \tag{111}$$

the limits of integration being such that $x=0$, $y=c$, when $\tau=0$; so that the axis of $x$ is the directrix of the curve, and at a distance $c$ below the lowest point of the curve; and the axis of $y$ passes through the lowest point. Also from (108) and (106) we have,

$$s = c\left\{\tan\tau+\frac{e\rho\omega cg}{2}(\tan\tau\sec\tau+\log(\tan\tau+\sec\tau))\right\}, \tag{112}$$

$$\mathrm{T} = \rho\omega cg\sec\tau. \tag{113}$$

If $\tau$ is eliminated by means of (110) and (111), the resulting equation is that of the string-curve: the expression however is so complicated that it is not worth while to write it at length. But in the case wherein $e$ is small, and the second and higher powers of it may be neglected without appreciable error, from (110) we have,

$$\sin\tau = \frac{e^{\frac{x}{c}}-e^{-\frac{x}{c}}}{e^{\frac{x}{c}}+e^{-\frac{x}{c}}}; \tag{114}$$

and therefore from (111)

$$y = \frac{c}{2}\left\{e^{\frac{x}{c}}+e^{-\frac{x}{c}}+\frac{e\rho\omega cg}{4}\left(e^{\frac{x}{c}}-e^{-\frac{x}{c}}\right)^2\right\}, \tag{115}$$

which is the equation to the catenary of slight extensibility. Also to determine the increase of the length of the arc, in this case we have from (101),

$$\begin{aligned} ds' &= \frac{ds}{1+e\mathrm{T}} \\ &= ds(1-e\mathrm{T}); \end{aligned}$$

therefore from (113), and neglecting terms in (111) involving $e$,

$$\begin{aligned} ds' &= ds-e\rho\omega cg\sec\tau\,ds \\ &= ds-e\rho\omega gy\,ds, \\ \therefore\quad ds-ds' &= e\rho\omega gy\,ds, \\ \therefore\quad s-s' &= e\rho\omega g\int y\,ds. \end{aligned} \tag{116}$$

Now if $\bar{y}$ is the distance from the directrix of the centre of gravity of the arc $s$,

$$\bar{y}s = \int y\,ds;$$

$$\therefore\quad s-s' = e\rho\omega gs\bar{y};$$

that is, the increase of the arc $s$ due to the tension varies as the distance from the directrix of the centre of gravity of the arc.

152.] Instead however of proceeding farther into the discussion of particular problems of elastic strings, elastic membranes, and springs, it is better to enter at once on the general investigation of the equations of equilibrium of elastic bodies: because these may be applied to the solution of particular cases, and because they, in an extended sense, will be required in a future part of our work. But to this end some of the more prominent physical properties of such bodies must be stated with greater precision than heretofore.

A body consists of material molecules, the relative positions of which are assigned by the action of certain forces of which the molecules are the subjects: and a body is supposed to be *rigid*, either when the relative position of the molecules is not changed, or, if changed, when the displacements of them are infinitesimally small, and not taken account of, by the action of the other external forces. The internal forces which act on the molecules are of two kinds: firstly, there is an attractive force arising from the mutual action of the molecules, and which is commonly called the force of cohesion: and, secondly, a repulsive force depending on the heat of the body; and such that, when the heat of the body is increased, the repulsive force is also increased: when these two forces balance each other, the molecules of the body are at relative rest. The action of the former is the subject of inquiry in the following Articles. And I shall suppose no change of heat in the body to take place owing to a change of temperature in the surrounding medium; and no increase of heat to be developed in the body itself by reason of the change of position of the constituent molecules: so that hereby our inquiry will be restricted to the *change only* of the molecular attractions or repulsions; to the change *only*, I say; because the original molecular forces and the heat-forces are in equilibrium of themselves. This hypothesis is doubtless only partially verified in nature: for no change of position of the molecules—and it is such a change which an alteration of the molecular forces necessitates—can take place without velocity being communicated and absorbed, and perhaps transmuted into heat: this change however of the heat-forces must be neglected at present.

153.] Imagine a body to be under the action of no external pressure, and its molecules to be acted on by no external forces, so that the molecules are in a state of relative rest from the action of the molecular and the heat- forces: also imagine, by reason of some external force, the distance between two molecules, which are so near as to attract each other, to be altered; then a new force will be brought into action, and this will be attractive or repulsive according as the original distance between the molecules is increased or diminished. Suppose the original distance to be $r$, and the increase or diminution of it, which I shall call the dilatation, to be $dr$: then the force thus brought into action is a function of the original distance $r$ and also of $dr$. We know that this force is zero, when $dr = 0$, whatever is the distance $r$; and therefore $dr$ must enter as a factor into its mathematical expression: and as $dr$ is infinitesimal, we may reasonably suppose that it enters in the first degree, and that all higher powers of it must be neglected; also the force decreases rapidly as $r$ increases, and vanishes when $r$ becomes finite. The law of it therefore may be expressed by

$$\mathrm{F}(r)\, dr, \tag{117}$$

where $\mathrm{F}(r)$ is a function of $r$ which vanishes when $r$ is finite. Of the form or properties of $\mathrm{F}(r)$, we can say no more at present.

The greatest value of $r$ for which the molecular forces act on a given molecule is called *the radius of the sphere of molecular activity:* so that the centre-molecule is acted on by all the molecules within the sphere of activity, and by none outside of it; the sphere therefore includes many molecules, and the radius of the sphere is infinitely greater than the distance between any two adjacent molecules, whatever is the molecular constitution of the body. Now imagine P to be a molecule of an elastic body, and imagine the sphere of activity to be described about P as a centre: let any plane be drawn through P, whereby the sphere is divided into two hemispheres: and at P in the dividing plane let an infinitesimal surface-element $\omega$ be taken, and imagine, on $\omega$ as a base, a cylinder to be taken which is bounded by the surface of the sphere of activity: let us confine our attention to one of the hemispheres. When the molecules are displaced by the action of external forces, the small cylinder standing on $\omega$ is displaced, and a force acts on $\omega$, which is proportional to $\omega$, and which may therefore be represented by $e\omega$, where $e$ is called the coefficient of elasticity, and $e\omega$ is the elastic force exercised by the hemisphere on $\omega$; the line of action of this elastic force is, in general, inclined to the normal of the plane of $\omega$, and the force may be either attractive or repulsive along its line of action: if the line of action of $e\omega$ is normal to $\omega$, the elastic force is an attraction or a repulsion according as it tends to draw $\omega$ from, or to press it towards, the dividing plane of the sphere. If the line of action of the elastic force is parallel to the plane of $\omega$, it tends to make the cylinder slide on the dividing plane, and it is then called the tangential elastic force.

In a similar way if the cylinder raised on $\omega$ is in the other hemisphere of the sphere of activity, the resultant of the forces which act on this cylinder is the elastic force $e'\omega$, which varies as $\omega$, and whose line of action may have any direction. If the plane $\omega$ is at rest, the elastic forces $e\omega$ and $e'\omega$ which act on it are equal, and have the same line of action, and act in opposite directions.

The elastic force $e\omega$, corresponding to $\omega$, will doubtless not be the same, throughout every body, as to intensity and line of action, even if $\omega$ remains equal and moves parallel to itself; also at a point P, it will generally vary as the direction of $\omega$ is changed, that is, as the position of the dividing plane of the sphere of activity is changed. In crystallised bodies, this is evidently the case: because such a body consists of molecules which have different attractive powers in different directions, and at their different faces, and admit of different compressions. And, similarly, if the texture of the body is fibrous, the elastic force in the direction of the fibre is doubtless different to that which is transverse to the fibre. Also in bodies of such textures the surface which limits the distance of molecular activity is generally not a sphere: a sphere however is sufficient for our definition of elastic force. Thus in the functional expression for the molecular force (117), $F(r)$ will vary as the direction in which $r$ is taken, and will therefore be a function of the direction-angles which determine $r$: and it will also generally change with the temperature, and as the temperature varies from one point to another, so will also $F(r)$.

Our problem is this: we shall consider a body of any material and form, the constituent molecules of which are subject to forces of the kind which we have stated. Other external forces act on it and produce displacements of the molecules, and hereby bring elastic forces into action: the effects of these forces are to be determined.

154.] Let $\omega$ be an infinitesimal surface-element in the interior of a solid and elastic body, and let the elastic force acting on it due to the action of given external forces be $e\omega$; suppose the direction-angles of the line of action of this elastic force to be $\alpha, \beta, \gamma$: so that the components of it along three rectangular axes of coordinates are

$$e\omega \cos\alpha, \quad e\omega\cos\beta, \quad e\omega\cos\gamma;$$

then, by the preceding Article, these quantities are functions of $x, y, z$, the coordinates of $\omega$, and of the direction-angles of the normal of $\omega$.

Let the solid body, the equilibrium of the elastic forces of which we are about to consider, be referred to a system of rectangular axes: and let the coordinates to a certain volume-element of it be $x, y, z$: let $\rho$ be the density of the volume-element: then the mass-element is $\rho\, dx\, dy\, dz$: also let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the components of the impressed external pressures acting on a mass-unit; then the components of the impressed pressures acting on the mass-element are

$$\rho\mathrm{X}\, dx\, dy\, dz, \quad \rho\mathrm{Y}\, dx\, dy\, dz, \quad \rho\mathrm{Z}\, dx\, dy\, dz: \tag{118}$$

and as the elastic forces, which these bring into action, are in equilibrium with these pressures, they must satisfy the six equations of equilibrium. Let us therefore investigate the elastic forces which are active on the mass-element.

Now $dy\, dz$ is the area of that face of the element which is perpendicular to the axis of $x$: let the elastic coefficient corresponding to it be $e_1$, and let the direction-angles of the elastic force be $\alpha_1 \beta_1 \gamma_1$, so that the components of the elastic force acting on the face $dy\, dz$ are

$$dy\, dz\, e_1 \cos\alpha_1, \quad dy\, dz\, e_1\cos\beta_1, \quad dy\, dz\, e_1\cos\gamma_1; \tag{119}$$

hence also the components of the elastic force acting on the opposite face of the elementary parallelepipedon are

$$dy\, dz\left\{e_1\cos\alpha_1 + \frac{d.e_1\cos\alpha_1}{dx}dx\right\}, \quad dy\, dz\left\{e_1\cos\beta_1 + \frac{d.e_1\cos\beta_1}{dx}dx\right\},$$
$$dy\, dz\left\{e_1\cos\gamma_1 + \frac{d.e_1\cos\gamma_1}{dx}dx\right\}; \tag{120}$$

and these of course act in directions opposite to those of (119), and the differences are

$$-dx\,dy\,dz\frac{d.e_1\cos\alpha_1}{dx},\quad -dx\,dy\,dz\frac{d.e_1\cos\beta_1}{dx},\quad -dx\,dy\,dz\frac{d.e_1\cos\gamma_1}{dx};\quad (121)$$

similarly, if the subscripts 2 and 3 refer to the faces perpendicular to the axes of $y$ and $z$, we have the components of the differences between the elastic forces on the opposite faces,

$$-dx\,dy\,dz\frac{d.e_2\cos\alpha_2}{dy},\quad -dx\,dy\,dz\frac{d.e_2\cos\beta_2}{dy},\quad -dx\,dy\,dz\frac{d.e_2\cos\gamma_2}{dy},\quad (122)$$

$$-dx\,dy\,dz\frac{d.e_3\cos\alpha_3}{dz},\quad -dx\,dy\,dz\frac{d.e_3\cos\beta_3}{dz},\quad -dx\,dy\,dz\frac{d.e_3\cos\gamma_3}{dz}.\quad (123)$$

And these forces are in the aggregate to balance the impressed pressures given in (118), so that

$$\left.\begin{aligned}\frac{d.e_1\cos\alpha_1}{dx}+\frac{d.e_2\cos\alpha_2}{dy}+\frac{d.e_3\cos\alpha_3}{dz}+\rho\textsc{x}&=0\\ \frac{d.e_1\cos\beta_1}{dx}+\frac{d.e_2\cos\beta_2}{dy}+\frac{d.e_3\cos\beta_3}{dz}+\rho\textsc{y}&=0\\ \frac{d.e_1\cos\gamma_1}{dx}+\frac{d.e_2\cos\gamma_2}{dy}+\frac{d.e_3\cos\gamma_3}{dz}+\rho\textsc{z}&=0\end{aligned}\right\}.\quad (124)$$

These equations however admit of simplification by reason of the equations of moments given in (95), Art. 57, which signify that the moment-axes of the couples turning the element about three axes parallel to the coordinate axes vanish. For suppose the three rotation-axes of these couples to pass through the centre of the element: then X, Y, Z act at the centre, and have no effects of rotation: similarly the lines of action of $e_1\cos\alpha_1$, $e_2\cos\beta_2$, $e_3\cos\gamma_3$, and of their increments pass through the centre of the element, and have no effects of rotation.

And as to the other faces; for the moment-axis of the couple whose rotation-axis is perpendicular to the face $dy\,dz$, we have no moment of rotation for $e_1\cos\beta_1$ and for $e_1\cos\gamma_1$ and their increments, because their lines of action pass through the centre of the face; nor for $e_2\cos\alpha_2$ and $e_3\cos\alpha_3$ and their increments, because their lines of action are parallel to the axis of $x$: so that there remain only $e_2\cos\gamma_2$ and $e_3\cos\beta_3$ and their increments; and in respect of these we have

$$e_2\cos\gamma_2\frac{dx\,dy\,dz}{2}+\left\{e_2\cos\gamma_2+\frac{d.e_2\cos\gamma_2}{dy}dy\right\}\frac{dx\,dy\,dz}{2}$$
$$-e_3\cos\beta_3\frac{dx\,dy\,dz}{2}-\left\{e_3\cos\beta_3+\frac{d.e_3\cos\beta_3}{dz}dz\right\}\frac{dx\,dy\,dz}{2}=0.$$

Therefore, neglecting infinitesimals of the higher orders,

$$e_2 \cos\gamma_2 - e_3 \cos\beta_3 = 0: \tag{125}$$

for a similar reason

$$e_3 \cos a_3 - e_1 \cos\gamma_1 = 0, \tag{126}$$

$$e_1 \cos\beta_1 - e_2 \cos a_2 = 0; \tag{127}$$

therefore of the nine elastic forces contained in (124), only six are independent.

Let the components of the elastic forces which are normal to the three sides of the parallelepipedon, respectively perpendicular to the axes of $x, y, z$, be $\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3$; and let

$$\left.\begin{array}{l} e_2 \cos\gamma_2 = e_3 \cos\beta_3 = \mathrm{T}_1 \\ e_3 \cos a_3 = e_1 \cos\gamma_1 = \mathrm{T}_2 \\ e_1 \cos\beta_1 = e_2 \cos a_2 = \mathrm{T}_3 \end{array}\right\}; \tag{128}$$

where $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$ are the tangential forces which tend to turn the element about axes through its centre severally parallel to the axes of $x, y, z$: so that (124) become

$$\left.\begin{array}{l} \dfrac{d\mathrm{N}_1}{dx} + \dfrac{d\mathrm{T}_3}{dy} + \dfrac{d\mathrm{T}_2}{dz} + \rho\mathrm{X} = 0 \\ \dfrac{d\mathrm{T}_3}{dx} + \dfrac{d\mathrm{N}_2}{dy} + \dfrac{d\mathrm{T}_1}{dz} + \rho\mathrm{Y} = 0 \\ \dfrac{d\mathrm{T}_2}{dx} + \dfrac{d\mathrm{T}_1}{dy} + \dfrac{d\mathrm{N}_3}{dz} + \rho\mathrm{Z} = 0 \end{array}\right\}; \tag{129}$$

which equations express the equilibrium of the elementary parallelepipedon.

155.] Thus also will the sum of them express the equilibrium of any finite portion of a solid body, whatever is its position and its form, provided that it is decomposable into rectangular parallelepipeda, the forces of which are parallel to the coordinate planes. If however the bounding facets of the body are inclined to the coordinate planes, the residue of the body over and above these parallelepipeda will consist of tetrahedra, the equilibrium of which under the action of forces acting on their surfaces requires further investigation.

The tetrahedra, according to the mode of resolution explained above, will have three faces at right angles to each other: let the fourth face, or the base, be inclined to the coordinate planes at the angles whose direction-cosines are $l, m, n$: so that these are also the direction-cosines of angles which its normal makes with the coordinate axes. Let the area of this face be $\omega$, and let the external forces acting on it be $\omega\mathrm{X}', \omega\mathrm{Y}', \omega\mathrm{Z}'$. Let the elastic forces acting on the three facets which are perpendicular to each other be expressed by the same symbols as in the last Article: and the area of these three facets are $\omega l$, $\omega m$, $\omega n$: so that the equations of equilibrium are

$$\left.\begin{aligned} l\mathrm{N}_1+m\mathrm{T}_3+n\mathrm{T}_2-\mathrm{X}' &= 0 \\ l\mathrm{T}_3+m\mathrm{N}_2+n\mathrm{T}_1-\mathrm{Y}' &= 0 \\ l\mathrm{T}_2+m\mathrm{T}_1+n\mathrm{N}_3-\mathrm{Z}' &= 0 \end{aligned}\right\}; \qquad (130)$$

the three equations of moment-axes have already been introduced into these equations by the same process as in the last Article.

156.] For the equilibrium of a finite portion of an elastic solid, (129) and (130) must be satisfied throughout the body and its whole surface respectively. Let us therefore multiply the first of (129) by $dx\,dy\,dz$, and integrate throughout the body, so that we have

$$\iiint \frac{d\mathrm{N}_1}{dx}\,dx\,dy\,dz + \iiint \frac{d\mathrm{T}_3}{dy}\,dx\,dy\,dz + \iiint \frac{d\mathrm{T}_2}{dz}\,dx\,dy\,dz$$
$$+\iiint \rho \mathrm{X}\,dx\,dy\,dz = 0. \quad (131)$$

In the first term of this expression, let the $x$-integration be performed; whereby we have

$$\iint \Big[\mathrm{N}_1\Big]_0^1 dy\,dz; \qquad (132)$$

$\Big[\mathrm{N}_1\Big]_0^1$ being the definite integral of $\mathrm{N}_1$, the limits of which are given by the bounding surface of the solid. Now if $\omega_1$ and $\omega_0$ are the areas of the facets of the small tetrahedra at these limits, and $l_1\, m_1\, n_1$, $l_0\, m_0\, n_0$ are the direction-cosines of the normals of these facets, since $dy\,dz = l\omega$, (132) is the difference between the values of $\mathrm{N}_1 \omega l$ at these two limits; and which we may represent by $\Sigma.\mathrm{N}_1\omega l$, wherein the sign of summation extends to the whole bounding surface of the body: and operating by a similar process on the second and third terms of (131) we have

$$\Sigma.\{\mathrm{N}_1 l+\mathrm{T}_3 m+\mathrm{T}_2 n\}\,\omega + \iiint \rho\mathrm{X}\,dx\,dy\,dz = 0; \qquad (133)$$

and therefore substituting from (130) we have

$$\Sigma.\mathrm{X}'\omega + \iiint \rho\mathrm{X}\,dx\,dy\,dz = 0,$$

and using the notation of Art. 134, Vol. II, and writing down the equations which follow from the second and third of (129), we have

$$\left.\begin{aligned}\iint \mathrm{X}'\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy+\iiint\rho\mathrm{X}\,dx\,dy\,dz&=0\\ \iint \mathrm{Y}'\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy+\iiint\rho\mathrm{Y}\,dx\,dy\,dz&=0\\ \iint \mathrm{Z}'\frac{\mathrm{Q}}{\mathrm{W}}\,dx\,dy+\iiint\rho\mathrm{Z}\,dx\,dy\,dz&=0\end{aligned}\right\};\qquad(134)$$

which equations express the conditions that the sum of all the components of all the forces along the three coordinate axes are equal to zero.

Again multiplying the second of (129) by $z$, and the third by $y$, and subtracting, we have

$$\frac{d(z\mathrm{T}_3-y\mathrm{T}_2)}{dx}+\frac{d(z\mathrm{N}_2-y\mathrm{T}_1)}{dy}+\frac{d(z\mathrm{T}_1-y\mathrm{N}_3)}{dz}+\rho(z\mathrm{Y}-y\mathrm{Z})=0,\quad(135)$$

the differentiations being partial. Now if we multiply this expression by $dx\,dy\,dz$, and integrate once each of the triple integrals, and follow a process similar to that pursued above, we shall have

$$\left.\begin{aligned}\iint\frac{\mathrm{Q}}{\mathrm{W}}(z\mathrm{Y}'-y\mathrm{Z}')\,dx\,dy+\iiint\rho(z\mathrm{Y}-y\mathrm{Z})&=0\\ \iint\frac{\mathrm{Q}}{\mathrm{W}}(x\mathrm{Z}'-z\mathrm{X}')\,dx\,dy+\iiint\rho(x\mathrm{Z}-z\mathrm{X})&=0\\ \iint\frac{\mathrm{Q}}{\mathrm{W}}(y\mathrm{X}'-x\mathrm{Y}')\,dx\,dy+\iiint\rho(y\mathrm{X}-x\mathrm{Y})&=0\end{aligned}\right\};\qquad(136)$$

the last two following severally from the third and first, and from the first and second, of (129). Thus there are six equations of equilibrium of an elastic solid body acted on by any forces.

157.] The preceding investigation shews that the equations of equilibrium of an elemental parallelepipedon, combined with those for the equilibrium of an elemental tetrahedron are sufficient for the equilibrium of the whole elastic body: we return now to the consideration of the element, and to the calculation of the elastic forces which act on it; the circumstances of the elements in the interior of the body thus come under consideration, as well as the determination of certain properties of the six elastic forces which are symbolized by the N's and the T's. Now the functions, which these symbols express, must satisfy the three equations (129), and also be subject to the conditions at the surface given by (130); but three partial differential equations of the first order will give us only three undetermined functions, and the three conditions at the surface can introduce only relations which these must satisfy; the six quantities, the N's and the T's, must depend therefore on three functions.

Let the elastic body be referred to a system of rectangular axes: let $(x, y, z)$ be the position of a molecule in it: let $(x', y', z')$ be the position of another molecule within the sphere of activity of the former: and let

$$x' = x+h, \qquad y' = y+k, \qquad z' = z+l: \tag{137}$$

and let $r$ be the distance between these two molecules: now by the action of external forces let the positions of these molecules be changed, so that $(x, y, z)$ becomes $(x+\xi, y+\eta, z+\zeta)$, and $(x', y', z')$ becomes $(x'+\xi', y'+\eta', z'+\zeta')$, and $r$ becomes $r+dr$: then, because $\xi, \eta, \zeta$ are functions of $x, y, z$, and $\xi', \eta', \zeta'$ are functions of $x', y', z'$, that is, of $x+h, y+k, z+l$, by (38), Art. 123, Vol. I, we have

$$\left.\begin{aligned} \xi' &= \xi + \left(\frac{d\xi}{dx}\right) h + \left(\frac{d\xi}{dy}\right) k + \left(\frac{d\xi}{dz}\right) l \\ \eta' &= \eta + \left(\frac{d\eta}{dx}\right) h + \left(\frac{d\eta}{dy}\right) k + \left(\frac{d\eta}{dz}\right) l \\ \zeta' &= \zeta + \left(\frac{d\zeta}{dx}\right) h + \left(\frac{d\zeta}{dy}\right) k + \left(\frac{d\zeta}{dz}\right) l \end{aligned}\right\}; \tag{138}$$

also

$$\begin{aligned} (r+dr)^2 &= (x'+\xi'-x-\xi)^2 + (y'+\eta'-y-\eta)^2 + (z'+\zeta'-z-\zeta)^2, \\ &= \left\{h + \left(\frac{d\xi}{dx}\right) h + \left(\frac{d\xi}{dy}\right) k + \left(\frac{d\xi}{dz}\right) l\right\}^2 + \dots + \dots \end{aligned}$$

and since $\xi, \eta, \zeta$ are infinitesimals in reference to $x, y, z$, neglecting infinitesimals of the higher orders, we have

$$\begin{aligned} r\,dr &= h^2\left(\frac{d\xi}{dx}\right) + k^2\left(\frac{d\eta}{dy}\right) + l^2\left(\frac{d\zeta}{dz}\right) \\ &+ kl\left\{\left(\frac{d\eta}{dz}\right) + \left(\frac{d\zeta}{dy}\right)\right\} + lh\left\{\left(\frac{d\zeta}{dx}\right) + \left(\frac{d\xi}{dz}\right)\right\} + hk\left\{\left(\frac{d\xi}{dy}\right) + \left(\frac{d\eta}{dx}\right)\right\}; \end{aligned} \tag{139}$$

let $\alpha, \beta, \gamma$ be the direction-angles of $r$, so that

$$h = r\cos\alpha, \qquad k = r\cos\beta, \qquad l = r\cos\gamma;$$

$$\begin{aligned} \therefore\ dr = r\Big[ & \left(\frac{d\xi}{dx}\right)(\cos\alpha)^2 + \left(\frac{d\eta}{dy}\right)(\cos\beta)^2 + \left(\frac{d\zeta}{dz}\right)(\cos\gamma)^2 \\ & + \cos\beta\cos\gamma\left\{\left(\frac{d\eta}{dz}\right) + \left(\frac{d\zeta}{dy}\right)\right\} + \cos\gamma\cos\alpha\left\{\left(\frac{d\zeta}{dx}\right) + \left(\frac{d\xi}{dz}\right)\right\} \\ & + \cos\alpha\cos\beta\left\{\left(\frac{d\xi}{dy}\right) + \left(\frac{d\eta}{dx}\right)\right\}\Big]; \end{aligned} \tag{140}$$

and this is the general value of the dilatation of the distance between the molecules, which are originally at a distance $r$ from each other.

If the two molecules are on the axis of $x$, $\alpha = 0$, $\beta = \gamma = 90^\circ$, and the dilatation $= r\left(\frac{d\xi}{dx}\right)$; similarly the dilatation of two molecules at a distance $r$ from each other on the axis of $y$ is $r\left(\frac{d\eta}{dy}\right)$; and on the axis of $z$, $r\left(\frac{d\zeta}{dz}\right)$; so that $\left(\frac{d\xi}{dx}\right)$, $\left(\frac{d\eta}{dy}\right)$, $\left(\frac{d\zeta}{dz}\right)$ are severally the dilatations of a line of an unit-length in the axes of $x$, $y$, and $z$, and are called *the linear dilatations.*

As these quantities lead to an expression which has been proved before, in Art. 200, Vol. II, by a different process, it is worth while to shew the identity of the results.

Under the action of the external pressures, lines which were originally $dx$, $dy$, $dz$, become

$$dx\left\{1+\left(\frac{d\xi}{dx}\right)\right\}, \quad dy\left\{1+\left(\frac{d\eta}{dy}\right)\right\}, \quad dz\left\{1+\left(\frac{d\zeta}{dz}\right)\right\}; \tag{141}$$

these are therefore the dilated values; and are in fact compressed values, if the dilatations are negative.

Hence also an element which was originally of the volume $dx\,dy\,dz$, becomes of the volume

$$dx\,dy\,dz\left\{1+\left(\frac{d\xi}{dx}\right)\right\}\left\{1+\left(\frac{d\eta}{dy}\right)\right\}\left\{1+\left(\frac{d\zeta}{dz}\right)\right\};$$

and neglecting infinitesimals of the highest orders becomes

$$dx\,dy\,dz\left\{1+\left(\frac{d\xi}{dx}\right)+\left(\frac{d\eta}{dy}\right)+\left(\frac{d\zeta}{dz}\right)\right\}. \tag{142}$$

And therefore the cubical dilatation of the volume-element is

$$dx\,dy\,dz\left\{\left(\frac{d\xi}{dx}\right)+\left(\frac{d\eta}{dy}\right)+\left(\frac{d\zeta}{dz}\right)\right\}. \tag{143}$$

158.] To return: in the expression for $dr$ in (140), if the position of the molecule, which was originally $(x, y, z)$, and is subsequently $(x+\xi,\ y+\eta,\ z+\zeta)$, remains the same, and the other molecule changes within the sphere of activity, then $\alpha, \beta, \gamma$ vary, but the coefficients of the cosines involving the partial derived functions remain the same. Hereby we are able to determine the general form of the components of the elastic forces acting on any small area $\omega$; for suppose two molecules to be at a distance $r$ from each other, and suppose this distance to be increased (or diminished) by $dr$: let the mass of one molecule be $m$ and of the other $m'$: and let the attractive or repulsive action due to the dilatation be symbolised as in Art. 153, then the attraction or repulsion of the two is (see Art. 170, equation (1))

$$m\,m'\,\mathrm{F}(r)\,dr; \tag{144}$$

and as the direction-angles of the line of action of this force are $\alpha, \beta, \gamma$, the components are easily found for this pair of particles: now if $m$ is the type-particle of the small cylinder standing on $\omega$, and $m'$ is the type-particle of the sphere of activity, the aggregate of all the corresponding components will yield the components of the elastic forces; and the results will evidently be of the form

$$\begin{aligned} \mathrm{N}_i &= \mathrm{A}_i\Big(\frac{d\xi}{dx}\Big) + \mathrm{B}_i\Big(\frac{d\eta}{dy}\Big) + \mathrm{C}_i\Big(\frac{d\zeta}{dz}\Big) \\ &\quad + \mathrm{D}_i\Big\{\Big(\frac{d\eta}{dz}\Big)+\Big(\frac{d\zeta}{dy}\Big)\Big\} + \mathrm{E}_i\Big\{\Big(\frac{d\zeta}{dx}\Big)+\Big(\frac{d\xi}{dz}\Big)\Big\} + \mathrm{F}_i\Big\{\Big(\frac{d\xi}{dy}\Big)+\Big(\frac{d\eta}{dx}\Big)\Big\}; \end{aligned} \tag{145}$$

$$\begin{aligned} \mathrm{T}_i &= \mathrm{A}_i'\Big(\frac{d\xi}{dx}\Big) + \mathrm{B}_i'\Big(\frac{d\eta}{dy}\Big) + \mathrm{C}_i'\Big(\frac{d\zeta}{dz}\Big) \\ &\quad + \mathrm{D}_i'\Big\{\Big(\frac{d\eta}{dz}\Big)+\Big(\frac{d\zeta}{dy}\Big)\Big\} + \mathrm{E}_i'\Big\{\Big(\frac{d\zeta}{dx}\Big)+\Big(\frac{d\xi}{dz}\Big)\Big\} + \mathrm{F}_i'\Big\{\Big(\frac{d\xi}{dy}\Big)+\Big(\frac{d\eta}{dx}\Big)\Big\}; \end{aligned} \tag{146}$$

and where $i$ is to be replaced in each expression by 1, 2, and 3.

Such are the forms of the general results for the particular kind of elastic matter which we suppose to be the subject of the forces, and these are arrived at without determining the particular form of $\mathrm{F}(r)$; each of the above contains six undetermined quantities, which are constant in the case of a solid of constant density and constant elastic force, and are functions of $x, y, z$ if the bodies are heterogeneous, and if they are crystals; and therefore there are in the whole thirty-six such quantities, each of which involves a definite integral, and which depends on the arrangement and other circumstances of the molecules of the elastic body.

159.] For the purpose of application let us consider the forms of (145) and (146) in the case of a solid, homogeneous and of uniform elasticity, and in which the quantities A, B, &c., are constant; and first let us investigate the elastic forces acting on that facet of the parallelepipedon which is perpendicular to the axis of $x$. $\mathrm{N}_1$, the normal component, evidently depends on $\xi$, the resolved part of the displacement along the axis of $x$, in a manner different to that in which it depends on

$\eta$ and $\zeta$; and by the principle of sufficient reason, it depends in the same manner on each of these last two; therefore in the expression for $N_1$, the coefficient A of $\left(\frac{d\xi}{dx}\right)$, is different to B the coefficient of $\left(\frac{d\eta}{dy}\right)$ and of $\left(\frac{d\zeta}{dz}\right)$; and for a similar reason $\left(\frac{d\eta}{dz}\right)+\left(\frac{d\zeta}{dy}\right)$ will have a coefficient D distinct from E, which is the common coefficient of the other two terms. Also in the expression for $T_1$, the parts of it corresponding to $\xi$ will be different to those corresponding to $\eta$ and $\zeta$, which latter will be the same. Also on account of the uniform density and elasticity of the body, the same results hold good of the other N's, and of the other T's. And we may collect the results in the following scheme:

| | $\left(\frac{d\xi}{dx}\right)$ | $\left(\frac{d\eta}{dy}\right)$ | $\left(\frac{d\zeta}{dz}\right)$ | $\left(\frac{d\eta}{dz}\right)+\left(\frac{d\zeta}{dy}\right)$ | $\left(\frac{d\zeta}{dx}\right)+\left(\frac{d\xi}{dz}\right)$ | $\left(\frac{d\xi}{dy}\right)+\left(\frac{d\eta}{dx}\right)$ |
|---|---|---|---|---|---|---|
| $N_1$ | A | B | B | D | E | E |
| $N_2$ | B | A | B | E | D | E |
| $N_3$ | B | B | A | E | E | D |
| $T_1$ | $A'$ | $B'$ | $B'$ | $D'$ | $E'$ | $E'$ |
| $T_2$ | $B'$ | $A'$ | $B'$ | $E'$ | $D'$ | $E'$ |
| $T_3$ | $B'$ | $B'$ | $A'$ | $E'$ | $E'$ | $D'$ |

(147)

wherein the several letters placed under the quantities in the upper row are the coefficients of the corresponding quantities in the expressions for the N's and the T's.

There are thus eight coefficients: an intelligible conception of a body of constant density, and of an elasticity invariable in all directions, will enable us to reduce them to a smaller number: for this purpose we assume the two following laws:

Firstly; imagine a solid body of the given molecular structure to be acted on by a pressure in the direction of the axis of $z$, so that the molecular displacements are in accordance with the three following equations,

$$\xi = 0, \qquad \eta = 0, \qquad \zeta = cz, \tag{148}$$

that is, the molecules suffer no displacements in lines parallel to the axes of $x$ and $y$, and that parallel to the axis of $z$ varies directly as the distance of the molecule from the plane of $xy$; this is therefore the case of a simple pulling force. Now for the purpose of determining the special forms of the above expressions, let us take a small surface-element $\omega$ perpendicular to the axis of $x$, raise on it a small cylinder, as in Art. 153, and investigate the elastic force acting on it. Now imagine, as in fig. 71, P to be a molecule of the cylinder whose base is $\omega$, and Q to be another molecule within the sphere of activity; and let us suppose P to be on the axis of $x$: also imagine Q′ to be another molecule at a distance from P equal to that of Q, and at the same distance below the axis of $x$, and suppose all these to be in the displaced position of the body: then the actions on P of Q and Q′ in directions of the axes of $y$ and $z$ neutralize each other; and in the direction of the axis of $x$ they act in the same direction; and therefore the resultant of the action of the pair of molecules on P is normal to the base $\omega$ of the cylinder: and as for all molecules in $\omega$ there are pairs of molecules within the sphere of activity, so will the aggregate of all the resultants give a force perpendicular to the plane of $\omega$, and thus there will be no force in that plane. Hence

$$\text{T}_2 = \text{T}_3 = 0. \tag{149}$$

Secondly; imagine a solid of elastic matter, such as we have supposed, to be twisted about the axis of $z$; and its nature to be such that no displacement of a particle takes place parallel to the axis of $z$, and that the displacements in directions parallel to the axes of $x$ and $y$ are proportional to (1) the distance of each particle from the plane of $xy$, (2) the distance of each particle from the axis of $z$: then, if $k$ is a constant,

$$\xi = -kyz, \qquad \eta = kxz, \qquad \zeta = 0. \tag{150}$$

Let us investigate the elastic forces acting on a cylinder perpendicular to, and raised on, a small plane area perpendicular to the axis of $x$. In fig. 72 let P be a molecule within the cylinder; and let Q and Q′ be two points equally distant from PA (which is parallel to the axis of $x$) and in the plane $xz$: also let Q be the projection of two molecules, $m$ and $m_1$, equally distant from the plane of $xz$, and on opposite sides of it: similarly let Q′ be the projection of two molecules $m'$ and $m'_1$, also at equal distances and on opposite sides of the plane of $xz$: and imagine all these to be in their several positions after the displacements have taken place, and the elastic forces have been brought into action; and these positions are possible by reason of the hypothesis which we have made as to the homogeneity of the body. Now according to the laws of displacement expressed by (150), the displacements of $m'$, $m'_1$ are less, and the displacements of $m$, $m_1$ are greater, than the displacement of P: therefore the distances of $m$ and $m'_1$ from P are increased by the torsion, while those of $m_1$ and $m'$ are diminished: also in the configuration of the system which we have imagined, the distances of the four molecules from P are all equal, and therefore the actions of them on P are also all equal; two however are repulsive, and two are attractive: and thus the total resultant in the plane perpendicular to PA is zero: and therefore

$$\text{N}_1 = 0, \qquad \text{T}_2 = 0, \qquad \text{T}_3 = 0. \tag{151}$$

In the plane however which is parallel to that of $xy$, the elastic force parallel to the axis of $y$ does not vanish, that is, of the three components of the elastic force on this plane $\text{N}_3 = 0$, $\text{T}_2 = 0$, $\text{T}_1$ is finite; and thus the elastic force which the torsion brings into action is tangential to the circle which it causes any molecule to describe: and the same result of course holds good of all the constituent molecules of the body.

It is also to be observed that as these results are true for the coordinate axes chosen arbitrarily, so are they also true of every system of axes. This may be proved by certain analytical transformations, which have been worked out by M. Lamé: but as they are only the mathematical interpretations of homogeneity and constant elasticity, it is unnecessary to introduce the investigations.

160.] Let us apply the results of the last Article to the simplification of the expressions in (147). From (148)

$$\left(\frac{d\xi}{dx}\right) = 0, \quad \left(\frac{d\eta}{dy}\right) = 0, \quad \left(\frac{d\zeta}{dz}\right) = c, \quad \left(\frac{d\xi}{dy}\right) = 0, \ldots\ldots$$

$$\therefore \quad \text{T}_2 = \text{B}'c, \qquad \text{T}_3 = \text{A}'c;$$

and therefore by reason of (149),

$$\text{A}' = \text{B}' = 0. \tag{152}$$

Again, from (150), and bearing in mind that $y = 0$, we have

$$\text{N}_1 = \text{D}kx, \qquad \text{N}_3 = \text{E}kx, \qquad \text{T}_2 = \text{T}_3 = \text{E}'kx;$$

and therefore by virtue of (151), and of the paragraph following it,

$$\text{D} = 0, \qquad \text{E} = 0, \qquad \text{E}' = 0. \tag{153}$$

Hence, if we replace as follows,

$$\text{B} = \lambda, \qquad \text{A} = \lambda + 2\mu, \tag{154}$$

$$\Omega = \left(\frac{d\xi}{dx}\right) + \left(\frac{d\eta}{dy}\right) + \left(\frac{d\zeta}{dz}\right), \tag{155}$$

where $\Omega$ is, by reason of equation (143), the dilatation of a cube whose volume is unity, we have from (147)

$$\text{N}_1 = \lambda\Omega + 2\mu\left(\frac{d\xi}{dx}\right), \quad \text{N}_2 = \lambda\Omega + 2\mu\left(\frac{d\eta}{dy}\right), \quad \text{N}_3 = \lambda\Omega + 2\mu\left(\frac{d\zeta}{dz}\right), \tag{156}$$

$$\text{T}_1 = \text{D}'\left\{\left(\frac{d\eta}{dz}\right) + \left(\frac{d\zeta}{dy}\right)\right\}, \quad \text{T}_2 = \text{D}'\left\{\left(\frac{d\zeta}{dx}\right) + \left(\frac{d\xi}{dz}\right)\right\}, \quad \text{T}_3 = \text{D}'\left\{\left(\frac{d\xi}{dy}\right) + \left(\frac{d\eta}{dx}\right)\right\}. \tag{157}$$

If however the coordinate axes are changed from one rectangular system to another, it will be found that the N's and the T's cannot continue of the same form as in (156) and (157), unless $\text{D}' = \mu$; so that we have ultimately

$$\text{N}_1 = \lambda\Omega + 2\mu\left(\frac{d\xi}{dx}\right), \quad \text{N}_2 = \lambda\Omega + 2\mu\left(\frac{d\eta}{dy}\right), \quad \text{N}_3 = \lambda\Omega + 2\mu\left(\frac{d\zeta}{dz}\right), \tag{158}$$

$$\text{T}_1 = \mu\left\{\left(\frac{d\eta}{dz}\right) + \left(\frac{d\zeta}{dy}\right)\right\}, \quad \text{T}_2 = \mu\left\{\left(\frac{d\zeta}{dx}\right) + \left(\frac{d\xi}{dz}\right)\right\}, \quad \text{T}_3 = \mu\left\{\left(\frac{d\xi}{dy}\right) + \left(\frac{d\eta}{dx}\right)\right\}; \tag{159}$$

wherein only two constants $\lambda$ and $\mu$ are involved; and these depend on the law of the elastic action of the body.

Let these expressions be substituted in the general equations (129), and let the simplification, given in (155), be introduced; then omitting the brackets indicative of partial differentiation, we have

$$\left.\begin{aligned}
(\lambda+\mu)\frac{d\Omega}{dx} + \mu\left\{\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} + \frac{d^2\xi}{dz^2}\right\} + \rho\text{X} &= 0 \\
(\lambda+\mu)\frac{d\Omega}{dy} + \mu\left\{\frac{d^2\eta}{dx^2} + \frac{d^2\eta}{dy^2} + \frac{d^2\eta}{dz^2}\right\} + \rho\text{Y} &= 0 \\
(\lambda+\mu)\frac{d\Omega}{dz} + \mu\left\{\frac{d^2\zeta}{dx^2} + \frac{d^2\zeta}{dy^2} + \frac{d^2\zeta}{dz^2}\right\} + \rho\text{Z} &= 0
\end{aligned}\right\}; \tag{160}$$

and the molecular displacements are subject to the laws expressed by these equations.

These equations have the following remarkable property: take the $x$-differential of the first, the $y$-differential of the second, the $z$-differential of the third, and add; and differentiating twice (155), we have

$$(\lambda+2\mu)\left\{\frac{d^2\Omega}{dx^2} + \frac{d^2\Omega}{dy^2} + \frac{d^2\Omega}{dz^2}\right\} + \rho\left\{\frac{d\text{X}}{dx} + \frac{d\text{Y}}{dy} + \frac{d\text{Z}}{dz}\right\} = 0; \tag{161}$$

and therefore, if the external forces are such that

$$\frac{d\text{X}}{dx}+\frac{d\text{Y}}{dy}+\frac{d\text{Z}}{dz}=0, \tag{162}$$

$$\left(\frac{d^2\Omega}{dx^2}\right)+\left(\frac{d^2\Omega}{dy^2}\right)+\left(\frac{d^2\Omega}{dz^2}\right)=0; \tag{163}$$

to the condition expressed in which equation the cubical dilatation is subject.

161.] As to the coefficients $\lambda$ and $\mu$, they are quantities of the same nature as the N's and the T's, that is, they are elastic forces, because they enter into the expressions (158) and (159) as coefficients of $\left(\frac{d\xi}{dx}\right)$ &c. which are numbers; they may be expressed therefore as ordinary pressures: their values for a solid of uniform elasticity and density and of a given nature are to be determined by experiment, and in the following manner:

Let the solid be subject to an uniform pressure P throughout its surface, so that its bounding form remains similar after the contraction or the dilatation. Suppose that the origin of coordinates remains fixed, so that

$$\xi = ax, \qquad \eta = ay, \qquad \zeta = az, \tag{164}$$

where $a$ is a constant depending on the nature of the body; therefore from (158) and (159)

$$\text{N}_1 = \text{N}_2 = \text{N}_3 = (3\lambda+2\mu)\,a, \tag{165}$$

$$\text{T}_1 = \text{T}_2 = \text{T}_3 = 0. \tag{166}$$

And as the normal pressure is constant throughout, and therefore at the surface it is equal to P,

$$\therefore \quad \text{P} = (3\lambda+2\mu)\,a.$$

Let $\alpha$ be the cubical compression or dilatation of the body due to an unit of pressure, so that by reason of (155) $\alpha = 3a$,

$$\therefore \quad \alpha = \frac{3}{3\lambda+2\mu}; \tag{167}$$

and therefore, if $\alpha$ is found by experiment, this equation gives a relation between $\lambda$ and $\mu$.

Again, suppose a body to be cylindrical or prismatic, and its generators to be parallel to the axis of $z$: and also suppose the terminal surface at each end to be perpendicular to the axis of $z$, and each unit surface at each end to be subject to a force P; then the displacement of each molecule in the plane parallel to that of $xy$ is uniform; and we have

$$\xi = ax, \qquad \eta = ay, \qquad \zeta = cz,$$

$$\therefore \quad \text{N}_1 = \text{N}_2 = (c+2a)\lambda + 2\mu a,$$

$$\text{N}_3 = (c+2a)\lambda + 2\mu c,$$

$$\text{T}_1 = \text{T}_2 = \text{T}_3 = 0;$$

but $\text{N}_1 = \text{N}_2 = 0$, since there is no lateral pressure on the sides of the solid; and $\text{N}_3 = \text{P}$.

$$\therefore \quad (c+2a)\lambda + 2\mu a = 0,$$

$$(c+2a)\lambda + 2\mu c = \text{P},$$

$$\therefore \quad c = \frac{\mu+\lambda}{\mu}\frac{\text{P}}{3\lambda+2\mu}, \qquad a = -\frac{\lambda}{2\mu}\frac{\text{P}}{3\lambda+2\mu};$$

$$\text{also} \qquad \Omega = 2a + c = \frac{\text{P}}{3\lambda+2\mu}.$$

Let $e$, as in Art. 149, be the dilatation of an unit-length of the prism, under the action of an unit force, and which is to be determined by experiment; and so that $\text{P}e = c$;

$$\therefore \quad e = \frac{\mu+\lambda}{\mu(3\lambda+2\mu)}; \tag{168}$$

which gives another relation between $\lambda$ and $\mu$: and from it and from (167) we have,

$$\mu = \frac{3}{9e-a}, \qquad \lambda = \frac{1}{a} - \frac{2}{9e-a}. \tag{169}$$

I am unable to proceed further into the general theory of elasticity of bodies, and must refer the reader who desires more information to (1) a memoir of M. Poisson in the XXth Cahier of the Journal of the Polytechnic School, Paris, 1831: (2) the treatise on Elastic Bodies by M. Lamé, Paris, 1852: (3) a memoir on Elasticity, &c. by Mr. J. Clerk Maxwell, in Vol. XX. of the Transactions of the Royal Society of Edinburgh, in which the results are tested by the phænomena of polarized Light: and (4) some memoirs by MM. Lamé and Clapeyron in Vol. VII. of Crelle's Journal. It is desirable however to apply the principles on which the preceding investigations are founded to the solution of some particular problems.

162.] And I will first consider the bending of an elastic lamina.

Imagine a rectangular plate of an uniform elastic action and of constant density, of a finite length and breadth, $a$ and $b$; and of infinitesimal thickness $2\tau$, which however is such as to develope forces of elasticity when the lamina is bent in the direction of its length by the action of certain external forces. Also imagine the plate to be resolved into a series of rods, all of which are parallel to the length $a$ of the plate, and are of infinitesimal depth $dz$; so that of each of these the thickness is $2\tau$ and the length is $a$. When the flexure takes place each of these rods may undergo three different kinds of change: (1) the length may be contracted or increased; (2) the absolute curvature may be altered; (3) one element of a rod may be twisted upon the consecutive element of the same rod: the first two effects I shall consider: the latter will not enter into the investigation, as the material is supposed to be of a non-crystalline texture, and, as such, to be incapable of developing forces which would cause the twisting.

Suppose the rectangular plate, fig. 73, to be that whose length is $a$ and breadth is $b$; and suppose it to be perpendicular to the plane of the paper, and in its original unbent form to pierce the plane of the paper along the axis of $x$: also suppose it to be fixed throughout its breadth at the extremity passing through O, so that when the plate is bent, that end of it which is intersected by the plane of the paper at O may be unchanged as to position; and suppose the end of the plate at A to be stiff throughout its breadth, so that the plate may be bent by a single force applied at that extremity; and thus that its surfaces, which were originally plane and parallel and at a distance $2\tau$ apart, may be the two surfaces of a cylinder: and thus all the rods, into which we have imagined the plate to be divided, will be rods, equal and similarly bent, of the form delineated in the figure; and where X and Y are the pressures parallel to the axes, applied at the extremity A and causing the flexure of the plate.

Let us consider the bent rod of infinitesimal depth $dz$, and whose under-surface in the figure is OAB; and let us assume that the molecules, which in the bent state are along the normal common to both surfaces, were originally in a line normal to the two plane faces; so that $\Pi P''P'$ is the common normal to the two curves $OP'$ and $BP''$; let another consecutive normal be drawn to these curves, and let it meet the former normal in $\Pi$, so that $\Pi$ is the centre of curvature. Again, let the rod be resolved into other smaller rods or fibres, the depth of each of which is the same as that of the rod, and the sum of the several breadths of which is $2\tau$: then each of those contained within the space $P'Q'Q''P''$ is of course parallel to $P'Q'$ and to $P''Q''$; and of these let us consider PQ. Π is the common centre of curvature of all: let $\rho$ be the radius of curvature of that one which is equally distant from P′ and P″, and which I shall call the mean fibre; and let $\sigma$ be the length of this mean fibre contained between P′P″ and Q′Q″, and let the angle at $\Pi = d\psi$; so that

$$\sigma = \rho\, d\psi. \tag{170}$$

Let $\sigma'$ be the length of PQ, and let PQ be at a distance $u$ from the mean fibre,

$$\therefore \quad \sigma' = (\rho + u)\, d\psi; \tag{171}$$

$u$ being positive or negative according as PQ is nearer to or farther from the convex side of the plate than the mean fibre.

Now in the process of bending, the fibres on the side towards the convexity of the bent plate will undergo dilatation, and those towards the concavity will undergo contraction. For assuming the coefficients of dilatation (or contraction, as the case may be,) to be different for different fibres, if $s$ is the original length of the fibre contained between P′P″ and Q′Q″, we have by Hooke's law,

$$\sigma = s(1+e), \qquad \sigma' = s(1+e'), \tag{172}$$

and which correspond to dilatation or contraction according as $e$ is positive or negative. Therefore from the last three equations

$$\frac{1+e'}{1+e} = \frac{\rho+u}{\rho};$$

whence, as $e$, $e'$, and $u$ are infinitesimal,

$$e' = e + \frac{u}{\rho}. \tag{173}$$

Whence it appears that if the length of the mean fibre is not changed, that is, if $e = 0$, then $e'$ and $u$ have the same sign; and therefore the fibres undergo dilatation or contraction according as they are on the side towards P′Q′ or P″Q″; and in either case the change of length is proportional to the distance from the mean fibre.

163.] And imagining the bent lamina to be in a rigid state under the action of the several forces, let us investigate the elastic forces which act on the part AQ′Q″ by means of the section Q″QQ′. Now as any fibre PQ has undergone expansion or contraction, so does it tend to contract or expand; let us suppose its force, referred to an unit of surface, to vary as the coefficient of elasticity; so that the force acting on an unit of surface $= ke'$; and let us suppose the thickness of the plate to be, with the exception of a variation infinitesimal in comparison with the thickness, the same as before the flexure, so that its thickness is $2\tau$ and its depth is $dz$; then if $\mathrm{T} =$ the whole force, and this acts in a line normal to $\mathrm{Q''QQ'}$,

$$\mathrm{T} = \int_{-\tau}^{\tau} ke'\,dz\,du; \tag{174}$$

and if we replace $e'$ by its value from (173), and integrate,

$$\mathrm{T} = 2ke\tau\,dz. \tag{175}$$

Also let $\mathrm{L}$ be the moment of these elastic forces about an axis perpendicular to the plane of the paper and passing through the mean fibre; then

$$\begin{aligned}\mathrm{L} &= \int_{-\tau}^{\tau} ke'\,dz\,u\,du \\ &= k\,dz\int_{-\tau}^{\tau}\left(e+\frac{u}{\rho}\right)u\,du \\ &= \frac{2k\tau^3 dz}{3\rho}.\end{aligned} \tag{176}$$

Hence it appears (1) that $\mathrm{T}$ varies as the contraction or expansion of the mean fibre, and is independent of its curvature; (2) that $\mathrm{L}$ is independent of the extension, and varies directly as the curvature of the mean fibre: (3) that $\mathrm{T}$ varies directly as the thickness, and $\mathrm{L}$ varies as the cube of the thickness.

Also when the length of the mean fibre is not changed by the bending, $e = 0$, $\mathrm{T} = 0$, and $\mathrm{L}$ remains the same.

And because similar results are true for each rod into which the plate is divided, so for a section parallel to the side whose length is $b$ and through the whole breadth of the plate,

$$\mathrm{T} = 2bke\tau, \tag{177}$$

$$\mathrm{L} = \frac{2kb\tau^3}{3\rho}; \tag{178}$$

and if $\omega$ is the area of the section, $\omega = 2b\tau$; therefore

$$\mathrm{T} = ke\omega, \tag{179}$$

$$\mathrm{L} = \frac{k\omega\tau^2}{3\rho}. \tag{180}$$

164.] The preceding investigations also enable us to find the equation to the curves which the fibres take. Since the force $\mathrm{T}$ (175) acts on the element $\mathrm{P'Q'Q''P''}$ at the side $\mathrm{Q'Q''}$, an equal and opposite force acts on the side $\mathrm{P'P''}$, because there is equilibrium and no other force acts. And as the same result is true for all the elements of the lamina, T is constant throughout, and is therefore equal to the parts of X and Y which are normal to the end at A; $e$ is also constant, and by virtue of equation (175) is proportional to this force, and is positive or negative according as the impressed forces act to dilate or contract the mean fibres.

Let $(x_0, y_0)$ be the point A at which are applied forces X, Y which cause the bending of the lamina: then these forces, together with those applied on that section of the plate whose intersection with the plane of the paper is Q′Q″, keep in equilibrium the part of the plate between A and Q′Q″. Now if $(x, y)$ is the point P, $x$ and $y$ will also, neglecting infinitesimals, be the coordinates to the point of intersection of the mean fibre and Q′Q″; and therefore, taking moments about that point, we have

$$\mathrm{L}+\mathrm{X}(y_0-y)-\mathrm{Y}(x_0-x) = 0; \tag{181}$$

and substituting for L from (178), and replacing $\rho$ by its equivalent expression, we have for the differential equation of the curve,

$$\frac{2kb\mathrm{T}^3}{3}\frac{d^2y}{dx^2}+\{\mathrm{X}(y_0-y)-\mathrm{Y}(x_0-x)\}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = 0; \tag{182}$$

the integral of which will contain two arbitrary constants: and these will be determined by the condition that $\frac{dy}{dx} = 0$, when $x = y = 0$, and by the length of the curve between the origin and $(x_0, y_0)$ being given.

If the lamina is not fixed at O, a force equal and opposite to the resultant of X and Y must be supplied at it.

165.] Let us consider more at length two particular cases of (182) in which the forces act (1) wholly perpendicular to, (2) along, the plate in its original unbent state. In the first case $\mathrm{X} = 0$; so that (182) becomes

$$\frac{2kb\mathrm{T}^3}{3}\frac{d^2y}{dx^2}-\mathrm{Y}(x_0-x)\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}} = 0; \tag{183}$$

Let $\frac{2kb\mathrm{T}^3}{3\mathrm{Y}} = c$, $\quad\therefore\quad c\frac{d^2y}{dx^2} = (x_0-x)\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$;

therefore integrating, see Vol. II, Ex. 3, Art. 373, and observing that at the inferior limit, $x = 0$, when $\frac{dy}{dx} = 0$,

$$2c\frac{dy}{dx} = (2x_0x - x^2)\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}; \tag{184}$$

$$\therefore \quad dy = \frac{(2x_0x - x^2)\,dx}{\{4c^2 - (2x_0x - x^2)^2\}^{\frac{1}{2}}}, \tag{185}$$

$$ds = \frac{2c\,dx}{\{4c^2 - (2x_0x - x^2)^2\}^{\frac{1}{2}}}; \tag{186}$$

neither of which expressions can be integrated further.

If however the elastic force of the lamina is very great compared with the deflecting force, $\frac{dy}{dx}$ is very small, and neglecting the second and higher powers of it we have from (184)

$$2c\,dy = (2x_0x - x^2)\,dx,$$

$$6cy = 3x_0x^2 - x^3; \tag{187}$$

hence when $x = x_0$, $y = y_0 = \frac{x_0^3}{3c}$;

and therefore replacing $c$ by its equivalent,

$$y_0 = \frac{x_0^3\,\mathrm{Y}}{2kb\tau^3};$$

that is, the distance through which the end of the lamina has been moved varies as the deflecting force, as the cube of the length (approximately), inversely as the breadth of the plate, and inversely as the cube of the thickness.

166.] In the second case, suppose $\mathrm{Y} = 0$, and $\mathrm{X}$ to act, like a crushing pressure, towards O: then if the lamina under the action of such a force is bent at all, its deflection from a straight line is very slight, and thus $\frac{dy}{dx}$ is very small: I shall neglect therefore the second and higher powers of it: also $y_0 = 0$, since the force X acts still along the axis of $x$: therefore (182) becomes

$$\frac{2kb\tau^3}{3}\frac{d^2y}{dx^2} + \mathrm{X}y = 0: \tag{188}$$

and if we put

$$\frac{3\mathrm{X}}{2kb\tau^3} = c^2, \tag{189}$$

we have

$$\frac{d^2y}{dx^2} + c^2y = 0;$$

and supposing the end of the plate to press against a rough fixed plane at O, but not to be fixed as heretofore, we have

$$\frac{dy^2}{dx^2} + c^2(y^2 - a^2) = 0;$$

so that $\frac{dy}{dx} = ca$, when $y = 0$, and where $a$ is undetermined: and integrating again

$$y = a \sin cx, \tag{190}$$

because $y = 0$, when $x = 0$. The plate therefore takes a corrugated form, the section of which, by a plane perpendicular to it, and parallel to its length, is the curve of sines. And from (189) $a$ is the greatest amplitude of this curve, and therefore is very small in comparison of the length of the curve, because the deflection of the plate from a plane is supposed to be very small.

If $a = 0$, the plate will continue plane, and its length will, by reason of Art. 149, be slightly diminished, and become $a\left(1 - e\frac{\mathrm{X}}{2b\tau}\right)$. If $a$ is not equal to zero, the plate, which is as a rectangular piece of watch-spring, takes the corrugated form, and the number of undulations on the cylindrical surface will depend on $c$; let $h$ be the distance OA, then since $y = 0$, when $x = h$, $c$ and $h$ must be related by the equation,

$$c = \frac{\pi i}{h},$$

where $i$ is any whole number;

$$\therefore \quad y = a \sin \frac{\pi i}{h} x; \tag{191}$$

and therefore also $y = 0$, when $x = \frac{nh}{i}$, where $n$ is any number from 0 up to $i$, so that the curve cuts the axis of $x$ in $i+1$ points, and therefore the surface has $i$ elevations or depressions. Also if $l$ is the length of the curve,

$$l = \int_0^h \left\{1 + \frac{\pi^2 i^2 a^2}{h^2}\left(\cos\frac{\pi i}{h}x\right)^2\right\}^{\frac{1}{2}} dx;$$

and omitting the fourth and higher powers of $a$, we have

$$l = h + \frac{\pi^2 i^2 a^2}{4h};$$

$$\therefore \quad a = \frac{2h}{\pi i}\left\{\frac{l}{h} - 1\right\}^{\frac{1}{2}}; \tag{192}$$

whereby $a$ is given in terms of $h$ and $l$.

167.] The greatest value of the compressing force X which can be applied at the end of a spring, and not bend it, is called *the vertical strength of the spring;* in this case $l = h$, $i = 1$: therefore from (189)

$$\begin{aligned} \mathrm{X} &= \frac{2kb\tau^3 c^2}{3}, \\ &= \frac{2kb\tau^3\pi^2}{3l^2}; \end{aligned}$$

and using for $l$ its approximate value $a$,

$$\mathrm{X} = \frac{2k\pi^2 b\tau^3}{3a^2}; \tag{193}$$

so that, other incidents being the same, the vertical strength of the spring varies inversely as the square of the length.

Hereby also are we enabled to calculate the greatest weight that a vertical pillar of a given form and height can bear without being bent by the weight.

Suppose the pillar to be of a height $h$ and its transverse section to be rectangular, the sides of the rectangle being $a$ and $b$; then the greatest weight which it will bear, without being bent perpendicularly to the side $b$, is

$$\frac{k\pi^2 a b^3}{12h^2};$$

and, without being bent perpendicularly to the side $a$, the greatest weight is

$$\frac{k\pi^2 a^3 b}{12h^2};$$

and if the transverse section is a square, $a=b$, and the strength of the beam perpendicularly to either of the sides is

$$\frac{k\pi^2 a^4}{12h^2}, \tag{194}$$

and varies therefore as the fourth power of the side.

Suppose a transverse section to be square and to be hollow, so that the side of the external square is $a$ and of the internal square $b$; then

$$\text{the vertical strength of the beam} = \frac{k\pi^2(a^4-b^4)}{12h^2}.$$

Also let us investigate the vertical strength of beams, the transverse sections of which are of forms other than rectangles; and let us assume, as the most probable hypothesis, that the mean fibre is that which passes through the centres of gravity of all similar transverse sections; then L must be calculated in each case, as in Art. 163, so that we may substitute in equation (181); and let $\xi$ and $\eta$ be the coordinates to any element of the area of the transverse section; and let us consider the following examples.

Ex. 1. The section of the beam is a circle, of which the radius is $a$.

Suppose the mean fibre of the cylindrical beam originally to be coincident with the axis of $x$; and ultimately, if bent, to be in the plane of $xy$, so that the bending takes place about an axis perpendicular to the plane of $xy$: let $\xi$ be taken in, and $\eta$ perpendicular to, the plane of $xy$: then

$$\begin{aligned} \text{L} &= \int_{-a}^{+a} 2k(a^2-\xi^2)^{\frac{1}{2}} e' \xi \, d\xi, \\ &= 2k \int_{-a}^{+a} (a^2-\xi^2)^{\frac{1}{2}} \left(e+\frac{\xi}{\rho}\right) \xi \, d\xi, \\ &= \frac{k\pi a^4}{4\rho}: \end{aligned}$$

so that (181) becomes

$$\frac{\pi k a^4}{4} \frac{d^2 y}{dx^2} + \text{X} y = 0; \qquad \therefore \quad \text{X} = \frac{\pi^3 k a^4}{4h^2};$$

a comparison of this result with (194) shews, that if the areas of the transverse sections are equal in the two cases, the vertical strengths of the square and the circular beams are as $\pi : 3$; there is therefore a small advantage in favour of the square beam.

Ex. 2. Let the beam be circular and hollow: let $a$ be the radius of the external, $b$ the radius of the internal surface: then by the last result, if X is the vertical strength,

$$\text{X} = \frac{\pi^3 k}{4h^2}(a^4-b^4).$$

Ex. 3. Let the transverse section of the beam be an isosceles triangle, of which the base is $a$ and the altitude $c$: then if the altitude lies in the plane of $xy$, and the base of the triangle becomes convex,

$$\text{L} = \frac{kac^3}{36\rho}; \qquad \therefore \quad \text{X} = \frac{k\pi^2 ac^3}{36h^2}.$$

168.] To conclude this part of the subject, let us consider the deflexion which a given rectangular and prismatic beam undergoes, when it is fixed in a horizontal position by one end, and is bent by its own weight.

Let $a$ be the length of the beam, $b$ its thickness, and $2\tau$ its depth; and let it be placed in a horizontal position, with the side $2\tau$ vertical: then as the deflexion is very small, we may, in equation (181), consider $x_0 = a$, $\text{X} = 0$, and $\frac{dy}{dx}$ to be very small, so that its second and higher powers may be neglected. Taking moments for the section Q′Q″, in fig. 73,

$$\text{L} = \frac{2kb\tau^3}{3} \frac{d^2 y}{dx^2};$$

and as the weight of that part of the beam which lies beyond Q′Q″ is in equilibrium with this force of elasticity, we have, if $\rho =$ the density of the beam, $g =$ the earth's attraction,

$$\frac{2kb\tau^3}{3}\,\frac{d^2y}{dx^2} = b\tau\rho g\,(a-x)^2;$$

$$\therefore\quad \frac{dy}{dx} = \frac{\rho g}{2k\tau^2}\{a^3-(a-x)^3\},$$

because $\frac{dy}{dx} = 0$, when $x = 0$; therefore

$$y = \frac{\rho g}{2k\tau^2}\left\{\frac{3a^2x^2}{2} - ax^3 + \frac{x^4}{4}\right\};$$

and therefore the whole deflexion at the extremity is

$$\frac{3\rho g a^4}{8k\tau^2};$$

that is, varies as the fourth power of the length of the beam, and inversely as the square of the depth of the beam.

Such are the principles on which is founded the mathematical investigation of the strength of materials: for a more complete investigation I must refer the reader to treatises wherein the subject is specially discussed; because the constants, which are left undetermined in the preceding expressions, are to be found by experiment; and particular and very delicate apparatus, the construction of which requires explanation, is needed for their determination.

# CHAPTER VI.

## ON ATTRACTIONS.

169.] In the following chapter, amongst many properties of matter which will be formally stated as axiomatic principles of the science of motion, will occur one which is called the law of Inertia, and which declares that matter has no power to change the state in which itself is; and experiment amply verifies it in the phænomena of nature: it is not however hence to be inferred that matter has no power of acting on, or of influencing, other matter: on the contrary, matter does act on other matter in the way of either attraction or repulsion, and according to certain laws: and this action is not impeded by the presence or the intervention of other matter; every particle of matter attracts or repels every other particle in the same way as if the two existed alone. As evidence of this active power of matter, Nature presents to us many phænomena. There is, in the first place, that universal law of gravitation, by reason of which every material particle of the celestial system exercises on every other particle a force which varies as the product of the masses of the particles, and inversely as the square of the distance between them; and which acts along the line joining the two particles, and tends to draw them nearer together. So again in the explanation of magnetic and electrical phænomena, there are doubtless two states in which particles active with the influence may be: and the attraction or repulsion which mutually acts between them varies as the product of the intensities of the two particles, and inversely as the square of the distance between them; and the force is attractive or repulsive according as the particles are in opposite or in the same magnetic states; and the line of action is that which joins the two particles. There are also other phænomena where the attraction varies inversely as the square of the distance, but where the line of action is not that which joins the two particles. These and similar cases require investigation, and for this reason: when two single material particles attract or repel each other, it is easy to conceive the force which mutually acts from one to the other; we can easily imagine the tendency of the one to move towards or from the other in the straight line which joins the two. But when one material particle is attracted simultaneously by many others, aggregated into a finite body of a given form and density, the determination of the intensity and of the line of action of the resultant force requires investigation; and perhaps also the density of the attracting body may vary, in which case the difficulty is increased. The following inquiry will be for the most part confined to the cases where the law of attraction is that of the product of the two attracting particles, and of the inverse square of the distance between them, because this is the law of gravitation, and generally rules in cosmical phænomena: but it will also embrace other laws; so that by operating with general laws we may determine the results which they necessitate, and by a comparison of these with the works of nature, may obtain a knowledge of the special laws which rule therein.

And the investigation is of great importance in its application to natural phænomena: the determination of the form of the earth and of the heavenly bodies depends on it. It underlies the theories of Electricity, Magnetism, and, in many cases, of Heat. Into these special studies I have not entered: but I believe that the general theorems which are investigated in the following Articles are sufficient for a full comprehension of those subjects.

170.] As to the attraction varying as the product of the masses of the attracting and the attracted particles: let there be two particles $m$ and $m'$ at a distance $r$ apart; and let the law of attraction which is a function of the distance between them be $f(r)$; so that the attraction of an unit-particle in the position of $m'$ on an unit-particle in the position of $m$ is $f(r)$: now $m'$ contains $m'$ unit-particles; and each one of these attracts the unit-particle in the position of $m$ with a force $f(r)$; therefore the whole force of $m'$ on the unit-particle in the position of $m$ is $m'f(r)$: but $m$ also contains $m$ unit-particles, and each of these is attracted with equal force by $m'$; therefore the whole attractive (or repulsive) force of $m'$ on $m$ is

$$m\,m'f(r). \qquad (1)$$

If the attraction varies inversely as the square of the distance, then

$$\text{the attractive force} = \frac{mm'}{r^2}; \tag{2}$$

and in all cases which we shall investigate, the line of action of the force lies along $r$.

Suppose now $m$ to be the mass of an attracted particle, and $d\mathrm{v}$ to be a volume-element of the attracting body, and $\rho$ to be the density of $d\mathrm{v}$, and $r$ the distance between $m$ and $d\mathrm{v}$, then the attraction of $d\mathrm{v}$ on $m$ is

$$m\rho\, d\mathrm{v}\, f(r); \tag{3}$$

and the attraction of the whole body on $m$ will be found by means of the Integral Calculus.

The course of our enquiry will be—(1) to solve certain particular problems of attractions of rods, plates, shells, and solids on given particles: and (2) to investigate some general properties of a function (the potential of attraction), and to shew that the complete solution of the problem depends on this function, by means of differentiation only. And let it be observed that whenever the law of attraction is not expressly stated, it is assumed to be that of gravitation.

171.] The attraction of a straight rod or wire of uniform thickness and density on an external particle. Fig. 74.

Let O be the attracted particle whose mass is $m$, and let AB be the attracting bar: of which let the density be $\rho$, and let the area of a transverse section be $\omega$; from O draw OC perpendicular to AB; let CA $= a$, CB $= b$: OC $= c$. Let PQ be a volume-element of the bar, CP $= y$, PQ $= dy$: therefore the mass-element at P $= \rho\omega dy$; and let the attractions be calculated along, and perpendicular to, OC; let the attraction of the bar on O along OC and towards C $=$ X, and let the attraction at right angles to OC and towards A $=$ Y. Then

$$\text{the attraction of P on O along OP} = \frac{m\omega\rho\, dy}{c^2+y^2};$$

therefore the attraction of P on O in the direction OC

$$= \frac{m\omega\rho\, dy}{c^2+y^2}\cos \text{POC},$$

$$= \frac{m\omega\rho c\, dy}{(c^2+y^2)^{\frac{3}{2}}}.$$

The attraction of P on O at right angles to OC

$$= \frac{m\omega\rho\, dy}{c^2+y^2} \sin \text{POC},$$

$$= \frac{m\omega\rho\, y\, dy}{(c^2+y^2)^{\frac{3}{2}}};$$

$$\therefore \quad \text{X} = \int_{-b}^{a} \frac{m\omega\rho\, c\, dy}{(c^2+y^2)^{\frac{3}{2}}}$$

$$= \frac{m\omega\rho}{c} \left[ \frac{y}{(c^2+y^2)^{\frac{1}{2}}} \right]_{-b}^{a}$$

$$= \frac{m\omega\rho}{c} \left\{ \frac{a}{(c^2+a^2)^{\frac{1}{2}}} + \frac{b}{(c^2+b^2)^{\frac{1}{2}}} \right\}$$

$$= \frac{m\omega\rho}{c} \left\{ \frac{\text{CA}}{\text{OA}} + \frac{\text{CB}}{\text{OB}} \right\}$$

$$= \frac{m\omega\rho}{c} \{\sin \text{AOC} + \sin \text{BOC}\}; \qquad (4)$$

$$\text{Y} = \int_{-b}^{a} \frac{m\omega\rho\, y\, dy}{(c^2+y^2)^{\frac{3}{2}}}$$

$$= m\omega\rho \left\{ \frac{1}{(b^2+c^2)^{\frac{1}{2}}} - \frac{1}{(a^2+c^2)^{\frac{1}{2}}} \right\}$$

$$= m\omega\rho \left\{ \frac{1}{\text{OB}} - \frac{1}{\text{OA}} \right\}. \qquad (5)$$

If $a = b$, so that C is the middle point of the bar, $\text{Y} = 0$, and $\text{X} = \frac{2m\omega\rho}{c} \sin \text{AOC}$; that is, the attraction of the bar acts only in a direction at right angles to its length.

172.] By the following geometrical construction we obtain a remarkable equivalent for the attraction of a rod on a particle outside of it, as in the last Article.

From centre O, fig. 75, and radius OC, describe an arc of a circle meeting OA, OB, OP, OQ in the points $a, b, p, q$; and suppose a bar of the same material, density, and thickness as AB to be bent into an arc of a circle, and to coincide with the arc $ab$; then the attraction of this bent bar on O is the same as that of the straight bar AB.

From O as a centre, and with the radius OP, describe a small arc PR; then

$$\frac{\text{PQ}}{pq} = \frac{\text{PR} \sec \text{QPR}}{pq} = \frac{\text{OP}}{\text{O}p} \sec \text{POC},$$

$$= \frac{\text{OP}^2}{\text{O}p^2};$$

$$\therefore \quad \frac{\text{PQ}}{\text{OP}^2} = \frac{pq}{op^2}.$$

Now the attraction of PQ on O in the direction OP $= \dfrac{\mu\omega\rho\,\text{PQ}}{\text{OP}^2}$, and therefore is equal to $\dfrac{\mu\omega\rho pq}{op^2}$; that is, is the same as that of the element $pq$ of the circular arc: and as a similar result is true for all the elements of the circular arc, so the total attraction of the bar AB on O is the same as that of the circular bar $ab$. If the angle AOB is bisected by the line OD, the line of action of the whole attraction of the bar $ab$ manifestly is OD: OD is therefore also the line of action of the whole attraction of the bar AB on O.

Hence it follows that if O is capable of moving towards AB, as each element of its path bisects the angle AOB, so will the path be an hyperbola of which A and B are the two foci, and the particle will ultimately meet the bar at a point, the difference between whose distances from A and B is equal to OA—OB. Thus if O is a particle of iron filings and AB is a magnetized bar, the path which O will take in moving towards AB is an hyperbola.

Hence also if three bars of the same thickness and density, and attracting with a force varying inversely as the square of the distance, are arranged as a triangle, a particle placed in the centre of the circle inscribed in the triangle is equally attracted in all directions.

The preceding process is also applicable when the density of the attracting bar is variable.

Also let the following results be proved:

(1) The attraction of a bar of uniform thickness and density, when the attraction varies directly as the distance, on a particle in contact with it at distances $a$ and $b$ respectively from the ends of the bar is

$$\frac{m\rho\omega}{2}(a^2-b^2);$$

and therefore if the attracted particle is placed at the end of a bar whose length is $a$, so that $b=0$,

$$\text{the attraction} = \frac{m\rho\omega a^2}{2},$$

and is the same as if the whole bar were condensed into its centre of gravity at the middle point of $a$.

(2) The attraction of a bar of uniform thickness and density,

on a particle in the same straight line with it, and at distances $a$ and $b$ severally from the ends of the bar, is

$$m\rho\omega\frac{a-b}{ab}.$$

(3) The attraction of two straight bars, each of which is of uniform thickness and density, on each other, in the same straight line, of the lengths $a$ and $b$, and at a distance $c$ apart, is

$$\rho\rho'\omega\omega'\log\frac{(a+c)(b+c)}{c(a+b+c)};$$

and this is of course the force which is required to keep the bars asunder.

Since the result involves the anharmonic ratio of the four points which are the ends of the bars, it follows that if AB and CD are the bars, and if through any point V lines VA, VB, VC, VD are drawn of any length, and any line A′B′C′D′ is drawn cutting them, the mutual attraction of A′B′ and C′D′ is the same as that of AB and CD.

(4) Two straight bars of lengths $2a$ and $2b$ and of constant thickness and density, and each particle of which attracts with a force varying inversely as the square of the distance, are placed parallel to each other at a distance $c$ apart, and so that the line joining their middle points is perpendicular to each of them: it is required to shew that the force necessary to keep them apart is

$$\frac{2\rho\rho'\omega\omega'}{c}\Big\{\{c^2+(b+a)^2\}^{\frac{1}{2}}+\{c^2+(b-a)^2\}^{\frac{1}{2}}\Big\}.$$

173.] The attraction of a bent rod of uniform thickness and density on a given particle.

Let us first investigate the attraction of a bar bent into the form of a circular arc on a particle at the centre.

Let $\rho =$ the density, $\omega =$ the area of a transverse section of the bar: $a =$ the radius of the circle, $2\alpha =$ the angle subtended at the centre by the bar; fig. 76. Now it is manifest that the resultant attraction acts along the line OC bisecting the subtended angle, for the resultant attraction which is perpendicular to that line vanishes. Let POC $= \theta$, AOC $=$ BOC $= \alpha$: then the attraction of the bar in the direction OC

$$= \frac{m\omega\rho}{a}\int_{-\alpha}^{\alpha}\cos\theta\, d\theta$$

$$= \frac{2m\omega\rho\sin\alpha}{a}, \qquad (6)$$

and therefore varies directly as the sine of half the subtended angle and inversely as the radius of the arc.

Hence the whole attraction of the bar AB, in Art. 171, on O, and along the line OD, see fig. 75, is

$$\frac{2m\omega\rho}{\text{OC}}\sin\frac{\text{AOB}}{2}.$$

Hereby we are enabled to solve the following problems:

Ex. 1. Three bars, each of which is of uniform density and thickness, form a triangle; find the position of a particle placed within the triangle which is equally attracted in all directions.

Let the densities of the bars be respectively $\rho$, $\sigma$, $\tau$, and let the transverse sections of all three be the same; let the perpendiculars from the attracted particle on the sides be $p, q, r$; and let the sides subtend at the attracted particle angles $2a, 2\beta, 2\gamma$; then the particle is kept at rest by the three forces

$$\frac{2m\omega\rho\sin a}{p}, \quad \frac{2m\omega\sigma\sin\beta}{q}, \quad \frac{2m\omega\tau\sin\gamma}{r},$$

the angles between the lines of action of which are $\beta+\gamma$, $\gamma+a$, $a+\beta$; or $180°-a$, $180°-\beta$, $180°-\gamma$, because $a+\beta+\gamma=180°$; and therefore by the triangle of forces, Art. 19, the forces are proportional to the sines of these angles; therefore

$$\frac{\rho}{p}=\frac{\sigma}{q}=\frac{\tau}{r}.$$

And if $\rho=\sigma=\tau$, $p=q=r$, and the attracted particle is at the centre of the circle inscribed in the triangle.

Ex. 2. Two bars CA and CB of the same constant thickness and density meet at right-angles and attract a particle placed at the foot of the perpendicular from C on AB; it is required to find the magnitude and the line of action of the whole attraction.

Let $\text{CA}=a$, $\text{CB}=b$, $a^2+b^2=c^2$; and let P be the position of the attracted particle. Then the attraction of CA on P in the line bisecting the angle APC is $\frac{2m\omega\rho c^2}{ab^2}\sin 45°$; and, similarly, the attraction of CB on P in the line bisecting the angle BPC is $\frac{2m\omega\rho c^2}{ba^2}\sin 45°$; and as these two lines of action are perpendicular to each other,

$$\text{the resultant attraction} = \frac{2^{\frac{1}{2}}m\omega\rho c^3}{a^2b^2};$$

and the line of action of it is inclined at 45° to each of the lines CA and CB.

From (6) it appears, that the attraction of a circular rod on a particle at its centre is the greatest when $a = 90°$, that is, when the arc is a semicircle; and if $a = 180°$, that is, if the circle is complete, the attraction vanishes.

Suppose however the ring to be complete, and the attracted particle to be in the plane of the ring, and at a small distance $x$ from the centre; it is required to find the force acting on the particle.

Let $\rho$ be the density, and $\omega$ the area of a transverse section of the ring: $a =$ the radius, fig. 77, $\text{CO} = x$, which is very small, and such that we shall neglect the third and higher powers of it; $\text{PCA} = \theta$, $\text{QCP} = d\theta$; $m =$ the mass of O, $\text{OM} = a\cos\theta - x$, $\text{OP}^2 = a^2 - 2ax\cos\theta + x^2$. It is manifest that the ring attracts O along the line COA alone; and

the attraction

$$= 2m\int_0^{\pi} \frac{\rho\omega a\, d\theta}{\text{OP}^2} \cos \text{POM}$$

$$= 2m\rho\omega a\int_0^{\pi} \frac{(a\cos\theta - x)\, d\theta}{(a^2 - 2ax\cos\theta + x^2)^{\frac{3}{2}}}$$

$$= \frac{2m\rho\omega}{a^2}\int_0^{\pi} (a\cos\theta - x)\left(1 - \frac{2x\cos\theta}{a} + \frac{x^2}{a^2}\right)^{-\frac{3}{2}} d\theta$$

$$= \frac{2m\rho\omega}{a^2}\int_0^{\pi} \left\{a\cos\theta + \frac{3\cos 2\theta + 1}{2}x + \frac{9\cos\theta + 15\cos 3\theta}{3a}x^2\right\} d\theta$$

$$= \frac{\pi m\rho\omega}{a^2}x,$$

and therefore the attraction varies directly as the distance of the particle from the centre of the ring.

174.] To find the attraction of a circular ring of uniform thickness and density on a particle ($m$) at a given distance from its plane, and in the line perpendicular to the plane and passing through the centre of the ring.

Let $a$ be the radius, $\rho$ the density, $\omega$ the area of a transverse section of the ring, $c$ the distance of the attracted particle from the plane of the ring; see fig. 78; suppose the plane of the ring to be perpendicular to that of the paper; let P be an element of it, and let the line PC drawn from P to C, the centre of the ring, make an angle $\theta$ with the plane of the paper; then the mass of the element at $\text{P} = \rho\omega a\, d\theta$; and the attraction of P on O along the line OC

$$= \frac{m\rho\omega a\, d\theta}{c^2 + a^2} \cos \text{POC};$$

therefore the attraction of the ring $= \dfrac{m\rho\omega ac}{(c^2+a^2)^{\frac{3}{2}}}\,2\pi.$

Hereby we can determine the attraction of a hollow cylindrical tube on a particle at a given point in its axis.

Let $r =$ the radius of the interior surface of the tube, $\tau =$ the thickness, $\rho =$ the density; and let the distances of the attracted particle $(m)$ from the ends of the axis of the tube be $a$ and $b$; and let the tube be resolved into a series of rings of infinitesimal depth by means of planes perpendicular to the axis of the tube: then if $x$ is the distance from $m$ of any ring whose thickness is $dx$,

the attraction of the whole tube

$$= 2\pi m\rho\tau r\int_{-b}^{a}\frac{x\,dx}{(r^2+x^2)^{\frac{3}{2}}}$$

$$= 2\pi m\rho\tau r\left\{\frac{1}{(r^2+b^2)^{\frac{1}{2}}}-\frac{1}{(r^2+a^2)^{\frac{1}{2}}}\right\}.$$

175.] The attraction of a thin circular plate on a particle $(m)$ in the line passing through the centre of the plate and perpendicular to it.

Let the attracted particle and the centre of the plate be in the plane of the paper, fig. 79; and let the plane of the plate be perpendicular to it. Let $\rho =$ the density, $\tau =$ the thickness of the plate: $a =$ the radius, and $\text{OC} = c$, the distance of the attracted particle from the plate. Resolve the plate into concentric circular rings, of which let the radius of that containing the element P be $r$, and the depth be $dr$; then if $\text{ACP} = \theta$, $\text{PCQ} = d\theta$,

the mass of the element at P $= \rho\tau r\,dr\,d\theta$;

therefore the attraction of the plate on O in the direction OC

$$= m\rho\tau c\int_{0}^{a}\int_{0}^{2\pi}\frac{r\,d\theta\,dr}{(c^2+r^2)^{\frac{3}{2}}}$$

$$= 2\pi m\rho\tau c\left[-\frac{1}{(c^2+r^2)^{\frac{1}{2}}}\right]_{r=0}^{r=a}$$

$$= 2\pi m\rho\tau\left\{1-\frac{c}{(a^2+c^2)^{\frac{1}{2}}}\right\}. \qquad (7)$$

The attraction of the plate in a direction at right-angles to OC is evidently zero.

Since in (7) $\dfrac{c}{(a^2+c^2)^{\frac{1}{2}}} = \cos \text{AOC}$, it follows that the attraction of all circular plates of the same thickness and density on a particle in the line passing through their centres and perpendicular to their planes is the same, if their diameters subtend the same angle at the attracted particle. Hence if a right cone is divided into a series of circular plates, all of which are of the same thickness, by means of planes perpendicular to the axis of the cone, the attraction of each of these on a particle at the vertex is the same.

In (7) if the radius of the plate is infinite, that is, if $a = \infty$,

$$\text{the attraction} = 2\pi m\rho\tau,$$

which is independent of the distance of the attracted particle from the attracting plate: therefore the attraction of a plate of infinite extent on a particle outside of it is the same, whatever is the distance from the plate at which the particle is placed.

Hence for particles near to the surface of the earth, the earth's attraction is constant; because the earth may be conceived to be divided into a series of thin plates by planes perpendicular to the vertical line passing through the attracted particle, the radius of each of which is very large in comparison of the distance of the particle.

If the law of attraction varies as the $n$th power of the distance, the attraction of the circular plate on a particle outside of it in the line passing through its centre, and perpendicular to its plane,

$$= \frac{2\pi m\rho\tau}{n+1}\left\{(c^2+a^2)^{\frac{n+1}{2}} - c^{n+1}\right\}; \tag{8}$$

which is the same as (7), if $n = -2$.

176.] To determine the attraction of a solid of revolution on a particle in its axis.

Let the solid, fig. 80, be resolved into circular slices of infinitesimal thickness by means of planes perpendicular to the axis of revolution. Let O be the attracted particle, of which the mass is $m$; and let $y = f(x)$ be the equation to the generating curve of the bounding surface of the solid.

Let OM $= x$, MP $= y$, OA $= a$, OB $= b$; and let the thickness of the circular slice PMP′ be $dx$; then by (7),

the attraction on O of the differential circular slice

$$= 2\pi m\rho\left\{1 - \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right\}dx; \tag{9}$$

therefore the attraction of the solid on $m$

$$= 2\pi m\rho \int_a^b \left\{1 - \frac{x}{(x^2+y^2)^{\frac{1}{2}}}\right\} dx; \qquad (10)$$

$y$ having been replaced by its equivalent value in terms of $x$ from the equation to the generating curve.

Similarly may the whole attraction be found from (8), when the attraction varies as the $n$th power of the distance.

Ex. 1. To find the attraction of a right cone of uniform density ($\rho$) on a particle at its vertex.

Let the vertical angle $= a$; so that the equation to the generating line is

$$y = x \tan a;$$

and let $a$ be the altitude of the cone: then from (10),

$$\text{the attraction} = 2\pi m\rho \int_0^a \{1 - \cos a\}\, dx$$
$$= 2\pi m\rho\,(1 - \cos a)\, a.$$

If the attraction varies as the $n$th power of the distance, the attraction of the cone on a particle at its vertex is

$$\frac{2\pi m\rho}{(n+1)(n+3)} \{(\sec a)^{n+1} - 1\}\, a^{n+3}.$$

Ex. 2. The attraction of a circular cylinder of length $a$ and radius $c$, whose density is constant, on a particle in the centre of its circular end, is, if the attraction varies as the $n$th power of the distance,

$$\frac{2\pi m\rho}{(n+1)(n+3)} \left\{(a^2+c^2)^{\frac{n+3}{2}} - a^{n+3} - c^{n+3}\right\}.$$

Ex. 3. To find the attraction of a sphere of uniform density on a particle external to it.

Let $a =$ the radius, $\rho =$ the density of the sphere; $m =$ the mass of the attracted particle; $c =$ the distance of the particle from the centre of the sphere, so that the equation to the generating circle of the sphere, is

$$y^2 + (x-c)^2 = a^2.$$

The attraction of the sphere on $m$

$$= 2\pi m\rho \int_{c-a}^{c+a} \left\{1 - \frac{x}{(a^2-c^2+2cx)^{\frac{1}{2}}}\right\} dx$$
$$= \frac{4\pi m\rho a^3}{3c^2} \qquad (11)$$
$$= \frac{m\text{M}}{c^2}, \qquad (12)$$

if M = the mass of the sphere: but as $c$ is the distance of the attracted particle from the centre of the sphere, this result expresses the attraction on each other of two particles $m$ and M at the distance $c$ apart: therefore the attraction of a sphere on a particle external to it is the same as if the mass of the sphere were condensed into its centre.

This result is physically of great importance; because in the investigation of the circumstances of a particle moving under the attraction of a sphere, every particle of which attracts it with a force varying inversely as the square of the distance, the attracting sphere may be supposed to be condensed into its centre; and the problem becomes reduced to that of the mutual attraction of two particles.

Also if two spheres attract each other, the action is the same as that of two particles whose masses are equal to those of the spheres, and placed at the centres of the spheres: and therefore the force which acts mutually on them is equal to the product of their masses divided by the square of the distance between their centres.

Ex. 4. To find the attraction of a homogeneous sphere on a particle on its surface.

In this case, $y^2 = 2ax - x^2$; therefore

$$\text{the attraction} = 2\pi m\rho \int_0^{2a} \left\{1 - \left(\frac{x}{2a}\right)^{\frac{1}{2}}\right\} dx$$

$$= 2\pi m\rho \frac{2a}{3}$$

$$= \frac{4\pi m\rho a}{3};$$

and therefore the attraction varies directly as the radius of the sphere.

Ex. 5. To find the attraction of a sphere of constant density on a particle within it.

If $c$ is the distance of the particle from the centre of the sphere, and $a$ is the radius of the sphere, the attraction of the larger segment of the sphere whose base is the plane through the attracted particle and perpendicular to the line joining it and the centre is

$$\frac{2\pi m\rho}{3c^2}\{a^3 + c^3 - (a^2 - c^2)^{\frac{3}{2}}\};$$

and the attraction of the lesser segment is

$$\frac{2\pi m\rho}{3c^2}\{a^3-c^3-(a^2-c^2)^{\frac{3}{2}}\},$$

and the attraction of the whole sphere, being the excess of the former of these over the latter, is

$$\frac{4\pi m\rho}{3}c;$$

and varies therefore as the distance of the attracted particle from the centre of the sphere. But by the last example this would be the case if the particle were on the surface of a sphere whose radius is $c$; therefore the spherical shell, of the thickness $a-c$, exerts no attraction on the particle.

Ex. 6. By similar processes let it be proved that the attraction of an oblate spheroid on a particle ($m$) at its pole is

$$\frac{4\pi\rho m b}{e^2}\left\{1-\frac{(1-e^2)^{\frac{1}{2}}}{e}\sin^{-1}e\right\},$$

where $b$ and $e$ are respectively the semi minor axis and the eccentricity of the generating ellipse. And that the attraction of a prolate spheroid on a particle ($m$) at its pole is

$$\frac{4\pi\rho m a(1-e^2)}{e^2}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\};$$

where $a$ and $e$ are respectively the semi major axis and the eccentricity of the generating ellipse.

177.] The Calculus of Variations enables us to solve the following problem.

To determine the form of the bounding surface of revolution of a homogeneous mass of given volume, so that the attraction of it on a particle in its axis may be a maximum.

Let $u$ be the attraction, and let $\pi c^3$ be the volume of the given mass, which is to be contained between $x_1$ and $x_0$; and thus, if $r^2=x^2+y^2$,

$$u = 2\pi m\rho\int_0^1\left\{1-\frac{x}{r}\right\}dx, \tag{13}$$

$$\pi c^3 = \pi\int_0^1 y^2\,dx; \tag{14}$$

$$\therefore\quad \delta u = 0 = \left[\left(1-\frac{x}{r}\right)\delta x\right]_0^1-\int_0^1\left\{\frac{xy}{r^3}\,dy\,\delta x-\frac{xy}{r^3}\,dx\,\delta y\right\}, \tag{15}$$

$$\delta.c^3 = 0 = \Big[y^2\,\delta x\Big]_0^1-\int_0^1\{2y\,dy\,\delta x-2y\,dx\,\delta y\}; \tag{16}$$

therefore, if $k^2$ is an arbitrary constant,

$$k^2\frac{xy}{r^3} = y; \qquad \therefore \quad k^2 x = (x^2+y^2)^{\frac{3}{2}},$$

$$y^2 = x^{\frac{2}{3}}(k^{\frac{4}{3}}-x^{\frac{4}{3}}). \tag{17}$$

Hence it appears that the curve, which by its revolution about the axis of $x$ generates the solid, cuts the axis of $x$ at the origin, and also when $x = k$; thus $x_0 = 0$, $x_1 = k$, and the integrated part of (15) shews that it cuts it in both points at right angles. Also substituting from (17) in (14) we have

$$\begin{aligned} c^3 &= \int_0^k (x^{\frac{2}{3}}k^{\frac{4}{3}}-x^2)\,dx \\ &= \frac{4}{15}k^3; \\ \therefore \quad k^3 &= \frac{15}{4}c^3: \end{aligned}$$

substituting from which in (17), the equation to the curve is completely determined. And the attraction of it on $(m)$

$$\begin{aligned} &= 2\pi m\rho\int_0^k \left\{1-\frac{x^{\frac{2}{3}}}{k^{\frac{2}{3}}}\right\}dx \\ &= \frac{4}{5}\pi m\rho k \\ &= \left(\frac{48}{25}\right)^{\frac{1}{3}}\pi m\rho c = \text{A}_1 \text{ (say)}. \end{aligned}$$

Now the attraction of a sphere whose mass is $\pi\rho c^3$ on a particle $(m)$ at its surface is

$$\left(\frac{4}{3}\right)^{\frac{2}{3}}\pi m\rho c = \text{A}_2 \text{ (say)}; \qquad \therefore \quad \frac{\text{A}_1}{\text{A}_2} = \left(\frac{27}{25}\right)^{\frac{1}{3}}.$$

Every particle on the surface of the solid of given mass and of greatest attraction attracts $(m)$ with equal force in the direction of the axis; for if $r$ is the distance of any particle on the surface from $(m)$, and if $\theta$ is the angle between $r$ and the axis, from the preceding equations we have,

$$\frac{\cos\theta}{r^2} = \frac{1}{k^2},$$

which is constant, and therefore is the same for all particles on the surface.

178.] To find the attraction of a spherical shell of infinitesimal thickness, and of constant density on an external particle, when the law of attraction is represented by $f$ (distance).

Let the centre of the shell be the origin; and let the shell be referred to that system of polar coordinates in space, which is explained in Art. 146, Vol. II, and let the attracted particle be on the axis of $z$ at a distance $c$ from the centre of the shell; let $r =$ the radius, $dr =$ the thickness, $\rho =$ the density of the spherical shell: so that

$$\text{the mass-element of the shell} = \rho r^2 \sin\theta\, dr\, d\theta\, d\phi:$$

let $u =$ the distance of this mass-element from the attracted particle $(m)$; then the attraction of the mass-element on the attracted particle along the line joining $(m)$ and the centre of the shell is

$$\frac{\rho m r^2 \sin\theta (c - r\cos\theta) f(u)}{u}\, dr\, d\theta\, d\phi; \tag{18}$$

and therefore the attraction of the shell on $(m)$

$$= \rho m r^2 dr \int_0^{\pi}\int_0^{2\pi} \frac{\sin\theta (c - r\cos\theta)}{u} f(u)\, d\phi\, d\theta$$

$$= 2\pi\rho m r^2 dr \int_0^{\pi} \frac{\sin\theta (c - r\cos\theta)}{u} f(u)\, d\theta. \tag{19}$$

But since $\qquad u^2 = r^2 - 2rc\cos\theta + c^2;$

$$\therefore \quad u\, du = rc\sin\theta\, d\theta,$$

and $\qquad 2c(c - r\cos\theta) = u^2 + c^2 - r^2,$

and when $\theta = \pi$, $u = c + r$, when $\theta = 0$, $u = c - r$; therefore substituting in (19), the attraction of the shell on $(m)$

$$= \frac{\pi\rho m r\, dr}{c^2} \int_{c-r}^{c+r} (u^2 + c^2 - r^2) f(u)\, du. \tag{20}$$

(1) Let the law of attraction be that of the inverse square of the distance,

$$\therefore \quad f(u) = \frac{1}{u^2};$$

$$\therefore \quad \text{the attraction} = \frac{\pi\rho m r\, dr}{c^2} \left[u - \frac{c^2 - r^2}{u}\right]_{c-r}^{c+r}$$

$$= \frac{4\pi\rho m r^2 dr}{c^2}; \tag{21}$$

and the mass of the shell $= 4\pi\rho r^2 dr = \text{M}$ (say),

$$\therefore \quad \text{the attraction of the shell} = \frac{m\text{M}}{c^2};$$

and is therefore the same as if the mass of the shell were condensed into its centre.

(2) Let the attraction vary directly as the distance: then $f(u) = u$, and

$$\text{the attraction of the shell} = \frac{\pi\rho m r\,dr}{c^2}\left[\frac{u^3}{3} + \frac{c^2-r^2}{2}u^2\right]_{c-r}^{c+r}$$
$$= 4\pi\rho m r^2 dr\, c \qquad (22)$$
$$= m\text{M}c;$$

and therefore the attraction is the same as if the mass of the shell were condensed into its centre.

Similarly may the attraction be determined for any other law of attraction.

179.] Hereby the attraction of a sphere on an external particle can be determined by considering it as resolved into a series of concentric spherical shells of infinitesimal thickness.

(1) Let the sphere be homogeneous, and let $a$ be its radius: then if the law of attraction is that of the inverse square of the distance, from (21),

$$\text{the attraction on } (m) = \frac{4\pi\rho m}{c^2}\int_0^a r^2 dr$$
$$= \frac{4\pi\rho m a^3}{3c^2}; \qquad (23)$$

and is the same as if the sphere were condensed into its centre: and if the particle $(m)$ is on the surface, $c = a$, and

$$\text{the attraction} = \frac{4\pi\rho m}{3}a, \qquad (24)$$

and varies directly as the radius of the sphere.

Let the law of attraction be that of the direct distance: then from (22),

$$\text{the attraction on } (m) = 4\pi\rho m c\int_0^a r^2 dr$$
$$= \frac{4\pi\rho m c a^3}{3}; \qquad (25)$$

and therefore is the same as if the sphere were condensed into its centre.

(2) Let us assume the density of a particle of the sphere to vary as some power of the distance of the particle from the centre: so that the sphere is composed of a series of homogeneous concentric shells, the density of which is different for different shells.

Thus suppose the density to vary inversely as the distance from the centre, and the law of attraction to be that of the inverse square of the distance; then by reason of (21), since $\rho = \frac{k}{r}$,

$$\text{the attraction on } (m) = \frac{4\pi\rho mk}{c^2}\int_0^a r\,dr$$

$$= \frac{2\pi\rho mka^2}{c^2}. \tag{26}$$

And if the attraction varies directly as the distance,

$$\text{the attraction on } (m) = 2\pi\rho mcka^2.$$

In each of the cases, (21) and (22), as the attraction of the shell on $(m)$ is the same as if the shell were condensed into its centre, so will the attraction of the whole full sphere be the same as if it were condensed into its centre.

Now in celestial mechanics this fact is of great importance: for the planetary bodies are nearly spherical, and the density of each of them is variable; and they are probably composed of concentric shells, each of which is of uniform density, and the density of which decreases as we pass from the centre to the surface. Thus by this property we can avoid the difficulty of investigating the attracting properties of them as solid bodies, and we can treat them as single attracting material particles.

From the preceding results also it follows that supposing the earth to be nearly a sphere, the attraction of it on particles external to it varies as the square of their distance from the centre of the earth. Thus if $g$ and $g'$ are the attractions of the earth on the same particle respectively at the mean surface which corresponds to the radius $r$, and on the top of a mountain whose height is $h$, then

$$\frac{g}{g'} = \frac{(r+h)^2}{r^2}.$$

180.] And the preceding results suggest another important question: Are there any other laws of attraction, besides those of the inverse square of the distance, and of the direct distance, for which the attraction of a spherical shell on a particle without it is the same as if the shell were condensed into its centre?

If $\rho =$ the density of the shell, and $c$ is the distance of its centre from $(m)$, the attraction of the shell, condensed into its centre, is $4\pi\rho mr^2 dr f(c)$; and as this is equal to its attraction in its actual form, we have from (20)

$$4\pi\rho mr^2 dr f(c) = \frac{\pi\rho mr\,dr}{c^2}\int_{c-r}^{c+r}(u^2+c^2-r^2)f(u)\,du, \tag{27}$$

where the form of $f$ is to be determined. Omitting common factors, and integrating by parts the right-hand member of (27),

$$4rc^2f(c) = \left[(u^2+c^2-r^2)\int f(u)\,du\right]_{c-r}^{c+r} - 2\int_{c-r}^{c+r}\left\{u\int f(u)\,du\right\}du. \quad (28)$$

Let $\int f(u)\,du = \phi(u)$, and let $\int u\phi(u)\,du = \psi(u)$; (29)

$$\therefore \quad \phi'(u) = f(u), \qquad \psi'(u) = u\phi(u); \quad (30)$$

and therefore from (28),

$$\begin{aligned}4rc^2f(c) &= \left[(u^2+c^2-r^2)\phi(u)-2\psi(u)\right]_{c-r}^{c+r}\\ &= 2c(c+r)\phi(c+r)-2c(c-r)\phi(c-r)-2\psi(c+r)+2\psi(c-r)\\ &= 2c^2\frac{d}{dc}\left\{\frac{\psi(c+r)-\psi(c-r)}{c}\right\};\end{aligned}$$

$$\therefore \quad 2rf(c) = \frac{d}{dc}\left\{\frac{\psi(c+r)-\psi(c-r)}{c}\right\}; \quad (31)$$

which is a functional equation to be satisfied by the form of *f*. Now expanding by Taylor's series, we have

$$2rf(c) = \frac{d}{dc}\cdot\frac{2}{c}\left\{\psi'(c)\frac{r}{1}+\psi'''(c)\frac{r^3}{1.2.3}+\dots\right\}, \quad (32)$$

and as no relation exists between $r$ and $c$, the coefficients of the several powers of $r$ must vanish separately; therefore

$$f(c) = \frac{d}{dc}\frac{\psi'(c)}{c}; \quad (33)$$

$$0 = \frac{d}{dc}\frac{\psi'''(c)}{c}; \text{ and so on}; \quad (34)$$

but from (30), $\psi'(c) = c\phi(c)$; therefore from (33),

$$\begin{aligned}f(c) &= \frac{d}{dc}\phi(c)\\ &= f(c),\end{aligned}$$

which is only an identity. Also since from (30)

$$\begin{aligned}\psi'(c) &= c\phi(c),\\ \therefore \quad \psi''(c) &= \phi(c)+c\phi'(c)\\ &= \phi(c)+cf(c), \text{ from (30)};\\ \therefore \quad \psi'''(c) &= \phi'(c)+f(c)+cf'(c)\\ &= 2f(c)+cf'(c);\end{aligned}$$

therefore from (34), if $3\text{A}$ is an arbitrary constant,

$$\begin{aligned}2f(c)+cf'(c) &= 3\text{A}c,\\ \therefore \quad 2cf(c)+c^2f'(c) &= 3\text{A}c^2;\\ c^2f(c) &= \text{A}c^3+\text{B},\end{aligned}$$

where B is another arbitrary constant: therefore

$$f(c) = \text{A}c+\frac{\text{B}}{c^2}; \quad (35)$$

and this value of $f(c)$ also makes to vanish the coefficients of all the other powers of $r$ in (32): it is therefore the complete solution of the equation (31).

Thus the only laws which satisfy the requirements of the problem are (1) that of the inverse square of the distance, when $\text{A} = 0$ and B is finite; (2) that of the direct distance, when $\text{B} = 0$ and A is finite; and (3) that of these laws in combination, when A and B are both finite.

These are of course the only laws of attraction for which a sphere can attract an external particle with the same force as if it is condensed into its centre; because the sphere may be resolved into a series of concentric homogeneous shells, each of which will attract with the same force as if it is condensed into its centre.

Hence also two homogeneous shells external to each other will attract each other with the same force as if each is condensed into its centre.

181.] To investigate the attraction of a homogeneous spherical shell of infinitesimal thickness on a particle ($m$) placed within it, when the law of attraction is represented by $f$ (distance).

Let all the quantities and symbols be the same as in Art. 178: in this case however $c$ is less than $r$, and the limits of integration in the expression corresponding to (20) are $r+c$ and $r-c$; so that the attraction of the shell on ($m$)

$$= \frac{\pi \rho m r\, dr}{c^2} \int_{r-c}^{r+c} (u^2 + c^2 - r^2) f(u)\, du. \tag{36}$$

(1) Let the law of attraction be that of the inverse square of the distance; so that $f(u) = \dfrac{1}{u^2}$;

$$\begin{aligned} \text{the attraction } &= \frac{\pi \rho m r\, dr}{c^2} \Big[ u + \frac{r^2 - c^2}{u} \Big]_{r-c}^{r+c} \\ &= \frac{\pi \rho m r\, dr}{c^2} \{2c - 2c\} \\ &= 0; \end{aligned} \tag{37}$$

therefore the attraction of the shell on an interior particle is zero, and the particle is equally attracted in all directions.

The geometrical proof of this proposition is so simple that it is desirable to insert it. In fig. 81, let the centre of the shell and the attracted particle be in the plane of the paper, and let the circular ring APBP′ be the section of the shell by the same plane. At O, the place of $(m)$, let solid angles be formed, which occupy all space about it: and let each be considered with reference to an equal opposite and vertical one; let $\omega$ be the area of a spherical surface, described from O at the radius = unity, which is intercepted by one of these solid angles: then the area of the spherical surface intercepted at the distance $r$ is $r^2\omega$: thus a mass-element of $\mathrm{P}pq\mathrm{Q}$ at the distance $r$ from O $= \rho r^2 \omega dr$; and as the attraction of this on $m$

$$= \frac{m}{r^2}\rho r^2 \omega dr = m\rho\omega dr,$$

so will the attraction of all the mass at $\mathrm{P}pq\mathrm{Q}$ be $m\rho\omega \times \mathrm{P}p$: similarly, the attraction at $\mathrm{P}'p'q'\mathrm{Q}'$ is $m\rho\omega \times \mathrm{P}'p'$: but by the geometry of the circle $\mathrm{P}p = \mathrm{P}'p'$; therefore the attractions of these masses are equal: and acting on O in opposite directions, they neutralize each other. And because the same result is true of every pair of such opposite small masses into which the whole shell may be divided, the attraction exercised by it on the particle $(m)$ at O is zero.

The shell has been considered to be of finite thickness, but it is obvious that the same result is true for a shell of infinitesimal thickness.

Hence it follows that the attraction of a full homogeneous sphere on a particle within it varies as the distance of the particle from the centre: for if a concentric spherical surface is described passing through the attracted particle, the shell lying outside of that sphere has no attraction on the particle; and it is attracted only by the mass lying within that sphere; and that varies directly as its radius: see equation (24).

If the sphere is composed of a series of concentric homogeneous shells, the density of which however varies, then the attraction of all those lying outside of the concentric sphere passing through the attracted particle is zero: and as the attraction of each of the others is the same as if it were condensed into its centre, so if $c =$ the distance of $(m)$ from the centre and $\mathrm{M} =$ the mass of all those shells lying nearer than $(m)$ to the centre,

$$\text{the attraction of the sphere on } (m) = \frac{m\mathrm{M}}{c^2};$$

and if the matter of M is homogeneous, so that M varies as $c^3$, the attraction varies directly as the distance from the centre.

(2) Let the attraction vary directly as the distance; then

$$f(u) = u,$$

and (36) gives

$$\text{attraction} = \frac{\pi\rho m r\,dr}{c^2}\left[\frac{u^4}{4} - \frac{(r^2-c^2)u^2}{2}\right]_{r-c}^{r+c}$$

$$= 4\pi\rho m r^2 dr\, c,$$

and which is the same as if the shell were condensed into its centre.

182.] The result in (37) leads us to inquire whether there are any other laws besides that of the inverse square of the distance, for which the attraction of a homogeneous spherical shell on a particle within it is zero.

In this case from (36),

$$0 = \int_{r-c}^{r+c} \{u^2+c^2-r^2\} f(u)\,du$$

$$= \left[(u^2+c^2-r^2)\int f(u)\,du\right]_{r-c}^{r+c} - 2\int_{r-c}^{r+c}\left\{u\int f(u)\,du\right\}du;$$

now making the substitutions of (29), we have

$$0 = \Big[(u^2+c^2-r^2)\,\phi(u) - 2\,\psi(u)\Big]_{r-c}^{r+c}$$

$$\therefore\quad 0 = c(r+c)\,\phi(r+c) + c(r-c)\,\phi(r-c) - \psi(r+c) + \psi(r-c)$$

$$= \frac{d}{dc}\left\{\frac{\psi(r+c)-\psi(r-c)}{c}\right\};$$

$$\therefore\quad 0 = \frac{d}{dc}\left\{\psi'(r) + \psi'''(r)\frac{c^2}{1.2.3} + \dots\right\},$$

and integrating,

$$\psi'(r) + \psi'''(r)\frac{c^2}{1.2.3} + \dots = \text{a constant} = \text{A}; \qquad (38)$$

and as no relation exists between $r$ and $c$,

$$\psi'(r) = \text{A}, \qquad (39)$$

$$\psi'''(r) = 0, \text{ and so on.}$$

And since from (30), $\psi'(r) = r\phi(r)$,

$$\therefore\quad r\phi(r) = \text{A}, \qquad \phi(r) = \int f(r)\,dr = \frac{\text{A}}{r};$$

$$\therefore\quad f(r) = -\frac{\text{A}}{r^2};$$

therefore the law of the inverse square of the distance is the only one for which the attraction of a homogeneous spherical shell on an internal particle is zero.

183.] Before I proceed to other parts of the subject, there are two observations to be made.

(1) In the preceding investigations, whenever the law of attraction has been, as to distance, that of the distance directly, the attraction of the body on the attracted particle has been the same as if the body were condensed into its centre of gravity. Now this property admits of generalization, so that in all cases, whatever is the form of the attracting body, the total attraction of it on a material particle is the same as if the body were condensed into its centre of gravity.

Let the centre of gravity be taken as the origin; $(a, \beta, \gamma)$ the position of $(m)$ the attracted particle; $(x, y, z)$ a point of the attracting body, at which let the element $dx\,dy\,dz$ of its volume abut: let $\rho$ be the density, and $r =$ the distance between $(m)$ and $dx\,dy\,dz$: then if $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are the resolved attractions along the three coordinate axes,

$$\begin{aligned}
\mathrm{X} &= m\iiint \rho r\,dx\,dy\,dz\,\frac{a-x}{r} \\
&= m\iiint \rho(a-x)\,dx\,dy\,dz \\
\mathrm{Y} &= m\iiint \rho(\beta-y)\,dx\,dy\,dz \\
\mathrm{Z} &= m\iiint \rho(\gamma-z)\,dx\,dy\,dz;
\end{aligned}$$

but because the origin is the centre of gravity,

$$\iiint \rho x\,dx\,dy\,dz = \iiint \rho y\,dx\,dy\,dz = \iiint \rho z\,dx\,dy\,dz = 0;$$

and if $\mathrm{M} =$ the mass of the attracting body,

$$\mathrm{M} = \iiint \rho\,dx\,dy\,dz;$$

$$\therefore \quad \mathrm{X} = m\mathrm{M}a, \qquad \mathrm{Y} = m\mathrm{M}\beta, \qquad \mathrm{Z} = m\mathrm{M}\gamma;$$

and these are the resolved parts of the attraction of $\mathrm{M}$, which is at the origin, on $m$ placed at the point $(a, \beta, \gamma)$; and therefore the proposition is proved.

(2) Some remarkable results arise, when a spherical shell or a sphere attracts a material particle, if the law of attraction is that of the inverse square of the distance; (1) the attraction of a spherical shell or of a sphere on an external particle is the same as if they were respectively condensed into their centres;

and (2) the attraction of a spherical shell on an internal particle is zero. Also we shall hereafter shew that, Kepler's laws being assumed to be true, the same law of attraction holds good in the motion of the celestial bodies; and also in electrical and magnetic phænomena, results can be explained and accounted for by the same law. Now is there for this law any *a priori* probability? Can we assign any reasons why the attraction should vary directly as the product of the attracting masses, and inversely as the square of the distance? Suppose $m$ to be the mass of an attracting particle, the influence of the attraction of which on other matter radiates from it in all directions; and which is such that none of its intensity is lost by the process of propagation; let $m$ be the vertex of a cone, whose vertical angle is (say) $2\alpha$; and let us consider the parts of spherical surfaces which are described from ($m$) as a centre with different radii, and are intercepted by the cone. The areas of these spherical segments vary as the squares of their radii, and the same amount of attracting influence is spread over each one; therefore the intensity of the amount of attraction on an unit surface varies inversely as the area of the spherical surface intercepted by the cone; and therefore varies inversely as the square of the distance of the unit surface from the centre of the cone, that is, from the position of $m$; and as the same result is true for each unit which is at the same distance from $m$, therefore, if there are $m'$ units, and $r$ is the distance,

$$\text{the attraction} = \frac{mm'}{r^2};$$

and thus, on the hypothesis that none of the attracting influence is lost by means of or during the propagation, the law of attraction is that of the inverse square of the distance.

184.] The attraction of a homogeneous ellipsoid.

Let the attracted particle ($m$) be at the origin, and let the coordinate axes be parallel to the principal axes of the ellipsoid, the centre of the ellipsoid being at $(\alpha, \beta, \gamma)$, so that the equation to its surface is

$$\frac{(x-\alpha)^2}{a^2} + \frac{(y-\beta)^2}{b^2} + \frac{(z-\gamma)^2}{c^2} = 1; \tag{40}$$

and let us refer it to the system of polar coordinates explained in Art. 146, Vol. II; so that, if

$$\left.\begin{array}{r}\left(\frac{\sin\theta\cos\phi}{a}\right)^2+\left(\frac{\sin\theta\sin\phi}{b}\right)^2+\left(\frac{\cos\theta}{c}\right)^2=\mathrm{A}\\ \frac{\alpha\sin\theta\cos\phi}{a^2}+\frac{\beta\sin\theta\sin\phi}{b^2}+\frac{\gamma\cos\theta}{c^2}=\mathrm{B}\\ \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1=\mathrm{C}\end{array}\right\}, \tag{41}$$

(40) becomes
$$\mathrm{A}r^2-2\mathrm{B}r+\mathrm{C}=0; \tag{42}$$
and if $r_1$ and $r_2$ are the roots of this equation, $r_1$ being the greater, and $r_2$ the less,
$$r_1=\frac{\mathrm{B}+(\mathrm{B}^2-\mathrm{AC})^{\frac{1}{2}}}{\mathrm{A}}, \qquad r_2=\frac{\mathrm{B}-(\mathrm{B}^2-\mathrm{AC})^{\frac{1}{2}}}{\mathrm{A}}. \tag{43}$$
If $\rho$ is the density of the ellipsoid,
$$\text{its mass-element} = \rho r^2\sin\theta\, dr\, d\theta\, d\phi;$$
and as the direction-cosines of $r$ are $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, and $\cos\theta$, so the resolved attractions on $(m)$ of the mass-element are
$$m\rho(\sin\theta)^2\cos\phi\, dr\, d\theta\, d\phi, \qquad m\rho(\sin\theta)^2\sin\phi\, dr\, d\theta\, d\phi,$$
$$m\rho\sin\theta\cos\theta\, dr\, d\theta\, d\phi;$$
the integrals of which for limits assigned by the geometrical conditions of the problem are the resolved parts of the total attraction.

Let X, Y, Z be the resolved parts of the attraction, and let us first consider the attracted particle to be within the ellipsoid; so that the limits of the $r$-integration are $r_1$ and $-r_2$; then
$$\mathrm{X}=\int_0^{\pi}\int_0^{\pi}\int_{-r_2}^{r_1} m\rho(\sin\theta)^2\cos\phi\, dr\, d\theta\, d\phi, \tag{44}$$
$$\mathrm{Y}=\int_0^{\pi}\int_0^{\pi}\int_{-r_2}^{r_1} m\rho(\sin\theta)^2\sin\phi\, dr\, d\theta\, d\phi, \tag{45}$$
$$\mathrm{Z}=\int_0^{\pi}\int_0^{\pi}\int_{-r_2}^{r_1} m\rho\sin\theta\cos\theta\, dr\, d\theta\, d\phi. \tag{46}$$
Of these three I shall consider the last, because it is the most simple; and results which are derived from it may be extended to the other two by an exchange of letters only.

Performing the $r$-integration, and replacing $r_1$ and $r_2$ by their values in (43),
$$\mathrm{Z}=\int_0^{\pi}\int_0^{\pi} m\rho\sin\theta\cos\theta\frac{2\mathrm{B}}{\mathrm{A}}\, d\theta\, d\phi. \tag{47}$$
Now
$$\frac{\sin\theta\cos\theta\,\mathrm{B}}{\mathrm{A}}=\frac{\sin\theta\cos\theta(b^2c^2\alpha\sin\theta\cos\phi+c^2a^2\beta\sin\theta\sin\phi+a^2b^2\gamma\cos\theta)}{(bc\sin\theta\cos\phi)^2+(ca\sin\theta\sin\phi)^2+(ab\cos\theta)^2};$$

and observing that the denominator is a rational function of $\sin\theta$ and $(\cos\theta)^2$, and that the first two of the three terms contained in the numerator are rational functions of the same quantities, and also observing that the limits of the $\theta$-integration are $\pi$ and 0, by virtue of the theorem in (15), Article 82, Vol. II, the integrals of the quantities corresponding to these two terms vanish, and we have

$$\text{z} = 2m\rho a^2 b^2 \gamma \int_0^{\pi}\int_0^{\pi} \frac{\sin\theta\,(\cos\theta)^2\, d\theta\, d\phi}{(bc\sin\theta\cos\phi)^2 + (ca\sin\theta\sin\phi)^2 + (ab\cos\theta)^2}, \quad (48)$$

and performing first the $\phi$-integration, we have

$$\text{z} = 2\pi m\rho ab\gamma \int_0^{\pi} \frac{\sin\theta\,(\cos\theta)^2\, d\theta}{\{c^2(\sin\theta)^2 + a^2(\cos\theta)^2\}^{\frac{1}{2}}\{c^2(\sin\theta)^2 + b^2(\cos\theta)^2\}^{\frac{1}{2}}};$$

and therefore by virtue of Theorem VII, Art. 82, Vol. II,

$$\text{z} = 4\pi m\rho ab\gamma \int_0^{\frac{\pi}{2}} \frac{\sin\theta\,(\cos\theta)^2\, d\theta}{\{c^2(\sin\theta)^2 + a^2(\cos\theta)^2\}^{\frac{1}{2}}\{c^2(\sin\theta)^2 + b^2(\cos\theta)^2\}^{\frac{1}{2}}}. \quad (49)$$

Let $\cos\theta = t$, $\therefore$ $\sin\theta\, d\theta = -dt$; and since $t = 0$, when $\theta = \frac{\pi}{2}$, and $t = 1$, when $\theta = 0$, we have, substituting in (49),

$$\text{z} = 4\pi m\rho ab\gamma \int_0^1 \frac{t^2\, dt}{\{c^2 + (a^2 - c^2)t^2\}^{\frac{1}{2}}\{c^2 + (b^2 - c^2)t^2\}^{\frac{1}{2}}}. \quad (50)$$

Similarly,

$$\text{x} = 4\pi m\rho bc\alpha \int_0^1 \frac{t^2\, dt}{\{a^2 + (b^2 - a^2)t^2\}^{\frac{1}{2}}\{a^2 + (c^2 - a^2)t^2\}^{\frac{1}{2}}}, \quad (51)$$

$$\text{y} = 4\pi m\rho ca\beta \int_0^1 \frac{t^2\, dt}{\{b^2 + (c^2 - b^2)t^2\}^{\frac{1}{2}}\{b^2 + (a^2 - b^2)t^2\}^{\frac{1}{2}}}; \quad (52)$$

which definite integrals represent the resolved attractions of a homogeneous ellipsoid on an internal particle.

Now these three expressions involve elliptic integrals which cannot be expressed in circular or logarithmic functions. The problem however is reduced to simple quadrature; and the required integration involves the summation of the attractions of a series of conical shells, whose common vertex is the attracted point, and the thickness of which increases directly as the distance from the vertex, because the $r$- and the $\phi$-integrations have been taken between limits which give a double conical shell; and therefore the element-function in (51) &c. is the attraction of such a double shell.

$t$, it will be observed, is the cosine of the semi-vertical angle of the cone: the axis of the cone being that principal axis of the ellipsoid parallel to which the attraction is resolved.

Jacobi has put the three preceding expressions for X, Y, Z under an elegant form by means of the following substitution:

$$\text{Let} \qquad t^2 = \frac{a^2}{u+a^2}; \qquad \therefore \quad dt = -\frac{a\,du}{2(a^2+u)^{\frac{3}{2}}};$$

$$\therefore \quad a^2+(b^2-a^2)\,t^2 = a^2\frac{u+b^2}{u+a^2},$$

$$a^2+(c^2-a^2)\,t^2 = a^2\frac{u+c^2}{u+a^2};$$

and when $t=0$, $u=\infty$; when $t=1$, $u=0$: therefore

$$\text{X} = 2\pi m\rho abc\alpha\int_0^\infty \frac{du}{(u+a^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}$$

$$= \frac{3\text{M}m\alpha}{2}\int_0^\infty \frac{du}{(u+a^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}},$$

if $\text{M}$ = the mass of the ellipsoid. Similarly,

$$\text{Y} = \frac{3\text{M}m\beta}{2}\int_0^\infty \frac{du}{(u+b^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}},$$

$$\text{Z} = \frac{3\text{M}m\gamma}{2}\int_0^\infty \frac{du}{(u+c^2)\{(u+a^2)(u+b^2)(u+c^2)\}^{\frac{1}{2}}}.$$

185.] It is to be observed that the values of X, Y, Z in (51) &c. are not changed, if the quantities $a$, $b$, $c$ are replaced by $ka$, $kb$, $kc$, where $k$ is any number: the attractions therefore are not changed by the addition or subtraction of a shell contained between two ellipsoidal surfaces concentric and similar, provided that the attracted particle is within the interior. Hence we infer that a homogeneous shell contained between two similar and concentric ellipsoids attracts a particle within it equally in all directions. This theorem is generally known by the name of Newton's theorem on attractions, and is proved synthetically in the Principia. To it also the geometrical method of Art. 181 is immediately applicable.

In fig. 82 let O be the attracted particle, and let the shell, of which the section through O and the centre by the plane of the paper is drawn in the figure be contained between two similar ellipsoidal surfaces concentric and similarly placed, and let us suppose the shell to be homogeneous. Consider O to be the vertex of solid angles which fill up the space around it; and to each one of these angles let the opposite and vertical angle be drawn as in the figure: let QPOP′Q′ be one of the lines of such angles made by the paper. Now of similar and similarly situated ellipsoids it is a property that PQ = P′Q′; let $\omega$ be the area of a spherical surface, described about O as a centre, with unity as the radius, which the cone intercepts: so that the volume, of which PQ$qp$ is the section made by the paper, consists of elements, each of which is equal to $\omega r^2 dr$, and the attraction of each of which on $(m)$ placed at O is

$$\frac{m\rho}{r^2}\,\omega r^2 dr = m\rho\omega dr;$$

and of which the sum is $m\rho\omega \times$ PQ; similarly the attraction of P′Q′$q'p'$, is $m\rho\omega \times$ P′Q′, which is equal to the preceding; therefore the two attractions acting in opposite directions neutralize each other: and as the same result holds true for all the solid angles at O, so the resultant attraction of the shell on O vanishes.

As this proposition is independent of the thickness of the shell, it is also true for a shell of infinitesimal thickness; and therefore is true also for a shell of any thickness, composed of homogeneous concentric, similar, and similarly-placed shells, the density of each of which varies according to any given law.

Hence also it appears that if, as in fig. 83, $(m)$ is without the shell, and from O a cone, intercepting a spherical area $(\omega)$ with a radius unity, is drawn, the attraction of the intercepted part of the shell at P is equal to that at Q; and as the same result is true for all similar cones, it follows that if O is considered to be a pole with reference to the exterior ellipsoid, the polar plane will divide the shell into two parts, the attractions of which on $(m)$ at the pole are equal.

186.] In the cases however of the bounding surfaces being surfaces of revolution, X, Y, Z in Art. 184 can be integrated again.

(1) Let the spheroid be oblate; that is, let $a = b$; and let $e$ be the eccentricity of the generating ellipse of the spheroid, so that

$$a^2 - c^2 = a^2 e^2, \qquad c^2 = a^2(1-e^2);$$

$$\therefore \quad \text{X} = \frac{4\pi\rho m c a}{a}\int_0^1 \frac{t^2 dt}{\{1-e^2 t^2\}^{\frac{1}{2}}}$$

$$= \frac{2\pi m\rho a(1-e^2)^{\frac{1}{2}}}{e^3}\{\sin^{-1}e - e(1-e^2)^{\frac{1}{2}}\}; \qquad (53)$$

similarly $$\text{Y} = \frac{2\pi m\rho\beta(1-e^2)^{\frac{1}{2}}}{e^3}\{\sin^{-1}e - e(1-e^2)^{\frac{1}{2}}\}, \tag{54}$$

$$\begin{aligned}\text{Z} &= 4\pi m\rho a^2\gamma\int_0^1 \frac{t^2\,dt}{c^2+a^2e^2t^2} \\ &= \frac{4\pi m\rho\gamma}{e^2}\left\{1-\frac{c}{ae}\tan^{-1}\frac{ae}{c}\right\} \\ &= \frac{4\pi m\rho\gamma}{e^2}\left\{1-\frac{(1-e^2)^{\frac{1}{2}}}{e}\sin^{-1}e\right\}.\end{aligned} \tag{55}$$

Hence it appears that the attraction depends solely on the eccentricity of the bounding spheroid, and is independent of its magnitude.

Thus if through the attracted particle a spheroidal surface is drawn similar to the given one, it will attract ($m$) with the same force as the given spheroid, and as any other similar concentric spheroid which includes ($m$) within its mass. Hence a spheroidal shell, the surfaces of which are similar and concentric, attracts a particle within it equally in all directions.

(2) Let the ellipsoid be a prolate spheroid: and let $b = c$; also let the eccentricity of the generating ellipse be $e$, so that

$$a^2 - b^2 = a^2e^2, \qquad b^2 = a^2(1-e^2);$$

then we have

$$\begin{aligned}\text{X} &= \frac{4\pi m\rho b^2 a}{a^2}\int_0^1 \frac{t^2\,dt}{1-e^2t^2} \\ &= \frac{4\pi m\rho a(1-e^2)}{e^2}\left\{\frac{1}{2e}\log\frac{1+e}{1-e}-1\right\},\end{aligned} \tag{56}$$

$$\text{Y} = \frac{2\pi m\rho\beta}{e^2}\left\{1-\frac{(1-e^2)^{\frac{1}{2}}}{2}\log\frac{1+e}{1-e}\right\}, \tag{57}$$

$$\text{Z} = \frac{2\pi m\rho\gamma}{e^2}\left\{1-\frac{(1-e^2)^{\frac{1}{2}}}{2}\log\frac{1+e}{1-e}\right\}. \tag{58}$$

As these expressions involve the eccentricity only, it follows that a spheroidal shell the bounding surfaces of which are similar and concentric prolate spheroids, attracts a particle within it equally in all directions.

(3) Let the bounding surface be a sphere: $a = b = c$;

$$\therefore \quad \text{X} = \frac{4\pi m\rho a}{3}, \qquad \text{Y} = \frac{4\pi m\rho\beta}{3}, \qquad \text{Z} = \frac{4\pi m\rho\gamma}{3}; \tag{59}$$

where $a, \beta, \gamma$ are the coordinates of the attracted particle from the centre of the sphere as origin.

187.] Returning now to the expressions (44) (45) and (46) with the object of applying them to the case of the ellipsoid attracting an external particle, the limits of the $r$-integration will be $r_1$ and $r_2$, so that

$$\mathrm{Z} = \int_0^{\pi}\int_0^{\pi}\int_{r_2}^{r_1} m\rho \sin\theta\cos\theta\, dr\, d\theta\, d\phi$$
$$= \int_0^{\pi}\int_0^{\pi} m\rho\sin\theta\cos\theta\,\frac{2(\mathrm{B}^2-\mathrm{AC})^{\frac{1}{2}}}{\mathrm{A}}\, d\theta\, d\phi;$$

but as this expression involves a radical quantity, it cannot be integrated further. We are therefore obliged to have recourse to a different method of investigation; and for which we are indebted to Mr. Ivory.

Let the centre of the ellipsoid be the origin, and $(\alpha, \beta, \gamma)$ the position of $(m)$ the attracted particle. Let $a, b, c$ be the semi-axes of the ellipsoid, of which the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1; \tag{60}$$

and let us consider first $\mathrm{X}$, which is the resolved attraction along the axis of $x$; now

$$\mathrm{X} = \iiint \frac{\rho m(\alpha-x)\,dx\,dy\,dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}; \tag{61}$$

and performing in order the $x$-, $y$-, $z$-integrations, we have for the whole ellipsoid,

$$\text{if} \quad \mathrm{X} = a\left\{1-\frac{z^2}{c^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}}; \tag{62}$$

$$\mathrm{Y} = \frac{b}{c}(c^2-z^2)^{\frac{1}{2}}, \tag{63}$$

$$\mathrm{X} = \rho m\int_{-c}^{+c}\int_{-\mathrm{Y}}^{\mathrm{Y}}\int_{-\mathrm{X}}^{\mathrm{X}}\frac{(\alpha-x)\,dx\,dy\,dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}. \tag{64}$$

Let the variables be changed by the following substitutions:

$$x = a\xi, \qquad y = b\eta, \qquad z = c\zeta;$$
$$\therefore \quad \xi^2+\eta^2+\zeta^2 = 1:$$

$$\text{let} \quad \Xi = (1-\eta^2-\zeta^2)^{\frac{1}{2}}, \qquad \mathrm{H} = (1-\zeta^2)^{\frac{1}{2}}; \tag{65}$$

so that (64) becomes

$$\mathrm{X} = \rho m abc\int_{-1}^{1}\int_{-\mathrm{H}}^{\mathrm{H}}\int_{-\Xi}^{\Xi}\frac{(\alpha-a\xi)\,d\xi\,d\eta\,d\zeta}{\{(\alpha-a\xi)^2+(\beta-b\eta)^2+(\gamma-c\zeta)^2\}^{\frac{3}{2}}}$$
$$= \rho m bc\int_{-1}^{1}\int_{-\mathrm{H}}^{\mathrm{H}}\Big[\{(\alpha-a\xi)^2+(\beta-b\eta)^2+(\gamma-c\zeta)^2\}^{-\frac{1}{2}}\Big]_{-\Xi}^{\Xi} d\eta\, d\zeta:$$

$$\text{let} \quad \left.\begin{array}{l}(\alpha-a\Xi)^2+(\beta-b\eta)^2+(\gamma-c\zeta)^2 = r_1^2\\ (\alpha+a\Xi)^2+(\beta-b\eta)^2+(\gamma-c\zeta)^2 = r_2^2\end{array}\right\}; \tag{66}$$

$$\therefore \quad \mathrm{X} = \rho m b c \int_{-1}^{1} \int_{-\mathrm{H}}^{\mathrm{H}} \left(\frac{1}{r_1} - \frac{1}{r_2}\right) d\eta\, d\zeta. \tag{67}$$

Suppose now that there is a second ellipsoid of the same density as the former, of which the semi-axes are $a_1, b_1, c_1$, and which attracts a particle ($m$) placed at $(\alpha_1, \beta_1, \gamma_1)$; then if $\mathrm{R}_1$ and $\mathrm{R}_2$ are the quantities analogous to $r_1$ and $r_2$ in (67),

$$\mathrm{X}_1 = \rho m b_1 c_1 \int_{-1}^{1} \int_{-\mathrm{H}}^{\mathrm{H}} \left(\frac{1}{\mathrm{R}_1} - \frac{1}{\mathrm{R}_2}\right) d\eta\, d\zeta; \tag{68}$$

and let the second ellipsoid be such, that $\mathrm{R}_1 = r_1$, and $\mathrm{R}_2 = r_2$, so that the definite integrals in (67) and (68) may be equal; then we have

$$\frac{\mathrm{X}}{\mathrm{X}_1} = \frac{bc}{b_1 c_1}; \tag{69}$$

also because $r_1 = \mathrm{R}_1$, and $r_2 = \mathrm{R}_2$, we have, replacing $x$ by its value,

$$\begin{aligned} &\alpha^2 + \beta^2 + \gamma^2 + a^2 - 2a\alpha\{1 - \eta^2 - \zeta^2\}^{\frac{1}{2}} \\ &\qquad - 2b\beta\eta - 2c\gamma\zeta - (a^2 - b^2)\eta^2 - (a^2 - c^2)\zeta^2 \\ &= \alpha_1^2 + \beta_1^2 + \gamma_1^2 + a_1^2 - 2a_1\alpha_1\{1 - \eta^2 - \zeta^2\}^{\frac{1}{2}} \\ &\qquad - 2b_1\beta_1\eta - 2c_1\gamma_1\zeta - (a_1^2 - b_1^2)\eta^2 - (a_1^2 - c_1^2)\zeta^2; \end{aligned}$$

and a similar equation, because $r_2 = \mathrm{R}_2$: therefore equating coefficients,

$$\alpha^2 + \beta^2 + \gamma^2 + a^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 + a_1^2, \tag{70}$$

$$a\alpha = a_1\alpha_1, \qquad b\beta = b_1\beta_1, \qquad c\gamma = c_1\gamma_1, \tag{71}$$

$$a^2 - b^2 = a_1^2 - b_1^2, \qquad a^2 - c^2 = a_1^2 - c_1^2, \tag{72}$$

$$\therefore \quad a_1^2 - a^2 = b_1^2 - b^2 = c_1^2 - c^2 = \omega \text{ (say)}. \tag{73}$$

From (72) it appears that the two ellipsoids are concentric and confocal, that is, their principal sections are confocal.

Also from (71)

$$\frac{\alpha_1}{a} = \frac{\alpha}{a_1}, \qquad \frac{\beta_1}{b} = \frac{\beta}{b_1}, \qquad \frac{\gamma_1}{c} = \frac{\gamma}{c_1}; \tag{74}$$

from which substituting for $\alpha, \beta, \gamma$, in (70), and observing that all the terms are multiplied by equal factors, see equation (73), we have

$$\frac{\alpha^2}{a_1^2} + \frac{\beta^2}{b_1^2} + \frac{\gamma^2}{c_1^2} = 1: \tag{75}$$

and $\therefore$ by (74)

$$\frac{\alpha_1^2}{a^2} + \frac{\beta_1^2}{b^2} + \frac{\gamma_1^2}{c^2} = 1; \tag{76}$$

whence it appears that the second ellipsoid passes through the point which is attracted by the first, and that the second attracted point lies on the surface of the first.

Now to determine the values of $a_1, b_1, c_1, \alpha_1, \beta_1, \gamma_1$; from (73) (74) and (75) we have

$$\frac{\alpha^2}{a^2+\omega}+\frac{\beta^2}{b^2+\omega}+\frac{\gamma^2}{c^2+\omega}=1; \tag{77}$$

which is a cubic equation in $\omega$, and has one positive real root: because when it is expanded, the last term is

$$a^2b^2c^2\left\{1-\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-\frac{\gamma^2}{c^2}\right\},$$

which is negative: and therefore if $\omega=\infty$, the result is positive, and if $\omega=0$, the result is negative: there is therefore one positive real root; and the other two roots of the equation are negative*, and give negative values to (say) $b_1{}^2$ and $c_1{}^2$; and therefore refer to two other surfaces of the second order which pass through the attracted particle, and are confocal with the ellipsoid, and which are therefore hyperboloids of one and of two sheets respectively: let us in this case take the real positive root: and we have

$$\left.\begin{array}{lll} a_1{}^2=a^2+\omega, & b_1{}^2=b^2+\omega, & c_1{}^2=c^2+\omega \\ \alpha_1{}^2=\dfrac{a^2\alpha^2}{a^2+\omega}, & \beta_1{}^2=\dfrac{b^2\beta^2}{b^2+\omega}, & \gamma_1{}^2=\dfrac{c^2\gamma^2}{c^2+\omega} \end{array}\right\}. \tag{78}$$

The attraction of the second ellipsoid on $(m)$ at $(\alpha_1, \beta_1, \gamma_1)$, which is an internal point, may be found as in Art. 184; and thus $\text{X}_1, \text{Y}_1, \text{Z}_1$ may be expressed as in (50), (51), (52); and we have

$$\text{X}=\frac{bc}{b_1c_1}\text{X}_1, \qquad \text{Y}=\frac{ca}{c_1a_1}\text{Y}_1, \qquad \text{Z}=\frac{ab}{a_1b_1}\text{Z}_1; \tag{79}$$

$a_1, b_1, c_1$ being given by (78): so that from (51) we have

$$\text{X}=\frac{4\pi m\rho abc\alpha}{(a^2+\omega)^{\frac{1}{2}}}\int_0^1\frac{t^2\,dt}{\{a^2+\omega+(b^2-a^2)t^2\}^{\frac{1}{2}}\{a^2+\omega+(c^2-a^2)t^2\}^{\frac{1}{2}}}; \tag{80}$$

and also similar values for Y and Z.

If the ellipsoid becomes a figure of revolution, we can by this process find the attraction of it on an external particle: thus for example let us take the oblate spheroid: here $a=b$; and if

$$a^2-c^2=(a^2+\omega)e'^2=(c^2+\omega)e''^2,$$

$$\begin{aligned}\text{X}&=\frac{4\pi m\rho a^2c\alpha}{(a^2+\omega)^{\frac{3}{2}}}\int_0^1\frac{t^2\,dt}{(1-e'^2t^2)^{\frac{1}{2}}}\\ &=\frac{4\pi m\rho a^2c\alpha}{(a^2+\omega)^{\frac{3}{2}}e'^3}\{\sin^{-1}e'-e'(1-e'^2)^{\frac{1}{2}}\};\end{aligned}$$

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* See a Memoir by Plana, On the Attraction of a homogeneous Ellipsoid, Crelle's Journal, Vol. XX. page 189.

similarly $$\text{Y} = \frac{4\pi m\rho a^2 c\beta}{(a^2+\omega)^{\frac{3}{2}} e'^3}\{\sin^{-1}e' - e'(1-e'^2)^{\frac{1}{2}}\},$$

$$\text{Z} = \frac{4\pi m\rho a^2 c\gamma}{(c^2+\omega)^{\frac{3}{2}} e''^3}\{e'' - \tan^{-1}e''\},$$

and where $\omega$ is the positive root of the quadratic equation

$$\frac{\alpha^2+\beta^2}{a^2+\omega} + \frac{\gamma^2}{c^2+\omega} = 1.$$

The point $(\alpha_1, \beta_1, \gamma_1)$ on the first ellipsoid is called the point *corresponding* to $(\alpha, \beta, \gamma)$ on the second ellipsoid: the definition of *correspondence* being that given in equations (74), which assign the ratio of the several coordinates to be the same as that of semi-axes of the ellipsoids which are parallel to them: thus if $(x, y, z)$ $(\xi, \eta, \zeta)$ are points on two ellipsoids whose semi-axes are severally $a, b, c,$ $\alpha, \beta, \gamma,$ the points are corresponding, when

$$\frac{x}{\xi} = \frac{a}{\alpha}, \qquad \frac{y}{\eta} = \frac{b}{\beta}, \qquad \frac{z}{\zeta} = \frac{c}{\gamma}; \tag{81}$$

and therefore we may enuntiate the theorem contained in (79), and which is known by the name of Ivory's theorem, in the following form:

The attractions with which two confocal homogeneous ellipsoids attract, parallel to each axis, equal particles placed at corresponding points on their surfaces, are as the products of the axes perpendicular to each component.

This theorem has also been extended by Poisson to the case in which the law of attraction is any function of the distance.

188.] Hence also may we derive the following theorem which was stated first by Maclaurin, but in only a particular form.

Two confocal homogeneous ellipsoids attract an external particle along the same line of action, and with forces proportional to their masses.

Let X, Y, Z be the components of the attraction of the ellipsoid $(a, b, c)$ on the particle $(m)$ placed at $(\alpha, \beta, \gamma)$; and let X′, Y′, Z′ be the components of the confocal ellipsoid $(a_1, b_1, c_1)$ passing through $(\alpha, \beta, \gamma)$ on the corresponding point on the first ellipsoid: and let also X″, Y″, Z″ be the components of the attraction of the second ellipsoid on the point $(\alpha, \beta, \gamma)$ on its surface. Then by (79),

$$\frac{\text{X}}{\text{X}'} = \frac{bc}{b_1 c_1}, \qquad \frac{\text{Y}}{\text{Y}'} = \frac{ca}{c_1 a_1}, \qquad \frac{\text{Z}}{\text{Z}'} = \frac{ab}{a_1 b_1};$$

But by reason of (51),

$$\mathrm{X}'' = 4\pi m\rho b_1 c_1 a\int_0^1 \frac{t^2\,dt}{\{a_1^2+(b_1^2-a_1^2)t^2\}^{\frac{1}{2}}\{a_1^2+(c_1^2-a_1^2)t^2\}^{\frac{1}{2}}},$$

$$\mathrm{X}' = \frac{4\pi m\rho b_1 c_1 a a}{a_1}\int_0^1 \frac{t^2\,dt}{\{a_1^2+(b_1^2-a_1^2)t^2\}^{\frac{1}{2}}\{a_1^2+(c_1^2-a_1^2)t^2\}^{\frac{1}{2}}};$$

$$\therefore\quad \frac{\mathrm{X}'}{\mathrm{X}''} = \frac{a}{a_1}, \qquad \frac{\mathrm{Y}'}{\mathrm{Y}''} = \frac{b}{b_1}, \qquad \frac{\mathrm{Z}'}{\mathrm{Z}''} = \frac{c}{c_1},$$

the last two being inferred by reason of the symmetry of the formulæ;

$$\therefore\quad \frac{\mathrm{X}}{\mathrm{X}''} = \frac{abc}{a_1 b_1 c_1} = \frac{\mathrm{Y}}{\mathrm{Y}''} = \frac{\mathrm{Z}}{\mathrm{Z}''}. \tag{82}$$

Let there be another homogeneous confocal ellipsoid of the same density, and of which the semi-axes are $a'$, $b'$, $c'$; and let $\mathrm{X}_1$, $\mathrm{Y}_1$, $\mathrm{Z}_1$ be the components of its attraction on the particle ($m$) placed at ($\alpha$, $\beta$, $\gamma$) without it; then

$$\frac{\mathrm{X}_1}{\mathrm{X}''} = \frac{a'b'c'}{a_1 b_1 c_1} = \frac{\mathrm{Y}_1}{\mathrm{Y}''} = \frac{\mathrm{Z}_1}{\mathrm{Z}''}; \tag{83}$$

$$\therefore\quad \frac{\mathrm{X}}{\mathrm{X}_1} = \frac{\mathrm{Y}}{\mathrm{Y}_1} = \frac{\mathrm{Z}}{\mathrm{Z}_1} = \frac{abc}{a'b'c'}; \tag{84}$$

which proves the theorem.

189.] I propose now to investigate certain properties of a general function from which by differentiation the components of attraction of a finite body on a given particle may be determined.

Let, as before, X, Y, Z be the components of attraction; ($\alpha, \beta, \gamma$) the position of ($m$) the attracted particle, ($x$, $y$, $z$) the place of a particle of the attracting mass, of which $dx\,dy\,dz$ is the volume, and $\rho$ is the density; and let the attraction vary directly as the product of the masses, and inversely as the square of the distance; so that

$$\left.\begin{aligned}
\mathrm{X} &= \iiint \frac{\rho m(\alpha-x)\,dx\,dy\,dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}\\
\mathrm{Y} &= \iiint \frac{\rho m(\beta-y)\,dx\,dy\,dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}\\
\mathrm{Z} &= \iiint \frac{\rho m(\gamma-z)\,dx\,dy\,dz}{\{(\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}
\end{aligned}\right\}; \tag{85}$$

and where the attractions tend to draw ($m$) towards the origin:

and the triple integrals extend to the whole mass of the attracting body.

Now suppose v to be the function which expresses the sum of every mass-element of the attracting body divided by its distance from the attracted particle, so that

$$\text{v} = \iiint \frac{\rho\, dx\, dy\, dz}{\{(a-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{1}{2}}}; \tag{86}$$

then $$\left(\frac{d\text{v}}{da}\right) = -\iiint \frac{\rho(a-x)\, dx\, dy\, dz}{\{(a-x)^2+(\beta-y)^2+(\gamma-z)^2\}^{\frac{3}{2}}}; \tag{87}$$

$$\therefore \quad \text{X} = -m\left(\frac{d\text{v}}{da}\right); \tag{88}$$

similarly $$\text{Y} = -m\left(\frac{d\text{v}}{d\beta}\right), \qquad \text{Z} = -m\left(\frac{d\text{v}}{d\gamma}\right); \tag{89}$$

and thus v has this remarkable property: the three partial derived functions of it with reference to the three coordinates of the attracted particle, when multiplied by ($m$), are the components along the coordinate axes of the attraction of the whole body on ($m$).

v has been named by Gauss *the Potential of the Mass* to which it refers: evidently if it can be found, the whole attraction may be easily found by means of its components, which are the derived functions of v. The potential, as it is here defined, is used in a restricted sense as applicable to the law of gravitation only: to other laws of attraction there are their own potentials, and it may generally be defined as that function, the partial derived functions of which with respect to the three coordinates of the attracted particle are the components of the whole attraction. In polar coordinates the potential for the law of gravitation, if the attracted particle is at the origin, is

$$\text{v} = \iiint \rho r \sin\theta\, dr\, d\theta\, d\phi. \tag{90}$$

Also if R = the whole force of attraction,

$$\text{R}^2 = m^2\left\{\left(\frac{d\text{v}}{da}\right)^2+\left(\frac{d\text{v}}{d\beta}\right)^2+\left(\frac{d\text{v}}{d\gamma}\right)^2\right\}. \tag{91}$$

190.] Suppose the point $(a, \beta, \gamma)$ to be on a curve of which $ds$ is a length-element, so that the direction-cosines of $ds$ are

$$\frac{da}{ds}, \qquad \frac{d\beta}{ds}, \qquad \frac{d\gamma}{ds};$$

then the resolved part of R along $ds$

$$= \mathrm{X}\frac{d\alpha}{ds} + \mathrm{Y}\frac{d\beta}{ds} + \mathrm{Z}\frac{d\gamma}{ds}$$

$$= -\frac{m}{ds}\left\{\left(\frac{d\mathrm{V}}{d\alpha}\right)d\alpha + \left(\frac{d\mathrm{V}}{d\beta}\right)d\beta + \left(\frac{d\mathrm{V}}{d\gamma}\right)d\gamma\right\}$$

$$= -m\frac{d\mathrm{V}}{ds}; \tag{92}$$

so that $-m\frac{d\mathrm{V}}{ds}$ is the resolved part of the total attraction along the line $ds$.

Suppose the line $s$ at the point $(\alpha, \beta, \gamma)$ to be such that the attraction of the body is wholly perpendicular to it; then

$$\frac{d\mathrm{V}}{ds} = 0, \tag{93}$$

$$\therefore \quad \mathrm{V} = \text{a constant}; \tag{94}$$

and as V is a function of $(\alpha, \beta, \gamma)$, this equation is that to a surface, to which at every point the line of action of the resultant attraction is perpendicular. Such a surface is called *a surface of equilibrium* (*surface de niveau*), and at every point of it a particle under the attraction of the attracting mass will remain at rest. The lines perpendicular to such a surface, being those along the tangents to which at each point of them the resultant attraction acts, are called *lines of force*. Of these we have had an instance in Art. 172: hyperbolæ are the lines of force of a straight bar: and spheroids are the surfaces of equilibrium. We shall hereafter see that of an ellipsoid attracting an external particle the confocal ellipsoid passing through the particle is the surface of equilibrium. For the same surface of equilibrium the constant in the right-hand member of (94) is the same; it changes however as we pass from one such surface to another.

If $\theta$ is the angle between $ds$ and the resultant attraction, by (92),

$$\mathrm{R}\cos\theta\, ds = -m\, d\mathrm{V},$$

$$\therefore \quad \int_0^1 \mathrm{R}\cos\theta\, ds = m(\mathrm{V}_0 - \mathrm{V}_1);$$

where $\mathrm{V}_0$ and $\mathrm{V}_1$ are the values of V for the limiting points of $s$; and therefore if $s$ is a closed line, and if the integration is extended through the whole line,

$$\int \mathrm{R}\cos\theta\, ds = 0.$$

191.] For a simple application of this theory of the potential, let us consider a sphere consisting of homogeneous concentric shells, the density of each of which varies as some function of the distance from the centre, and let the attracted particle ($m$) be on the axis of $z$ at a distance $\gamma$ from its centre. Let the radius of the sphere be $a$, and let the density of a shell, the internal radius of which is $r$, be expressed by $f(r)$; then

$$\begin{aligned} \mathrm{V} &= \int_0^a \int_0^\pi \int_0^{2\pi} \frac{r^2 f(r) \sin\theta \, d\phi \, d\theta \, dr}{(\gamma^2 - 2\gamma r \cos\theta + r^2)^{\frac{1}{2}}} \\ &= 2\pi \int_0^a \int_0^\pi \frac{r^2 f(r) \sin\theta \, d\theta \, dr}{(\gamma^2 - 2\gamma r \cos\theta + r^2)^{\frac{1}{2}}}. \end{aligned} \tag{95}$$

Here we have two cases, according as the attracted particle is external or internal; for an external particle $\gamma$ is greater than $r$, so that $(\gamma^2 - 2\gamma r + r^2)^{\frac{1}{2}} = \gamma - r$,

$$\begin{aligned} \therefore \quad \mathrm{V} &= \frac{4\pi}{\gamma} \int_0^a r^2 f(r) \, dr \\ &= \frac{\mathrm{M}}{\gamma}, \end{aligned}$$

if $\mathrm{M}$ is the mass of the sphere: taking the $\gamma$-differential of which, we have, by reason of equation (92), the attraction along the axis of $z$: and thus

$$\mathrm{Z} = -m \left(\frac{d\mathrm{V}}{d\gamma}\right) = \frac{\mathrm{M}m}{\gamma^2}, \tag{96}$$

and which therefore varies inversely as the square of the distance.

For an internal particle, the integral (95) must be divided into two parts, the former of which will correspond to the shell on the interior surface of which the attracted particle is, and the latter to the sphere on the surface of which the attracted particle is: so that

$$\begin{aligned} \mathrm{V} &= 2\pi \int_\gamma^a \int_0^\pi \frac{r^2 f(r) \sin\theta \, d\theta \, dr}{(\gamma^2 - 2\gamma r \cos\theta + r^2)^{\frac{1}{2}}} + 2\pi \int_0^\gamma \int_0^\pi \frac{r^2 f(r) \sin\theta \, d\theta \, dr}{(\gamma^2 - 2\gamma r \cos\theta + r^2)^{\frac{1}{2}}} \\ &= 4\pi \int_\gamma^a r f(r) \, dr + \frac{4\pi}{\gamma} \int_0^\gamma r^2 f(r) \, dr; \end{aligned}$$

therefore, by reason of the Theorems contained in (10) and (11), Art. 178, Vol. II,

$$\left(\frac{d\mathrm{V}}{d\gamma}\right) = -4\pi\gamma f(\gamma) + \frac{4\pi}{\gamma} \gamma^2 f(\gamma) - \frac{4\pi}{\gamma^2} \int_0^\gamma r^2 f(r) \, dr;$$

$$\therefore \quad \text{Z} = -m\left(\frac{d\text{V}}{d\gamma}\right) = \frac{4\pi m}{\gamma^2}\int_0^{\gamma} r^2 f(r)\,dr\,; \qquad (97)$$

and because $4\pi\int_0^{\gamma} r^2 f(r)\,dr$ is the mass of the sphere whose radius is $\gamma$, it follows that the shell lying outside of that sphere exercises no effect on the attracted particle.

For another application of the theory of the potential, let us investigate the components of attraction of a body of finite dimensions on a particle at a very great distance.

Let the centre of gravity of the attracting body be the origin; and $(\alpha, \beta, \gamma)$ the place of the attracted particle; then if $u =$ the distance between the attracted particle $(m)$, and $(x, y, z)$ the place of an element of the attracting body,

$$\begin{aligned} u^2 &= (\alpha-x)^2+(\beta-y)^2+(\gamma-z)^2 \\ &= (\alpha^2+\beta^2+\gamma^2)-2(\alpha x+\beta y+\gamma z)+x^2+y^2+z^2 \\ &= c^2-2(\alpha x+\beta y+\gamma z)+x^2+y^2+z^2, \end{aligned}$$

if $c$ is the distance of $m$ from the origin. Therefore expanding

$$\frac{1}{u} = \frac{1}{c} + \frac{\alpha x+\beta y+\gamma z}{c^3} + \frac{3(\alpha x+\beta y+\gamma z)^2-(x^2+y^2+z^2)\,c^2}{2c^5} + \dots \qquad (98)$$

$$\therefore \quad \text{V} = \iiint \frac{\rho\,dx\,dy\,dz}{u}$$

$$= \iiint \rho\left\{\frac{1}{c} + \frac{\alpha x+\beta y+\gamma z}{c^3} + \frac{3(\alpha x+\beta y+\gamma z)^2-(x^2+y^2+z^2)\,c^2}{2c^5} + \dots\right\} dx\,dy\,dz.$$

Of the terms in this series, the second disappears, because the origin is the centre of gravity: and the last may be omitted, because $c$ is assumed to be very much greater than any value of $x$, $y$, or $z$: therefore

$$\begin{aligned} \text{V} &= \iiint \frac{\rho\,dx\,dy\,dz}{c} \\ &= \frac{\text{M}}{\{\alpha^2+\beta^2+\gamma^2\}^{\frac{1}{2}}}, \end{aligned} \qquad (99)$$

if $\text{M} =$ the mass of the attracting body.

$$\therefore \quad \left.\begin{aligned} \text{X} &= -m\left(\frac{d\text{V}}{d\alpha}\right) = \frac{m\text{M}\alpha}{\{\alpha^2+\beta^2+\gamma^2\}^{\frac{3}{2}}} \\ \text{Y} &= -m\left(\frac{d\text{V}}{d\beta}\right) = \frac{m\text{M}\beta}{\{\alpha^2+\beta^2+\gamma^2\}^{\frac{3}{2}}} \\ \text{Z} &= -m\left(\frac{d\text{V}}{d\gamma}\right) = \frac{m\text{M}\gamma}{\{\alpha^2+\beta^2+\gamma^2\}^{\frac{3}{2}}} \end{aligned}\right\}; \qquad (100)$$

and therefore if R is the resultant of X, Y, Z,

$$\text{R} = \frac{m\text{M}}{\alpha^2+\beta^2+\gamma^2} = \frac{m\text{M}}{c^2},$$

and is therefore the same as if the whole attracting mass were condensed into a particle at its centre of gravity.

192.] I also propose to investigate the attraction of an ellipsoidal shell on a particle ($m$) external to it, by means of the potential, and according to the method which has been pursued by M. Chasles*. With this object let us first compare with each other the attractions of two concentric and confocal ellipsoidal shells of the same density on an external particle.

Let there be two ellipsoidal surfaces which are concentric, and the principal sections of which are confocal; and which we will call A′ and A: on A, the interior, let two points P, Q be taken, and on A′, the exterior, two corresponding points P′, Q′; then PQ′ = P′Q.

Let $x, y, z$ refer to the interior ellipsoid, of which let the semi-axes be $a, b, c$; and let similar letters with an accent refer to the exterior. Let Q be $(\xi, \eta, \zeta)$, Q′ be $(\xi', \eta', \zeta')$;

$$\therefore \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = \frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2} = 1,$$

$$\frac{x'^2}{a'^2}+\frac{y'^2}{b'^2}+\frac{z'^2}{c'^2} = \frac{\xi'^2}{a'^2}+\frac{\eta'^2}{b'^2}+\frac{\zeta'^2}{c'^2} = 1.$$

Also by reason of the points being corresponding, see equations (81),

$$\frac{x}{a}=\frac{x'}{a'}, \quad \frac{y}{b}=\frac{y'}{b'}, \quad \frac{z}{c}=\frac{z'}{c'}, \tag{101}$$

$$\frac{\xi}{a}=\frac{\xi'}{a'}, \quad \frac{\eta}{b}=\frac{\eta'}{b'}, \quad \frac{\zeta}{c}=\frac{\zeta'}{c'}; \tag{102}$$

also as the principal sections are confocal,

$$a'^2-a^2 = b'^2-b^2 = c'^2-c^2; \tag{103}$$

$$\therefore \quad (\text{P}'\text{Q})^2-(\text{P}\text{Q}')^2 = (x'-\xi)^2-(x-\xi')^2+\dots+\dots$$

$$= \left(\frac{a'}{a}x-\xi\right)^2-\left(x-\frac{a'}{a}\xi\right)^2+\dots+\dots$$

$$= (a'^2-a^2)\left(\frac{x^2}{a^2}-\frac{\xi^2}{a^2}\right)+(b'^2-b^2)\left(\frac{y^2}{b^2}-\frac{\eta^2}{b^2}\right)+(c'^2-c^2)\left(\frac{z^2}{c^2}-\frac{\zeta^2}{c^2}\right)$$

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* See Liouville's Journal, Tome V, 1840, and Comptes Rendus, Tome VI, p. 902, 1838.

$$(\mathrm{P'Q})^2-(\mathrm{PQ'})^2 = (a'^2-a^2)\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right\}$$
$$= (a'^2-a^2)\,\{1-1\}$$
$$= 0;$$
$$\therefore \quad \mathrm{P'Q} = \mathrm{PQ'}. \tag{104}$$

Again, by means of the relations (101),

$$\frac{dx\,dy\,dz}{abc} = \frac{dx'\,dy'\,dz'}{a'b'c'}, \tag{105}$$

where $(x, y, z)$, $(x', y', z')$ are two points on the surfaces of the interior and the exterior ellipsoids respectively.

Suppose now that there are two ellipsoidal surfaces A and B, concentric, similar, and similarly situated, and that their equations are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, \tag{106}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = n^2; \tag{107}$$

and suppose that to every point in the shell contained between them the corresponding point is taken: then the locus of all these corresponding points will be another shell contained between two concentric, similar, and similarly situated ellipsoids, which we will call A′ and B′, the ratio of whose homologous axes will be $n : 1$; and the exterior and interior of which will be confocal and concentric respectively with the exterior and interior surfaces of the former shell: and because (105) is true for each of the corresponding points, so will the volumes of the shells be in the ratio of $abc$ to $a'b'c'$.

Let $da$, $da'$ be the thicknesses of the shells at the extremities of the axes $a$ and $a'$; then if $a$ and $a'$ refer to the exterior surfaces and $na$, $na'$ to the interior,

$$da = a-na, \qquad da' = a'-na';$$

$$\therefore \quad \frac{da}{a} = \frac{da'}{a'}; \tag{108}$$

similarly, $$\frac{db}{b} = \frac{db'}{b'}, \qquad \frac{dc}{c} = \frac{dc'}{c}. \tag{109}$$

193.] Let $(\xi, \eta, \zeta)$, $(\xi', \eta', \zeta')$ be two fixed corresponding points Q and Q′ on the exterior bounding surfaces A and A′ of the two ellipsoidal shells; and let there be two other corresponding points P and P′, $(x, y, z)$ and $(x', y', z')$; and at these let the volume-elements $dx\,dy\,dz$, $dx'\,dy'\,dz'$ abut; then by reason of (104) and (105),

$$\frac{dx\,dy\,dz}{\text{PQ}'} = \frac{abc}{a'b'c'}\,\frac{dx'\,dy'\,dz'}{\text{P}'\text{Q}}; \tag{110}$$

and summing these expressions so as to include the two shells,

$$\iiint \frac{\rho\,dx\,dy\,dz}{\text{PQ}'} = \frac{abc}{a'b'c'}\iiint \frac{\rho\,dx'\,dy'\,dz'}{\text{P}'\text{Q}}; \tag{111}$$

that is, the potential of the inner shell on the particle Q′ exterior to it has to the potential of the exterior shell on the particle Q interior to it the ratio of the volumes of the shells.

By reason of (92), the component of the attraction in any direction may be obtained by taking the partial-derived function of the potential with respect to that direction. Now since

$$\iiint \frac{\rho\,dx'\,dy'\,dz'}{\text{P}'\text{Q}}$$

is the potential of the exterior shell on the particle Q placed within it, that potential is constant, see also (147) Art. 198, because the attraction of such a shell on a particle within it is zero, see Art. 185, and therefore its derived functions vanish. As the right-hand member of (111) is constant, so is also the left-hand member; and as the same result is true for all positions of the point Q′ on the exterior ellipsoid A′, so the potential of the interior ellipsoidal shell is constant for all points on the exterior surface of A′. A′ therefore is, by reason of equation (94), the equilibrium surface of the ellipsoidal shell: therefore

The equilibrium-surface of an infinitesimally-thin ellipsoidal shell attracting an external particle is a confocal and concentric ellipsoid passing through the attracted particle.

This result is of great importance: for since the exterior ellipsoid is the equilibrium-surface, it follows that the line of action of the resultant attraction of the infinitesimally-thin ellipsoidal shell is normal to it.

This line is also the principal interior axis of the cone whose vertex is at the attracted particle, and which circumscribes the ellipsoid. For if we employ Lamé's system of elliptical coordinates, and place the attracted particle at the common point of intersection of three concentric and confocal surfaces of the second order, their lines of intersection are the principal axes of the cone; and it is the normal to the ellipsoid along which the attraction acts.

Suppose also that there is another infinitesimally-thin ellipsoidal shell of the same density lying within the preceding shell A′, and that its principal semi-axes are $a_1, b_1, c_1$; then if $\text{P}_1\,(x_1, y_1, z_1)$ and $\text{Q}_1\,(\xi_1, \eta_1, \zeta_1)$ correspond respectively to $(x', y', z')$ and $(\xi', \eta', \zeta')$, we have analogously to (111)

$$\iiint \frac{\rho\, dx_1\, dy_1\, dz_1}{\text{P}_1\text{Q}'} = \frac{a_1 b_1 c_1}{a' b' c'} \iiint \frac{\rho\, dx'\, dy'\, dz'}{\text{P}'\text{Q}_1}; \tag{112}$$

and because $\text{Q}_1$ and Q lie within the exterior ellipsoid, and the attraction of the ellipsoidal shell on each of them is zero, therefore their potentials are equal, or

$$\iiint \frac{\rho\, dx'\, dy'\, dz'}{\text{P}'\text{Q}} = \iiint \frac{\rho\, dx'\, dy'\, dz'}{\text{P}'\text{Q}_1}; \tag{113}$$

therefore from (111) and (112),

$$\iiint \frac{\rho\, dx\, dy\, dz}{\text{P}\text{Q}'} = \frac{abc}{a_1 b_1 c_1} \iiint \frac{\rho\, dx_1\, dy_1\, dz_1}{\text{P}_1\text{Q}'}; \tag{114}$$

and therefore the potentials of two infinitesimally-thin confocal and concentric ellipsoidal shells of the same density on an external particle are to each other as the volumes or as the masses of the shells.

Let the potentials of these two shells be V and $\text{V}_1$; and let the shells be of the same density; then

$$\frac{\text{V}}{abc} = \frac{\text{V}_1}{a_1 b_1 c_1}; \tag{115}$$

and since the attraction of a mass on $(m)$ in any direction $s$ is

$$-m \frac{d\text{V}}{ds},$$

$$\therefore \quad \frac{d\text{V}}{ds} = \frac{abc}{a_1 b_1 c_1} \frac{d\text{V}_1}{ds};$$

and hence it follows that

If two ellipsoidal shells, infinitesimally thin, of the same density, concentric and confocal, attract an external particle $(m)$, the components of the attractions in any direction have the ratio of the masses of the shells: and therefore also the shells attract the particle in the same direction.

194.] The attraction of an ellipsoidal shell on an external particle is still required. The line of action of the resultant attraction has been found above; and in (111) the potential of an infinitesimally-thin shell on a particle is given in terms of the potential of a concentric and confocal infinitesimally-thin shell, on the exterior surface of which the attracted particle lies: our object therefore will be to investigate this latter attraction; and we shall shew that the attraction of an homogeneous and infinitesimally-thin shell, whose bounding surfaces are similar ellipsoids, on a particle $(m)$ at the exterior surface is equal to $4\pi\rho m\tau$, where $\tau$ is the thickness of the shell at the attracted point.

Let O be the position of $(m)$ on the exterior surface of the shell, a section of which by a plane through the normal OG and the centre C is delineated in fig. 84: then the line of action of the resultant attraction of the shell is the line OG. Let a series of very small solid angles originate at O: and let that one of which the section is $\mathrm{OP}p'q'\mathrm{Q}$ intercept an area $\omega$ of a spherical surface described from O as a centre with the radius = unity: so that the area intercepted at a distance $r = \omega r^2$: now the volume of each of the mass-elements of the shell thus intercepted and at a distance $r$ from $\mathrm{O} = \rho\omega r^2 dr$; and therefore its attraction on $(m) = m\rho\omega dr$; and therefore the attraction of OPQ on $m = m\rho\omega \times \mathrm{OP}$; and the attraction of $pp'q'q$ on $\mathrm{O} = m\rho\omega \times pp'$: but by reason of the similarity of the surfaces $\mathrm{OP} = pp'$: therefore the attraction on $(m)$ of the part of the shell intercepted by the cone $(\omega)$ in the direction OP,

$$= 2m\rho\omega \times \mathrm{OP};$$

and therefore the attraction in the direction OG

$$\begin{aligned} &= 2m\rho\omega\, \mathrm{OP} \cos \mathrm{POG} \\ &= 2m\rho\omega\, \mathrm{O}n \\ &= 2m\rho\omega\tau, \end{aligned}$$

if $\tau$ is the normal thickness of the shell. And since the surface of a hemisphere of radius = unity is $2\pi$, therefore the attraction of the whole shell $= 4\pi m\rho\tau$. This theorem was discovered by Laplace.

For $\tau$ we may substitute an equivalent expression. From C the centre of ellipsoid let the perpendicular $\mathrm{CY} = p$ be drawn to the tangent plane at O: then

$$\begin{aligned} \frac{\mathrm{O}n}{\mathrm{O}m} &= \cos \mathrm{COG} \\ &= \frac{\mathrm{CY}}{\mathrm{CO}}; \end{aligned}$$

$$\therefore\quad \mathrm{O}n = \tau = \frac{\mathrm{O}m}{\mathrm{OC}}\, p. \tag{116}$$

But by reason of the similarity of the bounding surfaces, if $a =$ the greatest semi-axis of the exterior surface, and if $da$ is the thickness of the shell at its extremity,

$$\frac{\text{om}}{\text{oc}} = \frac{da}{a} = \frac{db}{b} = \frac{dc}{c}, \tag{117}$$

the last two terms following from the symmetry.

Therefore the attraction of the shell, in the direction of the normal OG,

$$= \frac{4\pi m \rho p}{a} da. \tag{118}$$

Now if the equation to the exterior surface is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$\text{then,} \qquad \frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}; \tag{119}$$

and the direction-cosines of the normal are

$$\frac{px}{a^2}, \quad \frac{py}{b^2}, \quad \frac{pz}{c^2}; \tag{120}$$

therefore, if X′, Y′, Z′ are the components of the attraction of the shell on $m$, placed at the point $(x, y, z)$, along the three coordinate axes,

$$\left.\begin{aligned} \text{X}' &= \frac{4\pi m \rho p^2 x\, da}{a^3} \\ \text{Y}' &= \frac{4\pi m \rho p^2 y\, db}{b^3} \\ \text{Z}' &= \frac{4\pi m \rho p^2 z\, dc}{c^3} \end{aligned}\right\}. \tag{121}$$

Suppose now that there is a homogeneous infinitesimally thin shell, the bounding surfaces of which are two similar ellipsoids, the equation to the exterior one being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

and the corresponding semi-axes of the interior one being $a-da$, $b-db$, $c-dc$: and suppose that this shell attracts a particle $(m)$ at the point $(\alpha, \beta, \gamma)$: then if $m$ is considered to be placed on the exterior surface of a homogeneous infinitesimally thin shell contained between two bounding ellipsoids which are respectively concentric and confocal with those of the former shell, and if $a_1, b_1, c_1$ are the semi-axes of the exterior one, and $a_1-da_1$, $b_1-db_1$, $c_1-dc_1$ of the interior; then by reason of the last Article, if X, Y, Z are the components of the attraction of the former shell on $(m)$,

$$\text{X} = \frac{abc}{a_1 b_1 c_1} \frac{4\pi m \rho p_1^2 a\, da_1}{a_1^3}.$$

But by (108)

$$\frac{da_1}{a_1} = \frac{da}{a};$$

and $a_1, b_1, c_1$ are given by (78), where $\omega$ is a root of (77), and we have

$$\frac{1}{p_1^2} = \frac{\alpha^2}{a_1^4} + \frac{\beta^2}{b_1^4} + \frac{\gamma^2}{c_1^4}; \tag{122}$$

$$\therefore \quad \left.\begin{aligned} \text{X} &= \frac{bc}{b_1 c_1} \frac{4\pi \rho m p_1^2}{a_1^3}\, da.\alpha \\ \text{Y} &= \frac{ca}{a_1 c_1} \frac{4\pi \rho m p_1^2}{b_1^3}\, db.\beta \\ \text{Z} &= \frac{ab}{a_1 b_1} \frac{4\pi \rho m p_1^2}{c_1^3}\, dc.\gamma \end{aligned}\right\}. \tag{123}$$

The physical application of these results of the attraction of a thin ellipsoidal shell on an external particle is so elegant that I cannot refrain from inserting it. Let a conducting body whose bounding surface is of the form of the ellipsoid with three unequal axes be charged with electricity; then by the nature of electricity equilibrium is established in the interior of the body, and the excess of the fluid (if it is a fluid) lies on the surface of the body, and there forms a shell of infinitesimal, but variable, thickness, and which is in equilibrium under the pressure of the surrounding air, and its contact with the body. Thus we have an ellipsoidal shell of infinitesimal thickness, all the molecules of which have a force of attraction or repulsion, which varies inversely as the square of the distance, on a particle external to it, and which exhibits experimentally the following phænomena:

(1) The two surfaces which bound the shell of fluid are similar ellipsoids.

(2) The resultant of the attraction of the electrical fluid on the surface on a particle within the shell is zero.

(3) The repulsive action of electrical fluid on each particle of air which is in contact with it on the surface is along the normal to the external surface.

These laws of the distribution of electricity are the results partly of observation and partly of calculation, and we are able to deduce the following additional ones from our preceding investigations.

(4) The line of action of the whole attraction of the shell on an external particle $m$ is the normal to the ellipsoid which passes through the particle, and whose principal sections are confocal with those of the ellipsoid which is the internal surface of the shell.

(5) All ellipsoids, whose principal sections are confocal with those of the internal surface of the shell are equilibrium-surfaces.

(6) The force-line of any particle towards the shell under the action of its attraction is the line of intersection of the two hyperboloids, respectively of one and of two sheets, passing through the particle, and which are confocal with the given ellipsoid.

(7) If the attracted particle is in the plane of one of the principal sections of the electrised ellipsoid, the force-line is the hyperbola which is confocal with that principal section.

195.] From the expressions (123), we can deduce the components of the attraction of an ellipsoid resolved into homogeneous and similar shells. Let us investigate the $x$-component: and let $b, c, a_1, b_1, c_1, p_1$ be expressed in terms of $a$ alone, for this may be effected by means of the conditions of Art. 187. And let $\text{A}, \text{B}, \text{C}$ be the semi-axes of the bounding ellipsoid: so that the $x$-component of the attraction of the ellipsoid on $(m)$ will depend on the $a$-integral of the first of (123) after the substitutions, between the limits $\text{A}$ and 0.

Since the whole ellipsoid is resolved into shells, the bounding surfaces of which are similar to that of the whole ellipsoid, we have

$$\frac{a}{\text{A}} = \frac{b}{\text{B}} = \frac{c}{\text{C}}; \tag{124}$$

and let $\text{A}_1, \text{B}_1, \text{C}_1$ be the semi-axes of the ellipsoid, passing through $(m)$, concentric and confocal with the given ellipsoid; so that

$$\text{A}_1^2 - \text{A}^2 = \text{B}_1^2 - \text{B}^2 = \text{C}_1^2 - \text{C}^2 = \omega; \tag{125}$$

and
$$\frac{\alpha^2}{\text{A}_1^2} + \frac{\beta^2}{\text{B}_1^2} + \frac{\gamma^2}{\text{C}_1^2} = 1, \tag{126}$$

or
$$\frac{\alpha^2}{\text{A}^2+\omega} + \frac{\beta^2}{\text{B}^2+\omega} + \frac{\gamma^2}{\text{C}^2+\omega} = 1. \tag{127}$$

Also we have
$$a_1^2 - a^2 = b_1^2 - b^2 = c_1^2 - c^2; \tag{128}$$

$$\frac{\alpha^2}{a_1^2} + \frac{\beta^2}{b_1^2} + \frac{\gamma^2}{c_1^2} = 1. \tag{129}$$

Let $$v = \frac{\mathrm{A}_1}{\mathrm{A}}\frac{a}{a_1}; \quad \therefore \quad a_1 = \frac{\mathrm{A}_1}{\mathrm{A}}\frac{a}{v}; \tag{130}$$

therefore from (124) and (128),

$$b_1^2 = \frac{a^2}{\mathrm{A}^2}\{\mathrm{A}_1^2 v^{-2} - (\mathrm{A}^2 - \mathrm{B}^2)\},$$

$$c_1^2 = \frac{a^2}{\mathrm{A}^2}\{\mathrm{A}_1^2 v^{-2} - (\mathrm{A}^2 - \mathrm{C}^2)\}:$$

therefore (129) becomes

$$\frac{a^2 v^2}{\mathrm{A}_1^2} + \frac{\beta^2}{\mathrm{A}_1^2 v^{-2} - (\mathrm{A}^2 - \mathrm{B}^2)} + \frac{\gamma^2}{\mathrm{A}_1^2 v^{-2} - (\mathrm{A}^2 - \mathrm{C}^2)} = \frac{a^2}{\mathrm{A}^2}; \tag{131}$$

whereby $a$ is expressed in terms of $v$. Also

$$\frac{1}{p_1^2} = \frac{a^2}{a_1^4} + \frac{\beta^2}{b_1^4} + \frac{\gamma^2}{c_1^4};$$

$$\therefore \quad \frac{a^4}{\mathrm{A}^4 p_1^2} = \frac{a^2 v^4}{\mathrm{A}_1^4} + \frac{\beta^2}{\{\mathrm{A}_1^2 v^{-2} - (\mathrm{A}^2 - \mathrm{B}^2)\}^2} + \frac{\gamma^2}{\{\mathrm{A}_1^2 v^{-2} - (\mathrm{A}^2 - \mathrm{C}^2)\}^2}; \tag{132}$$

Now differentiating (131), and observing the value of $p_1$ in (132), we have

$$p_1^2\, da = \frac{\mathrm{A}_1^2 a^3}{\mathrm{A}^2 v^3}\, dv;$$

therefore from (123),

$$\mathrm{X} = \frac{4\pi\rho m a \mathrm{ABC}}{(\mathrm{A}^2+\omega)^{\frac{1}{2}}} \frac{v^2\, dv}{\{\mathrm{A}^2+\omega+(\mathrm{B}^2-\mathrm{A}^2)v^2\}^{\frac{1}{2}}\{\mathrm{A}^2+\omega+(\mathrm{C}^2-\mathrm{A}^2)v^2\}^{\frac{1}{2}}}; \tag{133}$$

which is the $x$-component of the attraction of an infinitesimally thin ellipsoidal shell on a particle ($m$) external to it. The $y$- and the $z$-components are of the same form and symmetrical.

The integral of the right-hand member of (133) between the right limits will give the $x$-component of the attraction on ($m$) of the full ellipsoid; and observing that $a = \mathrm{A}$ in (130), when $v = 1$, the limits of $v$ are 1 and 0; so that if X is the $x$-component of the attraction of the ellipsoid,

$$\mathrm{X} = \frac{4\pi\rho m a \mathrm{ABC}}{(\mathrm{A}^2+\omega)^{\frac{1}{2}}} \int_0^1 \frac{v^2\, dv}{\{\mathrm{A}^2+\omega+(\mathrm{B}^2-\mathrm{A}^2)v^2\}^{\frac{1}{2}}\{\mathrm{A}^2+\omega+(\mathrm{C}^2-\mathrm{A}^2)v^2\}^{\frac{1}{2}}}; \tag{134}$$

where $\omega$ is the positive real root of (127). Similar and symmetrical values are of course true for the other two components. This result is, it will be observed, the same as that deduced by means of Ivory's theorem: see equation (80).

196.] In the case of the ellipsoid being composed of a series of infinitesimally thin shells contained between two similar ellipsoids, each of which is homogeneous, but of which the density varies in passing from one to another, let us suppose the density to be a function of $\frac{a}{\text{A}}$; viz.,

$$\begin{aligned}\rho &= \text{F}\left(\frac{a}{\text{A}}\right)\\ &= \text{F}\left\{v\left(\frac{a^2}{\text{A}_1{}^2}+\frac{\beta^2}{\text{A}_1{}^2-(\text{A}^2-\text{B}^2)v^2}+\frac{\gamma^2}{\text{A}_1{}^2-(\text{A}^2-\text{C}^2)v^2}\right)^{\frac{1}{2}}\right\};\end{aligned}$$

which must be substituted for $\rho$ under the sign of integration in (134): and similar are the expressions for Y and Z.

If the density of each shell varies directly as $\frac{\text{A}}{a}$, a hypothesis which is made by many investigators of the figure of the earth, the integral admits of integration in finite terms.

In these and similar investigations accomplished mathematicians have almost revelled; and nothing short of absolute want of space hinders me from inserting processes which have been applied by them to the solution of the attraction of an ellipsoid on an external particle: I can only refer the reader to

(1) Theoria Attractionis Corporum Sphæroidicorum Ellipticorum Homogeneorum, methodo nova tractata; Auctore Carolo Friderico Gauss. Read to the Royal Society of Gottingen in the year 1813, and published in Vol. II. of the Transactions of the Society.

(2) Mémoire sur l'Attraction des Ellipsoides, par M. Chasles, and contained in Vol. IX. of the "Mémoires présentés par divers savants étrangères à l'Académie des Sciences," Paris, 1846, p. 629. This memoir contains a complete historical account of the problem, and of the several advances made by different mathematicians, and gives a synthetical solution of it.

(3) Nouvelle solution du problème de l'Attraction &c., par M. Chasles. Liouville's Journal, Vol. V, 1840, and Comptes rendus, Vol. VI, p. 902, 1838. It is from these papers, with some modifications, that the preceding solution of the attraction of an ellipsoidal shell has been in a great measure extracted.

(4) Mémoire sur l'Attraction d'un Ellipsoide, par M. Poisson, Mémoires de l'Académie Royale des Sciences de Paris, Tome XIII, p. 497. An abstract of this memoir is appended to Vol. II. of "Système du Monde, par M. De Pontécoulant," Paris, 1829.

(5) A memoir by C. F. Gauss in the "Resultate aus den Beobachtungen des magnetischen Vereins im Jahre 1839," Leipsic 1840, on attractive forces varying inversely as the square of the

distance. A translation of this memoir is inserted in Taylor's Translations of Foreign Scientific Memoirs, Vol. III, p. 153, 1843.

(6) Some papers by M. Liouville, and contained in his Journal, wherein he applies Lamé's method of elliptical coordinates.

197.] It remains for us to investigate another general property of the potential, which has been largely applied by Laplace and others in the discovery of the attraction of spheroids of small eccentricity on external and internal particles. Let the positions of the attracted and attracting particles be respectively $(a, \beta, \gamma)$ and $(x, y, z)$; and let $u =$ the distance between them: then

$$(a-x)^2+(\beta-y)^2+(\gamma-z)^2 = u^2; \tag{135}$$

$$\mathrm{V} = \iiint \frac{\rho\, dx\, dy\, dz}{u}; \tag{136}$$

$$\left(\frac{d\mathrm{V}}{da}\right) = -\iiint \frac{\rho\,(a-x)\, dx\, dy\, dz}{u^3},$$

$$\left.\begin{aligned} \left(\frac{d^2\mathrm{V}}{da^2}\right) &= \iiint \left\{\frac{3(a-x)^2}{u^5} - \frac{1}{u^3}\right\} \rho\, dx\, dy\, dz \\ \text{similarly}\quad \left(\frac{d^2\mathrm{V}}{d\beta^2}\right) &= \iiint \left\{\frac{3(\beta-y)^2}{u^5} - \frac{1}{u^3}\right\} \rho\, dx\, dy\, dz \\ \left(\frac{d^2\mathrm{V}}{d\gamma^2}\right) &= \iiint \left\{\frac{3(\gamma-z)^2}{u^5} - \frac{1}{u^3}\right\} \rho\, dx\, dy\, dz \end{aligned}\right\}; \tag{137}$$

therefore by addition

$$\left(\frac{d^2\mathrm{V}}{da^2}\right) + \left(\frac{d^2\mathrm{V}}{d\beta^2}\right) + \left(\frac{d^2\mathrm{V}}{d\gamma^2}\right) = 0: \tag{138}$$

this equation was first discovered by Laplace: it is only true however so long as the attracted particle is not a particle of the attracting mass: because if the particle is in the attracting mass, at that particular point $u = 0$, and V in (136) is infinite, and X, Y, Z assume indeterminate forms. To determine the value of (138) in this case, let us suppose a small sphere inclosing the attracted particle to be taken out of the attracting mass, and the radius of it to be so small, that the density may within the sphere be considered constant: let U be the potential of this small sphere, and U′ the potential of the whole excess of the attracting mass over the sphere: then by reason of (138),

$$\left(\frac{d^2\mathrm{U}'}{da^2}\right) + \left(\frac{d^2\mathrm{U}'}{d\beta^2}\right) + \left(\frac{d^2\mathrm{U}'}{d\gamma^2}\right) = 0.$$

And for the small sphere, let X′, Y′, Z′ be the components of its attraction on the particle placed at $(a, \beta, \gamma)$ which is within it: then by (59), if $\rho$ is the uniform density of the sphere,

$$\frac{\mathrm{x}'}{m} = -\left(\frac{d\mathrm{v}}{da}\right) = \frac{4\pi\rho a}{3}, \qquad \therefore \quad \left(\frac{d^2\mathrm{v}}{da^2}\right) = -\frac{4\pi\rho}{3};$$

similarly $\left(\frac{d^2\mathrm{v}}{d\beta^2}\right) = -\frac{4\pi\rho}{3}, \qquad \left(\frac{d^2\mathrm{v}}{d\gamma^2}\right) = -\frac{4\pi\rho}{3};$

$$\left(\frac{d^2\mathrm{v}}{da^2}\right) + \left(\frac{d^2\mathrm{v}}{d\beta^2}\right) + \left(\frac{d^2\mathrm{v}}{d\gamma^2}\right) = -4\pi\rho.$$

And thus if the attracted particle is a part of the attracting mass, since $\mathrm{V} = \mathrm{v} + \mathrm{v}'$,

$$\left(\frac{d^2\mathrm{V}}{da^2}\right) + \left(\frac{d^2\mathrm{V}}{d\beta^2}\right) + \left(\frac{d^2\mathrm{V}}{d\gamma^2}\right) = -4\pi\rho. \tag{139}$$

This Theorem was discovered by M. Poisson.

As the equivalents of the expressions (138) and (139) are often more convenient when expressed in terms of polar coordinates, let us investigate the transformation. Let the positions of the attracting and the attracted particles respectively be $(r', \theta'; \phi')$ $(r, \theta, \phi)$: and let $\zeta'$ and $\zeta$ be the curtate radii vectores: then, see Vol. II, Art. 146,

$$\left.\begin{aligned} z &= r'\cos\theta' \\ \zeta' &= r'\sin\theta' \end{aligned}\right\} \qquad \left.\begin{aligned} \gamma &= r\cos\theta \\ \zeta &= r\sin\theta \end{aligned}\right\}$$
$$\left.\begin{aligned} x &= \zeta'\cos\phi' \\ y &= \zeta'\sin\phi' \end{aligned}\right\} \qquad \left.\begin{aligned} a &= \zeta\cos\phi \\ \beta &= \zeta\sin\phi \end{aligned}\right\} \tag{140}$$

$$\therefore \ \mathrm{V} = \iiint \frac{\rho r'^2 \sin\theta'\, dr'\, d\theta'\, d\phi'}{\{r^2 - 2rr'(\sin\theta\sin\theta'\cos(\phi-\phi') + \cos\theta\cos\theta') + r'^2\}^{\frac{1}{2}}}; \tag{141}$$

and by Ex. 2, Art. 93, Vol. I,

$$\left(\frac{d^2\mathrm{V}}{da^2}\right) + \left(\frac{d^2\mathrm{V}}{d\beta^2}\right) + \left(\frac{d^2\mathrm{V}}{d\gamma^2}\right)$$
$$= \left(\frac{d^2\mathrm{V}}{dr^2}\right) + \frac{2}{r}\left(\frac{d\mathrm{V}}{dr}\right) + \frac{1}{r^2}\left(\frac{d^2\mathrm{V}}{d\theta^2}\right) + \frac{\cos\theta}{r^2\sin\theta}\left(\frac{d\mathrm{V}}{d\theta}\right) + \frac{1}{r^2(\sin\theta)^2}\left(\frac{d^2\mathrm{V}}{d\phi^2}\right)$$
$$= 0, \quad \text{or} \quad = -4\pi\rho, \tag{142}$$

according as the attracted particle is not or is part of the attracting mass.

198.] For a very simple example of this last formula, and for the purpose of indicating its use in the inquiry, let us apply it to a sphere. The object, be it observed, is to determine the potential, viz., V; and when it is impossible or difficult to do so directly, we attempt it indirectly; by the discovery, that is, of a function which will satisfy either of the equations (138) or (139), and (142) The last is known by the name of "The Differential Equation of Laplace's coefficients," being so called from the application which Laplace has made of it in the Mécanique Céleste, Tome II. The general integral of it has not yet been found.

Suppose the sphere, to which we are applying (142), to be composed of a series of concentric homogeneous shells, the density of each of which varies continuously as some function of the radius of its bounding surface: so that $\rho = f(r')$; let the origin be at the centre of the sphere: and let $r$ be the distance of the attracted particle from the centre: then $\text{v}$ is manifestly a function of $r$ only, and is independent of $\theta$ and $\phi$: so that (142) becomes

$$\frac{d^2\text{v}}{dr^2} + \frac{2}{r}\frac{d\text{v}}{dr} = 0, \quad \text{or} \quad = -4\pi\rho, \tag{143}$$

according as $m$ is without, or is a part of, the attracting sphere.

(1) Let $m$ be without the sphere,

$$\therefore \quad \frac{d^2\text{v}}{dr^2} + \frac{2}{r}\frac{d\text{v}}{dr} = 0;$$

$$\therefore \quad r^2\frac{d\text{v}}{dr} = -c; \tag{144}$$

where $c$ is an arbitrary constant.

$$\therefore \quad \text{v} = \frac{c}{r} + c'. \tag{145}$$

To determine $c$: if $m$ is placed at a great distance, the attraction is the same as if the sphere were condensed into its centre: in which case it would be

$$-\frac{\text{M}m}{r^2};$$

and since $-m\frac{d\text{v}}{dr}$ is the attraction of the sphere in the direction of $r$, therefore

$$c = \text{M}.$$

Also when $r = \infty$, the potential $= 0$, and therefore $c' = 0$: so that

$$\text{v} = \frac{\text{M}}{r}. \tag{146}$$

(2) If the attracted particle is within a shell: then, because the attraction vanishes when the particle is at the centre,

$$\frac{d\text{v}}{dr} = 0, \text{ when } r = 0;$$

therefore from (144), $c = 0$; therefore

$$\frac{dv}{dr} = 0, \qquad v = c': \tag{147}$$

for a similar reason will the potential have a constant value for all points within a shell whose bounding surfaces are similar and concentric ellipsoids: see Art. 193.

(3) Let $m$ be a particle of the attracting shell; so that (143) becoming

$$\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr} + 4\pi\rho = 0,$$

we have

$$r^2\frac{dv}{dr} + 4\pi\int r^2\rho\, dr = 0: \tag{148}$$

and let the radius of the interior surface of the shell be $r_0$, and that of the exterior surface $r_1$: then since for all values of $r$ between $r_1$ and $r$ the attraction of the shells is zero, we may take the limits of the $r$-integration to be $r$ and $r_0$.

Now $4\pi\int_{r_0}^{r} r^2\rho\, dr$ is the mass of the shell of which the exterior radius is $r$ and the interior is $r_0$; let this mass be $M_1$; and thus (148) becomes

$$r^2\frac{dv}{dr} + M_1 = 0;$$

$$\therefore \quad -\frac{dv}{dr} = \frac{M_1}{r^2}; \tag{149}$$

and therefore the attraction is the same as if $M_1$ were condensed into its centre. Also from (149), since $M_1 = \frac{4}{3}\pi\rho r^3$, if the sphere is homogeneous and full,

$$\therefore \quad \frac{dv}{dr} = -\frac{4\pi\rho r}{3}, \qquad \therefore \quad v - v_1 = -\frac{2\pi\rho}{3}(r^2 - r_1^2).$$

But from (146),

$$v_1 = \frac{4\pi\rho r_1^2}{3};$$

$$\therefore \quad v = 2\pi\rho r_1^2 - \frac{2\pi\rho r^2}{3}. \tag{150}$$

These results are evidently in accordance with those of the preceding Articles.

And here we bring to an end our Treatise on Statics; and the inquiry into the laws and circumstances of equilibrium of particles and bodies under the action of impressed pressures: and we shall resume the course of the treatise as it was originally laid down, and subsequently broken off at Art. 11; and shall proceed with a regular statement of the principles of the science of motion.

# ANALYTICAL MECHANICS.

## PART II.

## DYNAMICS; THE MOTION OF MATERIAL PARTICLES.

## CHAPTER VII.

### MOTION, ITS AFFECTIONS, ITS LAWS, AND ITS EQUATIONS.

199.] On resuming the course of our treatise of the science of motion which was interrupted at the end of Article 10, it is convenient to make some preliminary observations.

Mechanics is the science which treats of material particles and bodies, at rest and in motion; that part of it which relates to bodies at rest, that is, under the action of many forces in equilibrium, is called Statics, and has been discussed in the preceding part: and that part of it which relates to motion is called Dynamics, and will be developed in the following parts of the work: the passage from the latter to the former, and the process by which the principles of the latter include those of the former, as the general science includes its particular subordinate, will be investigated at the end of the present volume.

Dynamics, as it is intended to unfold the subject in the following pages, will be presented to the student in a twofold aspect: primarily and chiefly it will be considered as a positive and exact science, such as I have attempted to sketch it in the first chapter; and of that nature of which the pure sciences of number and geometrical space are supposed to be. Motion is the fundamental idea of it; that, viz., out of which spring all the truths of the science, and from axiomatic statements of which they are deductively inferred. Dynamics, as such, is a science of speculation and thought; doubtless in the construction of it experience may have *suggested* much, but the so called necessity of its principles is derived from another source. Secondarily it is my purpose to shew that the science is useful to explain phænomena of the world external to us: hence arises the necessity of proving that the axioms and the first statements of the pure science are true in the subject-matter of cosmical observation, and that the laws of natural phænomena are included within the range of the pure science. Now for this end large experience, in the way of observation and experiment, is frequently required. The operations of nature are complex, and it is only with deep searching that they allow themselves to be so far unravelled as to exhibit the laws they are subject to. In this respect then it is necessary to apply a limit to our inquiry; and I propose only to shew, and that concisely, that the axioms of the pure science, or the laws of motion, are true in cosmical matter; so that, thus far at least, it is likely that we are on the right road of *natural* philosophy. And it will also be desirable, here and there, to point out certain salient laws, such as the law of gravitation and Kepler's laws of planetary motion, that our attention may be directed to them rather than to others. As in the last Chapter it was beside our object to enter on the applications of attractions to the figures of the earth and the planets, to the theory of heat, and to magnetism and electricity, because such applications can be made only on certain hypotheses, and with the developement of functions in series involving infinitesimal terms, the knowledge of which belongs to the special subject: so in the following treatise I shall not enter on the planetary or lunar theories, on the phænomena of precession of the equinoxes and the nutation of the earth's axis, because such subjects require special knowledge, and belong to physical astronomy; neither shall I carry on the investigation into the theory of light: but the general equations of dynamics will be investigated in all their breadth, and will be brought down to that stage where these special sciences commence; and will not, except in very simple instances, be applied to cases or under circumstances wherein such special knowledge is required. Thus our science is a principal and normal one; *normal*, I say, because it is that to the rules of which each special subordinate science conforms: and the greater or less that conformity is, the more or less complete is that special science; and *principal*, because the laws of dynamics are those which the special science takes and applies, each in its form and degree; and they are so large, that many forms of them are included which observation has not yet shewn to exist in the material universe. The applied part also serves a moral purpose, insomuch that it enables man to fathom the depths of the laws of Cosmos, to express them in a concise form, and thus to study the works of God. It is for these reasons that the science of motion is the most perfect of the physical sciences.

Although philosophically perhaps it might be more correct separately to investigate these two branches of the subject, yet, as the treatise is didactic, it is more convenient to consider parts of one or the other, as they arise in the course of it.

The nature of the symbols which will be employed requires a remark; we shall have to speak of *time*, *space*, *velocity*, *matter*; these are heterogeneous quantities, and cannot be operated on so as to multiply time into space or mass into velocity; this is self-evident. But these quantities will be represented by symbols such as $t$, $dt$, $s$, $ds$, $v$, $dv$, $m$, $dm$; and these are *numbers*, and not the concrete things. Thus $t$ expresses the $t$ times an unit of time is taken; $dv$ the $dv$ times an unit of velocity is taken; and the numbers, of course, can be multiplied together, and the resultant of the operation is number of that kind which the symbols express before the operation. The unit of concretion however, which is to be introduced after the operation, may be different to that previous to the operation: see Art. 109. The concrete units are of course arbitrary, but remain unaltered during the whole of an operation. Sometimes a second, sometimes a year is taken as the unit of time; it varies according to the problem; and the circumstances of it will generally guide us to a judicious choice.

200.] *Motion* is the fundamental idea of mechanics; motion, that is, either real or virtual, either in act or in power; and therefore the science is more correctly termed *the science of motion.* Motion need not be defined: it is too general to be capable of useful expression by means of a more general term; it is a quality or a state: one result of it is change of position of the thing moving: I say, *thing* moving; for a necessary element in an adequate conception of motion as the fundamental idea of mechanics is that something moves: motion must exist in, and be of, something; and that something, in which it is and of which it is, is called *matter*. Motion, as it is scientifically treated of, must be clothed, and matter is that wherein it is clothed; motion therefore is a state of matter. Now when we speak of matter as the subject of the science, the term is not limited to the matter of the members of the solar system only; to that which has sensible properties, and which gravitates; but it embraces every thing that moves or is capable of motion; the particles of air of course are included; and they gravitate, and they are the subject matter by the vibrations of which sound is propagated; the particles of light which the emission-theory of light assumes, and the æthereal molecules of the undulatory theory, are also included. Matter is treated as the subject of motion; and when it is spoken of, it is supposed to have one essential property, and that is *mobility*.

Matter also admits of divisibility without limit: a very large quantity of it may have motion, or a very small, nay an infinitesimal, particle; such as is analogous to a geometrical point: and its other properties, mobility and such like, are independent of the quantity of it. This remark is important; because it will shortly be convenient to divide the subject, according as we consider the motion of a finite quantity of matter, which is supposed to consist of an infinite number of particles, and which is called a material body: or according as we consider that of an infinitesimal quantity, and which is called a material particle. The quantity of matter which a body or a particle contains is called its *mass*.

201.] Of matter, having the property of mobility, there are two other affections, which, by reason of their abstract nature, need not be defined: viz. *time* and *space*: it is sufficient for us to be able to form a notion of them, and to enuntiate of them such properties as are required for the science of motion. Space and time, like matter, are continuous and divisible; and these affections are without limit. Space may be very large, nay, infinite; we cannot fix the boundaries of that space in which the heavenly bodies are; and it may be very small, such as that occupied by a chemical atom or a material particle. Time also admits of degrees as to quantity; it may be an instant; such an infinitesimal, that the aggregate of an infinity will make only finite time: or it may reach through the present moment from ages bygone to ages to come. Motion, matter, time and space, stand to each other in the following relations.

Matter exists in space and time; all matter, even the minutest particle, occupies space. No two particles of matter and also no two bodies can occupy the same space at the same time; this property of matter is called its *impenetrability*. The same matter cannot be in two different places at the same time: hence a particle of matter or a body cannot pass from one position to another without lapse of time: time is consumed in the passage; and therefore a change of place requires time. And as a longer or a shorter time may be spent in the passage, so do we conceive of the rate or speed at which a particle or a body moves. And hence arises the quality of matter which is called velocity; velocity being the degree of swiftness or slowness with which matter moves. From these relations arises the necessity of measuring space and time, and of determining equal spaces and equal times. As material bodies exist in space, they have *volume* and *form;* volume depending on the quantity of space which they occupy, and form on the bounding terms of that space; but the knowledge of equal spaces must be found in an adequate knowledge of space. The method of measuring volume is founded on the geometrical principle of superposition, and two volumes are equal which occupy the same or equal spaces. The notion of equal times and also the measure of equal times arises out of the idea of time, and an idea of time is not adequate unless it has these notions; it is true that the passage of time is marked by events which take place in it; and equal times are marked by the regular recurrence of the same event, that is, by uniform motion; but equal times * are in themselves altogether independent of any particular kind of motion; they exist before it and they enable us to apprehend and to measure such a motion: equal times therefore must be deduced from the notion of time.

202.] The most simple motion which a material particle can have, is that in which it describes equal linear spaces in equal times; the motion of it is then said to be uniform, and the velocity to be constant; these two expressions in fact being equivalent; and when equal spaces are not described in equal times, the velocity is said to be variable. Such a velocity may vary continuously or discontinuously; but it will be necessary for us to consider only a continuously-varying velocity; because a discontinuous variation will be a succession of constant velocities, changing abruptly, and, as it were, by impulses.

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* M. Poisson, Traité de Mécanique, $2^{de}$ Ed. Tome I, p. 205, writes: "La notion des temps égaux, et la mesure du temps ne sont fondées nécessairement sur aucune loi particulière de mouvement, et l'on peut, en conséquence, les supposer dans la définition du mouvement uniforme et de toute autre sorte de mouvemens." Dr. Whewell, in his Treatise on Mechanics, Ed. 5, Art. 102, says: "Those intervals of time, in which there is no discoverable reason why they should be unequal, are supposed equal."

In the case of constant velocity, equal linear spaces are described in equal times; now although the velocity of a moving material particle is a quality or state of the particle itself and resides in it, and is that by which it differs from a particle at rest, and although no account more exact can be given of it, yet the velocity can be measured; and the measure is taken to be *the number of units of linear space passed through in an unit of time.* Suppose therefore a material particle to describe uniformly $v$ units of space (observing that linear space is meant) in one unit of time, $v$ is the measure of the velocity: and if $s$ represents the space passed through by the particle in $t$ units of time, then, bearing in mind the last clause in Art. 199,

$$s = vt, \tag{1}$$

and
$$v = \frac{s}{t}. \tag{2}$$

Thus velocity is linear space, and is the linear space described in one unit of time.

If the velocity continuously changes, equal spaces are not described by the particle in equal times, and the velocity becomes a function of the time. Let the time be resolved into infinitesimal elements, and, to fix our thoughts, let us suppose the particle to be moving along a straight path: and let us suppose it at the end of the time $t$ to be at a distance $s$ from an origin fixed on the line, and to be at that time moving with a velocity $v$: that is, if the particle were to move for one unit of time with the velocity which it has at $s$, it would describe $v$ units of space in that unit of time; and suppose $ds$ to be the space described in $dt$, the next element of $t$; then, if $v$ is the velocity at the beginning, and $v+dv$ is the velocity at the end, of $dt$, the *mean* velocity with which $ds$ has been described may be expressed by $v+\theta\, dv$, where $\theta$ is a proper fraction, and is positive or negative according as the velocity is increasing or decreasing: therefore by reason of (1),

$$ds = (v+\theta\, dv)\, dt;$$

and neglecting the infinitesimal of the second order, as by the principles of infinitesimal calculus we are obliged to do, we have

$$ds = v\,dt; \tag{3}$$

that is, $ds$ units of space are described in $dt$ units of time by the particle moving with the velocity $v$ at the beginning of $dt$; and therefore dividing through by $dt$, we have $\frac{ds}{dt}$ equal to the space described in one unit of time; and this is velocity; and thus we have

$$\frac{ds}{dt} = v. \tag{4}$$

In the cases therefore, both of constant and of varying velocity, velocity is the space described in an unit of time, and is, by reason of (2) and (4), the ratio of the space described to the time during which it is described; and is in the latter case the ratio of two infinitesimals.

It will be observed that $\theta$ has disappeared: now as it is upon the sign of $\theta$ that an increasing or decreasing velocity depends, so are the results (3) and (4) true in both cases.

203.] We have been speaking of varying velocity of a moving particle, with reference to that circumstance alone, and not with reference to any cause of it *; an important question now arises; according to the conception of matter, as the subject of motion, has it any power of changing its state; has it when at rest a power of putting itself into motion? has it when in motion a power of itself either of increasing or of diminishing its velocity? An adequate conception of matter involves a negative reply to these questions. *Matter is inert;* it has no power of acting on itself or of changing its own state. If it is at rest, it will remain at rest: if it is moving with a given velocity along a rectilineal path, it will continue to move with that velocity along that path: there is no more reason why it should change its course towards one side of that line than towards the other: this is equivalent to saying that lapse of time does not affect matter's state as to rest or motion. And not only does matter remain as it is, unless acted on by some source of velocity external to itself, but it also passively submits to external influence: whatever effect is communicated to it, that is also developed in it. Now I am not saying that matter does not act on other matter, for the matter of Cosmos does so act: thus leaden balls attract each other: particles of air repel each other: but it does not change its own state. Whenever therefore—and this is most important—matter's state is changed either from rest to motion, or vice versa, or when its velocity is increased or diminished, that change is due to some adequate cause, and velocity is communicated to it from some source external to itself. This source is called *Force;* and force is either accelerating or retarding according as the velocity of matter is, by its action, increased or diminished: a more exact definition will be given hereafter when we inquire into the correct measure of force by means of the effect which it produces. From the principle of inertia will be inferred the first equations, or propositions, of the science. The principle when stated in the following form is commonly called the first Law of Motion:

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* A distinction has been made, first, I think, by M. Ampére, between the results of motion, as they arise from the simple consideration of motion itself, or from the consideration of motion as the effect of certain acting causes. The former subject has been called Cinematics, and to the latter the term Mechanics has been applied. Our investigations thus far have been cinematical. The generation of curves by mechanical appliances belongs to cinematics.

*Matter at rest remains at rest, and matter in motion continues to move in the same line and direction, and with unvaried velocity, unless acted on by some force external to itself.*

This principle of inertia is axiomatic, and is the first axiom in the construction of the science; it rules that when a change of state takes place in matter, that change is due to the action of some cause external to the matter.

204.] And before we proceed further, it is desirable briefly to inquire how far these properties of matter are true in that of the earth and of the several bodies of the Solar System.

As to Mobility; the fact is shewn by daily observation: bodies falling towards the earth, particles of matter constantly moving in the air and as seen in a sunbeam, the waters of the sea never at rest, the motion of the moon and of the planets, the motion of particles of air in the wind, all bear evidence to this property: nothing is seen quiescent, every thing is in motion.

As to Inertia: terrestrial matter seldom changes its state without our being able to assign the cause; and hence we inductively infer, that the cause could always be assigned, if our knowledge of the moving matter and its circumstances was perfect. Consider a particle of iron, placed on a smooth table; relatively to the table it is at rest: but let a magnet be placed so that the particle of iron is within its influence; the particle will begin immediately to move towards it; and the longer the space is through which the particle moves, the greater will be its velocity; thus the magnet is the cause of the motion of the particle at first, and also of its subsequently increasing velocity. Now let another magnet be introduced of the same power as the former, and acting along the same line of action, and in an opposite direction, so that the action of the former magnet on the particle of iron is neutralized: then it is found that the iron-particle will continue to move with the velocity which it has at the time when the neutralizing magnet is introduced: that is, the velocity which it has at that instant is a quality residing in it, and which it has of itself no power to annihilate: its velocity will, it is true, during the subsequent motion become less and less; yet it appears that such a loss of velocity is caused by the friction against the table, the resistance of the air, and so on: for if these impediments are diminished, the particle continues to move with a velocity less rapidly decreasing: and hence we infer that if they are entirely removed, there will be no diminution of the iron's velocity.

So again if a ball is projected along a level surface, such as a bowlinggreen, the rougher the surface is the more impediment does it offer to the ball's motion, and the sooner is the ball reduced to rest: but if the surface is smooth, as a pavement, or smoother still, as a plate of glass, or as ice, the longer will the ball continue to move; eventually, however, it will be reduced to rest, because it is impossible to remove all the impediments which are continually acting on it as retarding forces, and are thereby withdrawing velocity from it.

Again, if a suspended pendulum oscillates, the time ere its motion ceases will be longer if it vibrates on a knife-edge than if it is suspended by a spring, because the resistance of the former is less than that of the latter; and if it oscillates in the exhausted receiver of an air-pump, the time ere its motion ceases will be longer than if the oscillations take place in air. From experiments such as these, it is inductively inferred, that if all the hinderances are removed, and if the moving matter does not receive velocity from any other source, it has in itself no power either to increase or to diminish its own velocity.

The uniform periods of the planets, and the never-varying length of the mean sidereal day, in a similar manner shew that the same law is true in the matter of which the bodies of the solar system consist.

The principle of inertia was first recognised by Galileo: mechanicians had before his time failed in a correct exposition of the principles of mechanics.

205.] Matter therefore can neither generate velocity for itself out of its own resources, neither can it absorb into itself velocity which it has, or velocity which is communicated to it: it is alike "natural" to it to be at rest and in motion; whenever therefore its state changes, some source external to itself is the origin of the change; if the velocity is increased, some velocity has been communicated to it; if it is diminished, velocity has been abstracted from it: whatever causes a change of velocity is called *Force*, and the word Force will be henceforth used in Dynamics in this meaning only.

And here a distinction must be made, which will necessitate a division of the subject. If matter is in motion, and one or more forces are also acting on it at the same time, their lines of action may be either that along which the body moves, or inclined at a certain angle to that line; in the latter case the matter will be deflected from its rectilinear path, and its path may henceforth not be straight; these circumstances of motion will be investigated in Chap. IX. In the former case the moving matter will be accelerated or retarded in its own line of motion. For suppose a material particle to be moving with a constant velocity $v$, and two forces $f$ and $f'$ to act on it, the effects of which are severally to produce velocities $u$ and $u'$ in one unit of time; and suppose each of these forces to act for $t$ units of time: then the velocity of the particle at the end of $t$ units of time will be $v+ut+u't$, since by the law of inertia, and by its consequence, that matter is passively receptive of velocity from an external source, the velocities arising from many forces acting simultaneously along the same line of action are simply additive. If one of the forces, say $f'$, act in a direction contrary to that of the particle's motion, it will abstract velocity, and the velocity of the particle will, at the end of $t$ units of time, be $u+ft-f't$.

Hence if two forces are capable of communicating equal velocities to the same body in equal infinitesimal elements of time, the two forces are said to be equal, and are such, that when applied to the same body in opposite directions along the same line of action, they neutralize each other, and do not change the body's state. This is the definition of equal forces. Similarly, forces which in equal infinitesimal elements of time will produce in a given body, twice, thrice, &c. the velocity which another force will, are estimated as double, triple, &c. of this latter force.

206.] When a force acts on matter, and communicates velocity to it, the communication may take place in two ways: in equal successive elements of time, either equal or unequal velocities may be communicated: in the former case, the force is called constant: in the latter, variable, and the law according to which the velocity is communicated is called *the law of force.* Let us consider the effect of these forces, as they are exhibited in the altered velocity of the moving matter. The velocity which a force transfers to a body is called *the impressed velocity:* and the velocity which is developed in the moving body is called *the expressed velocity;* in the case of a single material particle they are equal: but, for reasons which will be explained hereafter, they are generally unequal in the case of a particle which is a component element of a moving body: when they are equal, it is to be observed that they are the same thing under different aspects.

Now, to fix our thoughts, let us suppose a material particle to receive velocity: then a constant force being such that equal velocities are communicated in equal times, let $f$ represent the velocity communicated in one unit of time; (the velocity being represented by units of space described in one unit of time, and force being therefore also represented by units of space:) and let us suppose the force to act for $t$ units of time; then $ft$ is the velocity which will be communicated in the time. Suppose $v$ to represent the velocity which the moving particle has acquired in the time $t$ during which the force has been in action on it; that is, let $v$ be the expressed velocity of the particle; then if all the velocity which has been impressed by the force is also expressed in the moving particle,

$$v = ft. \tag{5}$$

Also if a particle is moving with a velocity $u$ before the force $f$ acts, and if the force acts for $t$ units of time along the line of motion, and in the direction in which the particle moves, then the velocity at the end of the time $t$ is

$$u+ft.$$

And if the force acts in a direction contrary to that in which the particle is moving, the velocity at the end of the time $t$ is

$$u-ft.$$

A variable force is that by which unequal velocities are communicated in equal successive times; in this case, the time must be resolved into its infinitesimal elements, and the increments of the velocity are to be considered as they are impressed by the force, and expressed in the moving particle. Suppose the force to be such that at the time $t$, a velocity $f$ would be impressed by it in one unit of time, if the force were constant during that unit: and to be such that at the time $t+dt$, a velocity $f+df$ would (under the same supposition) be impressed by it in one unit of time; let $v$ be the velocity of the particle at the time $t$, and $v+dv$ at the time $t+dt$; then, if $\theta$ is a proper fraction, $f+\theta\, df$ will express the mean value of the velocity communicated during the time $dt$; the term *mean value* implying that, if $f+\theta\, df$ is uniformly impressed during the time $dt$, the velocity $dv$ will be impressed on the moving particle. Hence, by reason of (5),

$$dv = (f+\theta\, df)\, dt; \tag{6}$$

and therefore neglecting $df \times dt$ which is an infinitesimal of the second order,

$$dv = f dt; \tag{7}$$

that is, the velocity in $dt$ units of time is increased by $f dt$; and is therefore increased by $f$ in one unit of time; $f$ therefore or $\frac{dv}{dt}$ is the velocity expressed in one unit of time; and therefore the unit of force is that which impresses an unit of velocity on the moving particle in an unit of time.

Hence then we have a measure of force; viz. the velocity which the force will impress on matter in one unit of time; and as in the case of a single material particle the expressed velocity is equal to the impressed, we may take the expressed velocity to be a correct and adequate measure of the force; and thus although $f$ may be called a force, yet it is really the *increment of velocity* in one unit of time, and thus is only the measure of the force. And it is also to be borne in mind, that the force is assumed to act invariably, that is, velocity is supposed to be uniformly communicated by it during that unit of time.

From the preceding equations we have the following results:

$$v = \frac{ds}{dt};$$

$$f = \frac{dv}{dt} = \frac{d}{dt} \cdot \frac{ds}{dt}$$

$$= \frac{d^2s\,dt - d^2t\,ds}{dt^3}; \tag{8}$$

and therefore if $s$ is equicrescent,

$$f = -\frac{ds\,d^2t}{dt^3}; \tag{9}$$

and if $t$ is equicrescent,

$$f = \frac{d^2s}{dt^2}. \tag{10}$$

We shall suppose $t$ to be an equicrescent-variable throughout the whole treatise, unless it is stated expressly that it is not so.

207.] Understanding then by force its measure or its effect, which is expressed velocity-increment, there are three different expressions for it: these require explanation.

Let a particle be moving, and let it describe the space $s$ in the time $t$:

(1) Let us suppose the space and the time to be resolved into corresponding infinitesimal increments, so that neither all the $dt$'s nor all the $ds$'s are equal: in which case neither $t$ nor $s$ is equicrescent; and thus (8) correctly represents the expressed velocity-increment due to one unit of time: but the expression is unnecessarily complicated, and is therefore of little practical use.

(2) Let the time be resolved into equal elements, that is, let $t$ be equicrescent: then $d^2t = 0$, and (10) expresses the expressed velocity-increment. Now if the velocity is constant, all the corresponding elements of space will be equal: that is, all the $ds$'s will be equal and $d^2s = 0$: there will, in this case, be no velocity-increment, and therefore no force is acting. If the velocity is not constant, the $ds$'s corresponding to equal $dt$'s will not be equal; there will be an excess of one $ds$ over the preceding or succeeding $ds$, and thus there will be a $d^2s$: as $dt$ is constant, let us assume it to be the unit of time: then $ds$ is the velocity; and $d^2s$ is the velocity-increment; and therefore measures the accelerating force. It is also to be observed, that if the velocity-increment is constant, the accelerating force is also constant; $d^2s$ is constant, and therefore $d^3s = 0$: but if on

the other hand the accelerating force is variable, the velocity-increment is also variable, and the $d^2s$'s vary, and $d^3s$ is not equal to zero: similarly we might proceed, and shew under what circumstances $d^3s$ would be constant, and therefore $d^4s = 0$.

(3) Let the space be the equicrescent variable; in which case, if the velocity is constant, the $dt$'s corresponding to the $ds$'s are equal, and $d^2t = 0$; but if the velocity is not constant, equal $dt$'s do not correspond to equal $ds$'s, and therefore $d^2t$ will not be equal to zero: in this case (9) is the expression for the velocity-increment; $f$ being affected with a negative sign because the velocity-increment becomes greater, as the time to which it is due becomes less; in this case therefore the $dt$'s, to which equal successive $ds$'s are due, are decreasing, and therefore $d^2t$ is negative.

208.] Force, therefore, such as we have considered it, impresses finite velocity in a finite time; and the effects of it have been resolved into elements corresponding to infinitesimal elements of time. Thus if a force acts for a finite time, and if the law of the force is given, the total velocity impressed by it during the whole time may be found by integration, and the whole velocity will be the measure of the force's action. A force of this kind is commonly called *a finite accelerating* (or retarding) *force*. But suppose a force to act, and to communicate a very great velocity in a very short time, (such as the explosive force of gunpowder, which will impress a very great velocity on a cannon ball in the very short time during which the ball is passing along the bore of the gun,) then doubtless if the law of the communication of the velocity is known, the whole velocity may be found as in the former case, and will be the measure of the action of the force; but if the law of the force is not known, and the force acts for a short time and then ceases, the whole velocity which is impressed by it may be taken as the measure of its action. A force of this kind is called an *Impulsive* or *Instantaneous Force*. This force does not, it is to be observed, differ in kind from finite accelerating force; the communication and the developement is just as gradual in one case as in the other; the difference consists in the mode of measurement of its effect: in the former case the law of force is given; and the total action of the force is determined by integration: in the latter case, whether the law of force is known or not, the action of the force is measured by the whole velocity which has been communicated by it.

209.] Hitherto motion and velocity have been considered independently of the quantity of matter of which they are. Velocity and its properties have been discussed as existing in a mass, and, to fix our thoughts, we have assumed in some cases that a material particle has been moving; but it has been unnecessary to introduce any reference to the quantity of matter, because the velocity of a material particle and of a mass of large dimensions may be the same: and inertia as a property of matter does not require any conditions as to the quantity of matter: it is true equally of a particle and of a large body. But now it is necessary to consider quantity of matter or mass: because the equations of motion of moving matter, from which all the theorems of dynamics will be deduced, are formed by comparing *the velocity impressed with the velocity expressed*; and therefore a question arises, whether two bodies having equal velocities impressed on them will move with equal velocities, whatever are their masses? No doubt, by the principle of sufficient reason, if their masses are equal, the expressed velocities will also be equal: but what will be their expressed velocities, if the masses are unequal? Hence arises the necessity of defining equality of mass; and we must be on our guard against an argument in a circle: equal masses must not be defined to be those on which equal forces act and impress equal velocities, when equal forces are defined to be those which impress equal velocities on equal masses. Suppose two masses to have the form of spheres, and to be moving with their centres along a straight line, and in opposite directions, then if they impinge on each other, and if each is by the collision brought to rest, these masses are said to be equal: so that equal masses are defined in the following terms: "*Two masses are equal which moving with equal velocities along the same straight line, in opposite directions, and impinging on each other, are reduced to rest by the collision.*"

When many masses have by this process been determined to be equal to each other, we may collect two or more into one mass, and thus obtain masses which shall be any multiple of a given mass: and by a reverse process we may obtain masses which are submultiples of another mass; and thus we may obtain masses which bear any ratio to each other. Thus if $m$

equal masses are collected into one mass, and $m'$ into another, the ratio of these collected masses will be to each other as $m$ to $m'$.

As matter which occupies finite space is called a finite body, so matter which occupies an infinitesimal space-element is called a material particle. Thus a finite body consists of many material particles. The motion of a material particle is of a kind more simple than that of a body, and for this reason: if a body moves, its particles may all of them describe equal and parallel paths, in which case the body is said to have only motion of translation; or the particles of the body may revolve one about another, without the relative positions of them being changed, in which case the body has the motion of rotation: a full discussion of these circumstances will be the subject of Part III of the Treatise: for the present we shall consider the motion of material particles only: and as each one occupies space equal to an infinitesimal geometrical point, the motion of rotation may be neglected, and we shall consider motion of translation only.

Experience teaches us, in the case of terrestrial matter, that if two particles are at rest, and if it is required to make them move with equal velocities, a much greater force must act on one than on the other; and the reason assigned is, that one has much greater *force of inertia* than the other. Now this is an inaccurate mode of expression. *Inertia* * has no force: it neither destroys nor generates velocity; motion and rest are equally natural to matter. The true reason is, one mass is greater than another, and therefore has a greater quantity of matter for velocity to be communicated to.

210.] Let two material particles be in motion; of which I will suppose one to be the unit-mass, for the unit is arbitrary, and the other to contain $m$ unit-masses; and let them move with equal constant velocities $v$: if the $m$ unit-masses of the larger mass are separate, each would move with the same velocity $v$, and therefore the sum of all the particles moving separately would be $mv$; and by the principle of inertia the sum of the velocities is not changed when all the particles are collected into one common mass, and therefore the quantity of velocity which is expressed in the moving mass $m$ is $mv$: that is, is in quantity $m$ times the velocity of the unit-mass. Although therefore the velocity of both the masses is the same as to intensity, yet in *quantity or amount of velocity*, that of the mass $m$ is $m$ times that of the unit-mass.

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* Many writers on mechanics use the expression "force of inertia;" and lately $\frac{d^2s}{dt^2}$ has been called "force of inertia;" the expression is surely inaccurate and unphilosophical, if the words are used in the senses which I have assigned to them: and I cannot but refer to M. Poisson, who is no mean authority on such questions, for a corroboration of the view of the subject here taken. Traité de Mécanique, 2^de^ Edition, Tome I, Art. 120.

If therefore the particle $m$ is originally at rest, $mv$ is the quantity of velocity which is expressed in it: and by the law of inertia this must have been impressed on it: these two quantities are therefore equal, and we have

$$\begin{aligned}\text{impressed quantity of velocity} &= \text{expressed quantity of velocity} \\ &= mv. \qquad (11)\end{aligned}$$

As we shall frequently speak of this "quantity of velocity," it is convenient to assign to it a distinctive name: it is, as explained above, the product of the numbers expressing the mass and the velocity, and has been ordinarily called *momentum* or quantity of motion; although the term is somewhat inaccurate, yet, to avoid the inconvenience of new nomenclature, I shall use it, and shall signify by it the quantity of velocity which exists in moving matter: and shall henceforth signify by the term "velocity," velocity as to intensity. The increments of these will be called respectively *the momentum-increment* and *the velocity-increment*; and of these, when expressed, the mathematical equivalents will be $m\frac{d^2s}{dt^2}$ and $\frac{d^2s}{dt^2}$, see equation (10), if $t$ is the equicrescent variable. And thus (11) becomes

$$\begin{aligned}\text{impressed momentum} &= \text{expressed momentum} \\ &= mv. \qquad (12)\end{aligned}$$

Hence if a particle of $m$ unit-masses moves with a velocity $v$, its momentum is $mv$; and if all the momentum is transferred to another particle $m'$, the velocity $v'$ of this latter particle will be such that

$$mv = m'v'. \qquad (13)$$

Thus if a particle (as a small ball) of mass $= 3$, moves uniformly with a velocity $= 4$, its momentum $= 12$; and if it impinges directly on another particle of mass $= 2$, and by means of the impact is reduced to rest, the whole of the momentum will be transferred, and the latter particle will move with a velocity $= 6$:

6 therefore will be the expressed velocity, and 12 will be the expressed momentum of this latter particle.

Hence also momentum is the measure of the pressure of percussion of a moving mass.

So if two particles, moving along the same line and in opposite directions with velocities which are inversely proportional to the masses, impinge directly on each other, they will be reduced to rest by the collision.

Again, if a cannon ball of mass $= 10$ is fired from a gun, and emerges from the bore with a velocity $= 250$, the momentum of the ball is 2500, and this will be the measure of the explosive force of the gunpowder.

Hence also it appears that whenever momentum is impressed on a mass by means of matter acting upon it, it is withdrawn from some other source, or an equivalent momentum is simultaneously produced in an opposite direction. Hence also we infer that the whole amount of momentum is always the same. Momentum cannot be created: it can only be transferred. It may perhaps be thought that momentum can be generated by muscular action, say, that a stone may be thrown, and thus receive momentum, by the muscular action of the arm: we must not however be deceived by appearances: let a person stand in a frame suspended as the scale of a balance, and which is capable of moving freely: if he impresses momentum on any body, as, for instance, if he throws a stone, it will be found that he moves in a direction directly opposite: and the product of his mass and the velocity with which he moves in the scale will be equal and opposite to that which he has given to the stone: the apparent creation then of momentum in one direction is accompanied by the creation of an equal quantity in the opposite direction. A similar effect takes place when momentum is imparted to a mass by means of a pressure against the earth.

The distinction which is drawn between velocity as to intensity and velocity as to quantity, or momentum, may be illustrated by the following analogies. Suppose two masses of the same substance, one of which is ten times as large as the other, to be in the same state of *temperature*, and suppose both to be heated so as to be of the same higher temperature: then to these masses heat has been transferred from some external source; and to the larger mass ten times as much as that to the smaller one; and thus, although both are of the same heat as to intensity, yet the quantity of transferred heat in one is ten times as great as that in the other. A thermometer measures temperature, that is, heat as to intensity; in the same manner does the space described in one unit of time measure velocity as to intensity. Again, suppose a certain quantity of light from a given source to be received by a table or a given area; the smaller the area is, if it receives the whole light, so much more intense will the illumination be. But if a given area is illuminated equally throughout its surface, the greater the surface is, the greater also will be the quantity of light received by it. Thus light does as to intensity vary inversely as the area over which it is spread: but the quantity of light received by a surface illuminated equally throughout varies directly as the surface.

211.] These questions must also be investigated with reference to a continuously increasing or decreasing velocity: let the mass of the moving particle be $m$; and let the expressed velocity-increment of it in one unit of time be $\frac{d^2 s}{dt^2}$: then its expressed momentum-increment $= m\frac{d^2 s}{dt^2}$; and this quantity is equal to the impressed momentum-increment: therefore

$$\begin{aligned}\text{impressed momentum-increment} &= \text{expressed momentum-increment}\\ &= m\frac{d^2 s}{dt^2}. \qquad (14)\end{aligned}$$

The source of the impressed momentum-increment is called *moving force;* and as by (14) the impressed momentum-increment is equal to the expressed momentum-increment, that is, to $m\frac{d^2 s}{dt^2}$, we may take this latter to be the measure of the moving force: and as the accelerating force is measured by the expressed velocity-increment $\frac{d^2 s}{dt^2}$, it follows that the measure of the accelerating force is the same as that of the moving force acting on an unit-mass. Hence also the unit moving force is that which impresses an unit of velocity on an unit of mass in an unit of time. Equation (14) is called an equation of motion.

Let us exemplify this result: suppose a particle $m$ to be falling towards the earth: it is found by experiment that the earth's attraction is an uniformly accelerating force, (see also Art. 226), which impresses on the falling particle a velocity-increment of 32 feet (approximately) in one second of time; let a second therefore be the time-unit, and let us represent the number 32 by $g$ (= gravity): then $mg$ is the impressed momentum-increment, and $m\frac{d^2s}{dt^2}$ is the expressed momentum-increment: therefore, by reason of (14),

$$mg = m\frac{d^2s}{dt^2}; \tag{15}$$

$$\therefore \quad g = \frac{d^2s}{dt^2}.$$

The illustration may be more correctly represented when we take account of the mass of the earth. For since the attraction between the earth and the particle is mutual, the particle attracts the earth while the earth attracts the particle: if therefore $m$ and $\text{M}$ are the masses of the particle and of the earth respectively, the velocity-increments of the particle and earth in an infinitesimal element of time are inversely as the masses.

Similar too is the mutual attraction of the earth and moon: the expressed velocity-increments with which they move towards each other are inversely as their masses. Hence it follows that their centre of gravity would remain at rest, if the earth and moon had no other motion than that which is due to their mutual attraction: but owing to the action of the sun, and the motion of each in space, the centre of gravity describes an ellipse with the sun in one of the foci.

Hence then it follows that (1) moving forces do not impress equal velocities on different masses, unless they are proportional to the masses; (2) the velocities expressed in equal masses are proportional to the moving forces; (3) the velocities expressed in unequal masses by equal moving forces are inversely proportional to the masses. Hence also we infer that when a moving force impresses velocity on a mass, the velocity expressed varies directly as the moving force and inversely as the mass. This last proposition has been commonly called the third Law of Motion, and is enuntiated in a form such as,

*When moving force produces velocity in a given mass, the velocity produced is inversely proportional to the mass.*

Sir Isaac Newton calls the following proposition the third law of motion: "Action and reaction are equal and opposite." This however is no more than a statement in plain language of (12) and (14); and it is necessary to explain the meaning of the terms *action* and *reaction*, and how they are measured.

And here we have come to the second axiomatic principle which is necessary to the construction of our science: when the state of matter, as to motion, changes, a measure of the change is hereby given: the product of the mass and of the velocity which is expressed in a given time is the measure of the force which has caused the change, and is by the first principle, that viz. of inertia, see Art. 203, equal to the impressed momentum. From this equation all the results of dynamics will be deduced.

212.] The truth of the preceding theorems connecting moving force, mass, and expressed velocity, in the case of terrestrial matter is proved by Attwood's machine, for a full description of which I must refer the reader to Attwood's original treatise on rectilineal motion, and to other treatises on experimental mechanics, but of which a concise account is given in Section 3 of the succeeding Chapter. It is shewn by numerous experiments made with it that the expressed velocity-increment in a second of time varies directly as the moving force and inversely as the whole mass moved; and therefore the product of the mass and the velocity-increment varies as the moving force, and may be taken to be a correct measure of it. The same theorem is also proved by the following experiment: it has been shewn in the preceding Chapter that the earth's attraction on bodies near to its surface is constant; and it will be shewn in Section 3 of the following Chapter that, when bodies move under the action of the constant accelerating force of gravity, the expressed velocities, due to given vertical distances through which the force acts, vary as the square roots of those distances. Suppose two spherical balls $m$ and $m'$ to be suspended from two points in the same horizontal line, and by strings of lengths such that the balls when at rest may just touch, and also have their centres in the same horizontal plane: let these balls be moved in circular arcs; then the velocities acquired by them as they fall from rest to the lowest point vary as the square roots of the versed-sines of the arcs through which they descend; and as the versed-sine varies as the square of the corresponding chord, so the velocities acquired in the descent vary as the chords. If therefore the two balls are raised through arcs, the chords of which are inversely as the masses of the balls, the velocities at the lowest points will also be inversely as the masses; and it is found by experiment that balls which have fallen through arcs, the chords of which are inversely as their masses, and which impinge on each other at the lowest point, are by the collision brought to rest: and therefore the momenta of them must have been equal, and thus being in opposite directions along the same line of action have neutralized each other. This then is an experimental proof that the momentum is equal to the product of the mass and of the velocity. It is also found that, if the arc through which one of the balls moves is greater than that determined above, when the balls come into contact, they are not reduced to rest, but move in the direction of the motion of that which has fallen through the proportionally greater arc.

It is also found by experiment that if two balls of unequal masses are placed in contact, and have a spring so arranged that when the spring is set free it exerts an equal action against both of them, the velocities which are expressed in them are respectively inversely as their masses.

213.] When the matter on which moving-force acts rests on a surface, the normal to which is along the line of action of the moving force, the effect is not *velocity* but *pressure:* for the infinitesimal element of velocity, which the moving force would impress in an infinitesimal element of time, is destroyed by the resistance of the surface. But if the surface were removed it would be expressed in the moving matter, and the elements of velocity being added to each other, a finite velocity would be expressed. When therefore a moving force impresses velocity, and the velocity is expressed, the elements of it are added to each other, and the resultant is the whole expressed velocity: but when the elements of velocity are destroyed as soon as they are communicated, the result is pressure. Hence it follows that two pressures are to each other as the product of their masses and the infinitesimal elements of velocity which would be expressed in them in an infinitesimal element of time if they were free. At this point therefore statics becomes a branch of dynamics, and on this principle, which is called the Principle of Virtual Velocities, the theorems of the latter science are applied to those of the former.

We are hereby supplied with a method, which is in practice most convenient, for determining the mass of terrestrial matter. Observation shews that at the same place all bodies, whatever are their substances, acquire the same velocity in falling in vacuo in the same time. The earth's attraction therefore is an accelerating force which acts independently of the particular kind and quantity of the matter which moves, and is therefore the same for all matter. Consequently the pressures of bodies under the attraction of the earth vary as their masses: these pressures are the *weights* of the bodies, and therefore the weights at the same place vary as the masses of the bodies; and as the balance affords an easy mode of comparing weights, we can hereby deduce the relative proportions of the masses.

It may probably be thought that this method of determining mass is more simple than that explained in Art. 209; and practically for the matter of the earth it is: but there are objections to it, so far as the principles of the pure science of motion are concerned: (1) it experimentally assumes the relation between mass, moving force or its measure momentum-increment, and accelerating force or its measure velocity-increment; and this it is adduced to prove: (2) only terrestrial matter can be compared by it, whereas the principles of the science of motion should be laid in breadth sufficient to include matter of all kinds: (3) M. Poisson writes in the Traité de Mécanique, Art. 62, Ed. $2^{de}$: "Toutefois, on doit avoir un idée préalable de l'égalité et du rapport des masses, indépendamment de la pesanteur, qui n'est qu'une propriété sécondaire des corps, puisqu'elle deviendrait tout-à-fait insensible, sans que les masses eussent changé, en les transportant à une distance suffisamment grande de la terre." Thus M. Poisson thinks that such a mode of determining mass would not be sufficiently general for even terrestrial matter.

# CHAPTER VIII.

## THE RECTILINEAR MOTION OF PARTICLES.

### SECTION 1.—*Direct impact and collision.*

214.] We proceed now to the application of the principles and equations which have been investigated in the preceding Chapter: and we shall begin with the most simple case, that of the direct impact and collision of two material particles. To fix our thoughts, however, I shall consider these particles to be spherical homogeneous balls of finite dimensions, which move so that all the particles describe equal and parallel paths, and the balls have therefore no motion of rotation; the velocities also of the balls will be supposed to be uniform both before and after collision, and the paths along which they move are supposed to be rectilineal; also the line of action of the mutual pressure of the balls during the collision is supposed to pass through their centres; and if this line is that in which the balls are moving the impact is said to be *direct;* but if either of the balls moves in a line not coincident with this line of action the impact is called *oblique*. We shall now investigate the former case: the latter will be considered in Section 1, Chap. X.

Let the masses of the two balls or material particles be $m$ and $m'$; and, to fix our thoughts, let us suppose them to be moving with uniform velocities in the same direction along the straight line OA, fig. 85, say, from left to right: let $v$ and $v'$ be their respective velocities, and let us suppose $v$ to be greater than $v'$, so that $m$ overtakes and impinges on $m'$: the momenta of the two balls are respectively $mv$ and $m'v'$.

Now no matter is perfectly rigid; all is more or less extensible, compressible, and also elastic; see Arts. 148, 152, 153, and the account of elasticity there given. Thus when $m$ impinges on $m'$, a compression of the particles of the two balls at, and about, the point of contact takes place: a change of form of the balls thus takes place, and the molecules of them move one relatively to another: velocity therefore has been impressed on them. The disturbance of the relative positions of the elements of the bodies also brings elastic forces of restitution into action: for the effects of the impact are supposed not to be such that the balls are broken or crushed by them: and the greater the disturbance of the particles is, the greater is this elastic force: now although according to the configuration of the balls which we have imagined, the velocity of *m* is greater than that of *m′*, yet during the collision momentum is being withdrawn from *m* and is transferred (1) to *m′* through the means of all its particles, whereby the velocity of *m* is increased; and (2) to the particles which are disturbed in and about the point of contact: the measure of this latter transferred momentum is the elastic force; this transference of momentum continues until *m* and *m′* move with the same velocity; which circumstance eventually occurs: for so long as the velocity of *m* is greater than that of *m′*, the change of the forms of the balls is increased, whereby the elastic force is also increased; and as this is in a greater proportion than the compressing force, the two balls must ultimately move with the same velocity: at this stage of the process, the compression is, it is to be observed, a maximum.

As soon however as the balls move with the same velocity, there is no mutual pressure between them: there is then no force to counteract the elastic forces which have been brought into action by the compression, and these therefore begin to produce their effects. Now the common velocity of *m* and *m′* at the instant, when the compression is a maximum, is from O towards A; and the effect of the elastic forces in the restitution of the figure is to increase the velocity of *m′* and to diminish that of *m* in that direction. In other words, during the compression, momentum of the balls is changed into elastic moving force: and in the restitution, this elastic force again produces momentum: and in both processes momentum is abstracted from *m* and is given to *m′*.

A question however arises: What relation does the momentum impressed by the elastic force during the restitution of the forms bear to that which was lost during the compression? Here, in our ignorance of the constitution of bodies and of their molecular action, we are obliged to have recourse to experiment; and it is found that in two balls of given substances there is always a certain definite ratio between the momentum spent in producing a certain compression, and that acquired during the restitution; the latter quantity being always less than the former: the ratio is called the measure of the elasticity of the bodies, and is symbolised by $e$; the limiting values of $e$ are 0 and 1: the former being its value for substances perfectly inelastic, and perfectly hard, because if a body is perfectly hard there is no compression, and therefore there is no elastic force: and the latter being the value of $e$ when the bodies are perfectly elastic, and when the momentum impressed during the restitution is equal to that spent in producing the compression. Mr. Hodgkinson has not found in the course of his experiments (see British Association Reports, Vol. III, p. 534) any matter perfectly fulfilling these conditions. Hence the value of $e$ for all known substances is a positive proper fraction. If therefore $\text{P}$ represents the momentum impressed during the compression, $e\text{P}$ is that acquired during the restitution.

215.] Let $m$ and $m'$ be the masses of the two balls, which move in the same direction along the straight line OA, see fig. 85, with uniform velocities $v$ and $v'$: and let us suppose $v$ to be greater than $v'$, so that $m$ overtakes and impinges on $m'$: let $u$ be the common velocity of the two balls when the compression is a maximum: let $\text{P}$ represent the momentum spent in producing the compression, and $e\text{P}$ that acquired in the restitution of the form of the bodies. Let $\text{V}$ and $\text{V}'$ be the velocities of $m$ and $m'$, when the collision ceases; and which are their uniform velocities after the collision has taken place. We shall consider the circumstances of the balls as they are (1) at the instant when collision begins, (2) at the instant when the compression is a maximum, (3) when the collision has ceased. Now

$mv$ = the momentum of $m$ at the beginning of the collision,

$\text{P}$ = the momentum spent in producing compression,

$mu$ = the momentum of $m$ when the compression is a maximum;

therefore, by reason of (12), Art. 210,

$$mv = mu + \text{P}; \tag{1}$$

$m'v'$ = the momentum of $m'$ at the beginning of the collision,

$m'u$ = the momentum of $m'$, when the compression is a maximum;

$$\therefore \quad m'v' = m'u - \text{P}: \tag{2}$$

and at the instant when the collision ceases, we have by a similar process,

$$m\text{V} = mu - e\text{P}, \tag{3}$$

$$m'\text{V}' = m'u + e\text{P}: \tag{4}$$

and therefore adding (1) and (2), and (3) and (4),

$$u = \frac{mv+m'v'}{m+m'} = \frac{m\mathrm{v}+m'\mathrm{v}'}{m+m'}; \qquad (5)$$

$$\therefore \quad mv+m'v' = m\mathrm{v}+m'\mathrm{v}': \qquad (6)$$

therefore the sums of the momenta before and after impact are equal.

From (1) and (5) we have

$$\mathrm{P} = \frac{mm'}{m+m'}(v-v'); \qquad (7)$$

therefore the momentum spent in producing the compression varies as the difference between the velocities before impact.

Substituting in (3) and (4) from (5) and (7), we have

$$\mathrm{v} = \frac{mv+m'v'}{m+m'} - \frac{em'}{m+m'}(v-v'); \qquad (8)$$

$$\mathrm{v}' = \frac{mv+m'v'}{m+m'} + \frac{em}{m+m'}(v-v'); \qquad (9)$$

whence the velocities of the balls after collision are known in terms of their masses, the elasticity, and their velocities before impact.

The momentum which is impressed on $m$ in a direction opposite to that of its motion, by the elastic force $e\mathrm{P}$ during the restitution of the form of the balls, may be such as either wholly to neutralize the velocity of $m$ and thus to bring it to rest, or to impress on it a velocity in the opposite direction. In this latter case, $\mathrm{v}$ will have a negative sign, and we shall have

$$\frac{v}{v'} \text{ greater than } \frac{(1+e)m'}{em'-m}.$$

If $m'$ before impact moves in a direction opposite to that which we have imagined, and so as to meet $m$, $v'$ must be affected with a negative sign in all the preceding formulæ; in which case if $mv = m'v'$, that is, if the momenta of the impinging balls are equal, $u = 0$, and the balls are at rest at the instant when the compression is a maximum; and after the restitution has taken place, $\mathrm{v} = -ev$, and $\mathrm{v}' = ev'$, and thus the balls move in opposite directions.

The product of the mass of a particle (and more generally too of a body which has only motion of translation) and of the square of its velocity, that is, of the numbers expressing these quantities, is called the *Vis Viva* of the mass. The sum of the *vires vivæ* of the balls before collision is $mv^2+m'v'^2$, and after collision $m\mathrm{v}^2+m'\mathrm{v}'^2$: and by (8) and (9) we have

$$m\mathrm{v}^2+m'\mathrm{v}'^2 = mv^2+m'v'^2 - \frac{(1-e^2)mm'}{m+m'}(v-v')^2; \tag{10}$$

and therefore in the case of imperfectly elastic balls, when $e$ is less than unity, vis viva is lost by collision.

Also since the balls after impact move with constant velocities v and v′, they in $t$ units of time severally pass over $\mathrm{v}t$ and $\mathrm{v}'t$ units of distance, see equation (1), Art. 202: and therefore the distance between them $= (\mathrm{v}'-\mathrm{v})t$

$$= e(v-v')t.$$

216.] Let us consider some special cases of the preceding results.

(1) Let the elasticity be perfect: $e = 1$: then

$$\mathrm{v} = v - \frac{2m'}{m+m'}(v-v'), \tag{11}$$

$$\mathrm{v}' = v' + \frac{2m}{m+m'}(v-v'); \tag{12}$$

and also if $m = m'$, $\mathrm{v} = v'$, $\mathrm{v}' = v$; that is, when a perfectly elastic ball impinges on another equal and perfectly elastic ball, each after impact moves with the velocity of the other before impact; if therefore one is at rest before impact, the impinging ball remains at rest after impact, and the other will move with the velocity of the impinging ball. Hence if there is a row of equal and perfectly elastic balls in a straight line; and if the first ball moves in that line with a velocity $v$, and impinges on the second, the first will be brought to rest, and the second will move on with the velocity $v$: similarly will it after impact on the third ball be brought to rest, and the third ball will move with the velocity $v$; and so on through all the balls, until finally the last ball moves with a velocity $v$ and all the others are reduced to rest. Now as this result does not depend on the distances between the balls, it will be true if the balls touch each other; and therefore if there is a row of equal and perfectly elastic balls in a straight line, which touch each other, if one of the extreme balls moves with the velocity $v$, and impinges on the next ball with a velocity $v$ in the direction of the row of balls, the intermediate balls will not be disturbed, and the last will move with the velocity of the impinging ball.

Also if $e = 1$,

$$m\mathrm{v}^2+m'\mathrm{v}'^2 = mv^2+m'v'^2; \tag{13}$$

that is, the sum of the vires vivæ is the same before and after impact. This is a particular instance of a general law of dynamics which will be explained hereafter.

(2) If the bodies are wholly inelastic, $e = 0$; also if they are perfectly hard, so that no change of form is caused by the impact, then no elastic force is brought into action, and $e = 0$. In these cases

$$\mathrm{v} = \mathrm{v}' = \frac{mv + m'v'}{m+m'}; \tag{14}$$

that is, the balls after impact move together, and of course with the same velocity.

(3) If $m'$ is infinitely greater than $m$, and if $v' = 0$, or, which is the same thing, if $m$ impinges on a fixed obstacle, as on a plane, then

$$\mathrm{v} = -ev; \tag{15}$$

that is, the ball rebounds with a velocity which is $e$ times that of impact, and in an opposite direction.

And if the elasticity is perfect, $e = 1$, and

$$\mathrm{v} = -v; \tag{16}$$

that is, the velocity of rebound is equal and opposite to that of impact.

And if $e = 0$, $\mathrm{v} = 0$, and the ball remains in contact with the plane.

217.] The velocity of the centre of gravity is not changed by the alteration which the velocities of the balls undergo by reason of the impact.

Let $x$ and $x'$ be the distances of the centres of $m$ and $m'$ from O, fig. 85, at the time $t$: so that their velocities along OA at that time are $\frac{dx}{dt}$ and $\frac{dx'}{dt}$, and thus

$$v = \frac{dx}{dt}, \qquad v' = \frac{dx'}{dt}.$$

Let $\bar{x}$ be the distance from O of their centre of gravity: then

$$(m+m')\,\bar{x} = mx + m'x';$$

$$\therefore \quad (m+m')\frac{d\bar{x}}{dt} = m\frac{dx}{dt} + m'\frac{dx'}{dt}$$

$$= mv + m'v'; \tag{17}$$

and is therefore equal to the sum of the momenta of the balls: but by (6) the sum of the momenta is the same before and after impact: therefore the velocity of the centre of gravity is also the same before and after impact. The same property is also true of any number of balls directly impinging on each other in a straight row.

218.] Examples illustrative of the preceding equations.

Ex. 1. Determine the velocity of a given ball ($m$) which impinges on another equal ball moving with a given velocity, when the impinging ball remains at rest after the collision.

Here $m' = m$, and $\mathrm{v} = 0$: therefore from (9),

$$v = -\frac{1+e}{1-e} v'.$$

Ex. 2. To determine the mass of a ball $m_2$, which, interposed between $m_1$ and $m_3$ is such that the velocity of $m_3$, which is originally at rest, may after impact from $m_1$ through the intervention of $m_2$ be a maximum.

Let $v$ be the velocity of $m_1$ at first: then

$$\text{the vel. of } m_2 \text{ after impact from } m_1 = \frac{(1+e)m_1 v}{m_1+m_2};$$

$$\therefore \quad \text{the vel. of } m_3 \text{ after impact from } m_2 = \frac{(1+e)^2 m_1 m_2 v}{(m_1+m_2)(m_2+m_3)}$$

$$= f(m_2), \text{ say}:$$

$$\therefore \quad f'(m_2) = \frac{(1+e)^2 m_1 (m_1 m_3 - m_2^2) v}{(m_1+m_2)^2 (m_2+m_3)^2} = 0,$$

if $m_2 = (m_1 m_3)^{\frac{1}{2}}$, and changes sign from + to −: therefore the value of $f(m_2)$ is a maximum, if $m_2$ is a mean proportional between the two extreme balls.

Ex. 3. $n$ balls $m_1, m_2, m_3, \dots m_n$, perfectly elastic, are placed in a row: find the ratio of their masses, when a momentum $m_1 v$ impressed on the first, is after impact equally divided amongst the $n$ balls.

$$\text{vel. of } m_1 \text{ after impact on } m_2 = \frac{m_1-m_2}{m_1+m_2} v; \tag{18}$$

$$\text{vel. of } m_2 \text{ after impact from } m_1 = \frac{2m_1 v}{m_1+m_2};$$

$$\text{vel. of } m_2 \text{ after impact on } m_3 = \frac{m_2-m_3}{m_2+m_3} \frac{2m_1 v}{m_1+m_2}; \tag{19}$$

$$\text{vel. of } m_3 \text{ after impact from } m_2 = \frac{2m_2}{m_2+m_3} \frac{2m_1 v}{m_1+m_2};$$

$$\text{vel. of } m_3 \text{ after impact on } m_4 = \frac{(m_3-m_4)\, 4 m_1 m_2 v}{(m_3+m_4)(m_2+m_3)(m_1+m_2)}; \tag{20}$$

and so on: therefore from (18), (19), and (20),

$$\frac{m_1 v}{n} = \frac{m_1-m_2}{m_1+m_2}\, m_1 v = \frac{2m_1m_2(m_2-m_3)v}{(m_2+m_3)(m_1+m_2)}$$

$$= \frac{4m_1m_2m_3(m_3-m_4)v}{(m_3+m_4)(m_2+m_3)(m_1+m_2)} = \dots$$

$$\therefore \quad \frac{m_1-m_2}{m_1+m_2} = \frac{1}{n}, \quad \frac{m_2-m_3}{m_2+m_3} = \frac{1}{n-1}, \quad \frac{m_3-m_4}{m_3+m_4} = \frac{1}{n-2}, \dots$$

$$\therefore \quad \frac{m_1}{n+1} = \frac{m_2}{n-1}, \quad \frac{m_2}{n} = \frac{m_3}{n-2}, \quad \frac{m_3}{n-1} = \frac{m_4}{n-3}, \dots$$

$$\therefore \quad \frac{m_r}{n-r+2} = \frac{m_{r+1}}{n-r};$$

which gives the ratio of every two successive balls.

219.] The theory of impact and collision which is explained in the preceding Articles may be applied to the solution of the following problem:

By the law of inertia, a material particle or body which has a certain momentum continues to move in a rectilinear path, and with a constant velocity, unless it is acted on by some force; that is, unless momentum is abstracted from it or is communicated to it. Now if a particle or body moves in a vacuum, its velocity is not affected by any action of the medium through which it passes: but if the particle moves in a resisting medium, such as air or water, or in any other medium gaseous or liquid, whose density is finite, the particles of the medium are to be displaced, to allow the particle to pass through the medium; that is, the particles must move, and must therefore have momentum communicated to them; and this will be abstracted from the moving body; hence it loses momentum, the amount of which it is our object now to determine. And a loss of momentum will arise not only from the displacement of the particles which the passage of the body through the medium requires, but also from the action of the particles on each other, and from their friction against the surface of the moving body: the latter causes of loss of momentum, involving data extraneous to the present subject, we cannot now determine; but the former cause, and which is the principal one, we can determine, at least approximately.

Let the moving mass present to the resisting medium a plane face, whose area is $\omega$, and the plane of which is perpendicular to the line of motion of the body: let $m$ be the mass of the moving body, $\rho =$ the density of the resisting medium, and let the plane face $\omega$ be at the time $t$ at a distance $x$ from a fixed point in the line of its motion: let $dx$ be the distance through which $\omega$ moves in the time $dt$, and let $v$ be the velocity of the body: so that $dx = v\,dt$. In the time $dt$, the plane face $\omega$ will have passed over a space equal to $dx$, and will have impinged upon, and communicated momentum to, all the particles of the medium contained within a small cylindrical surface, of which $\omega$ is the base and $dx = v\,dt$ is the altitude; then as $\rho$ is the density, and as the particles move with a velocity $v$, so as to allow the body to pass through, a velocity $v$ will have been communicated to the mass $\rho\omega dx$, that is, to $\rho\omega v\,dt$; and as this moves with a velocity $v$ its momentum is $\rho\omega v^2 dt$; and this has been abstracted from the moving mass; therefore by reason of (12) Art. 210,

$$-m\,dv = \rho\omega v^2 dt;$$

$$\therefore \quad m\frac{dv}{dt} = -\rho\omega v^2: \tag{21}$$

the resistance of the medium therefore will have caused to the moving body a loss of momentum which varies as the density of the medium, as the plane area of the body on which the medium acts, and as the square of the velocity with which the body moves.

If however this result does not exactly correspond with experiment, let it be remembered that we have neglected some circumstances which affect the loss of momentum; and thus the result which is only approximately true may be estimated at its value.

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## SECTION 2.—*Rectilinear motion of particles under the action of uniformly accelerating force.*

220.] It may be observed that the results of this section are true also of the motion of a finite body, if all the particles of the body describe equal and parallel straight lines, and therefore move with the same velocity.

Let $m$ = the mass of the moving particle; and let a point O, fig. 86, in its line of motion be taken as the origin: let P be its position at the time $t$, let OP $= x$, and let PQ $= dx$ be the space-element described in the time $dt$: so that if $v$ is the velocity of $m$ at the time $t$,

$$\frac{dx}{dt} = v, \tag{22}$$

$$dx = v\,dt: \tag{23}$$

let $f$ be the impressed velocity-increment due to, and the measure of, the accelerating (or retarding) force: then $mf$ is the impressed momentum-increment of $m$ in an unit of time.

Let $dv$ be the expressed velocity-increment due to the time $dt$; therefore $m\,dv$ is the expressed momentum-increment due to the same time; and $m\frac{dv}{dt}$ is the expressed momentum-increment due to one unit of time: therefore by reason of (14), Article 211,

$$m\frac{dv}{dt} = mf;$$

$$\therefore \quad f = \frac{dv}{dt} \tag{24}$$

$$= \frac{d}{dt}\cdot\frac{dx}{dt}$$

$$= \frac{d^2x}{dt^2}, \tag{25}$$

if $t$ is the equicrescent variable: $f$ also is to be affected with a positive or negative sign according as from (24) the action of the force makes the velocity increase or decrease as the time increases. To fix our thoughts, let $f$ be positive, therefore

$$\frac{d^2x}{dt^2} = f. \tag{26}$$

Now suppose the circumstances of motion to be such that the velocity of $m = u$, when $t = 0$; then, integrating between limits thus assigned, we have

$$d.\frac{dx}{dt} = f\,dt;$$

$$\therefore \quad \frac{dx}{dt} - u = ft; \tag{27}$$

that is, the *increase* of velocity in the time $t$ is $ft$: $u$ is called the initial velocity.

Again, integrating, and supposing the particle to be at A (OA $= a$), when $t = 0$, we have from (27),

$$dx = u\,dt + ft\,dt,$$

$$\therefore \quad x - a = ut + \frac{ft^2}{2}, \tag{28}$$

$$\therefore \quad x = a + ut + \frac{ft^2}{2}. \tag{29}$$

If $m$ is at the origin, when $t = 0$, $a = 0$, and

$$x = ut + \frac{ft^2}{2}; \tag{30}$$

and also if the particle starts from rest, then $u = 0$, and we have

$$x = \frac{ft^2}{2}; \tag{31}$$

in this last equation $x$ is called the space due to $f$ during the time $t$: and $t$ is called the time to which $x$ is due under the action of $f$.

Again, multiplying both sides of (26) by $2\,dx$, we have

$$d.\frac{dx^2}{dt^2} = 2f\,dx;$$

and supposing that the velocity of the particle is $u$ when $x = 0$, so that $u$ and 0 respectively are the inferior limits of the definite integrals of the sides of the equation, we have

$$\frac{dx^2}{dt^2} - u^2 = 2fx; \tag{32}$$

and if the velocity of the particle $= 0$, when $x = 0$, then $u = 0$, and we have

$$\frac{dx^2}{dt^2} = 2fx;$$

$$\therefore \quad \text{the velocity} = (2fx)^{\frac{1}{2}}. \tag{33}$$

Hence (27) gives the relation between the velocity and the time, (33) that between the velocity and the space, and (30) or (31) that between the space and the time. And it appears that when a particle moves under the action of a finite accelerating force,

($\alpha$) The velocity acquired during a given time varies as the time.

($\beta$) The velocity acquired by the particle during its motion through a certain space varies as the square root of the space.

($\gamma$) The space through which the particle passes varies as the square of the time.

If the force is retarding, $f$ must be affected with a negative sign, and we have

$$\frac{dx}{dt} = \text{velocity} = u - ft, \tag{34}$$

$$\left(\frac{dx}{dt}\right)^2 = u^2 - 2fx, \tag{35}$$

$$x = a + ut - \frac{ft^2}{2}. \tag{36}$$

Also if the initial velocity is in a direction the opposite of that in which the force acts, then $u$ is negative, and the necessary changes must be made in the preceding formulæ.

And if the particle is projected with a velocity $u$ from o in a direction contrary to that in which the accelerating force acts, it comes to rest when $\frac{dx}{dt} = 0$; that is, when

$$t = \frac{u}{f}, \quad \text{and} \quad x = \frac{u^2}{2f}.$$

221.] The product of the numbers expressing the mass and the square of the velocity being called the *vis viva*, (see Article 215), equation (33) gives,

$$\text{vis viva} = mv^2 = 2mfx; \tag{37}$$

that is, the vis viva varies as the distance through which the force has acted on the particle, and has thereby increased its momentum.

For many years, and even after the principles of mechanical science had been correctly explained, a controversy was waged with much acrimony as to the correct measure of a constant accelerating force: the disputants on one side urging that vis viva was the correct measure, and those on the other, that the force's effect was measured by the momentum which it impressed and not by the vis viva. The dispute however is one of *words* only, and if the combatants had at the outset defined what they meant by "effect," the controversy would have been concluded; for let vis viva and let momentum be resolved into their elements, and they are the sums of an infinite number of infinitesimals, and it would have been perceived that the elements are identical, although the resolution of the former would have led to elements corresponding to infinitesimal space-elements, and that of the latter to elements corresponding to infinitesimal time-elements. And thus as to the word "effect:" if the effect of a constant accelerating force is to vary with the time during which it acts, then momentum becomes the measure of that effect: if it is to vary with the space through which it has acted, then vis viva is the measure of the effect.

222.] If a particle moves from rest, the space described in $t$ units of time is given by (31), and we have

$$x = \frac{ft^2}{2}.$$

Let $x_1, x_2, \ldots x_n$ be the spaces described in the first, second, ... $n$th units of time: then we have

$$\left.\begin{array}{rlrl} x_1 &= \frac{f}{2}, & \therefore\quad x_1 &= \frac{f}{2}\times 1 \\ x_1+x_2 &= 4\frac{f}{2}, & x_2 &= \frac{f}{2}\times 3 \\ x_1+x_2+x_3 &= 9\frac{f}{2}, & x_3 &= \frac{f}{2}\times 5 \\ \cdots & & \cdots & \\ x_1+x_2+\ldots+x_n &= n^2\frac{f}{2}, & x_n &= \frac{f}{2}(2n-1) \end{array}\right\}; \qquad (38)$$

that is, the spaces described in the first, second, ... $n$th units of time are as the numbers $1, 3, 5, \ldots (2n-1)$, and increase therefore in an arithmetical ratio, the common difference of which is $f$.

As a sufficient comprehension of these results of uniformly accelerating forces is indispensable for a complete knowledge of subsequent results, let us consider the subject in its more elementary form.

Let $m$ start from o, fig. 86, from rest: and let the time of its motion be resolved into equal infinitesimal elements, each of which we shall represent by $\tau$: and let $x_1, x_2, x_3, \ldots x_n$ be the spaces which it describes in the first, second, ... $n$th time-elements; then since $f$ is the velocity which the accelerating force impresses in an unit of time, the velocities of the particle at the end of the first, second, ... $n$th time-elements will be

$$f\tau, \quad 2f\tau, \quad 3f\tau, \quad \ldots \quad nf\tau.$$

Now imagine each successive space-element to be described in the same time $\tau$, and with an uniform velocity: then if $\theta$ is a symbol for a positive proper fraction, these successive uniform velocities will be

$$\theta_1 f\tau, \quad (f+\theta_2 f)\tau, \quad (2f+\theta_3 f)\tau, \ldots \{(n-1)f+\theta_n f\}\tau;$$

and because the space is equal to the product of the time and the velocity, see (1) Art. 202,

$$\begin{aligned} x_1 &= \theta_1 f\tau^2 \\ x_2 &= (1+\theta_2)f\tau^2 \\ & \cdots \\ x_n &= (n-1+\theta_n)f\tau^2, \end{aligned}$$

$$\begin{aligned} \therefore \text{ the whole space} &= x_1+x_2+\ldots+x_n \\ &= \{1+2+\ldots+(n-1)\}f\tau^2 \\ &\qquad + (\theta_1+\theta_2+\ldots+\theta_n)f\tau^2 \quad (39) \\ &= \frac{n(n-1)}{2}f\tau^2 + (\theta_1+\theta_2+\ldots+\theta_n)f\tau^2. \quad (40) \end{aligned}$$

Let the whole space described by the particle $= x$, and let the whole time $= t$: then $t = n\tau$: and since $\tau$ is an infinitesimal time-element, during which we may without sensible error suppose the velocity to be uniform, $n$ is an infinity of that order of which $\tau$ is an infinitesimal, and we have

$$n = \frac{t}{\tau};$$

therefore from (40),

$$x = \frac{t(t-\tau)}{2} f + f\tau^2 \Sigma . \theta;$$

and omitting the infinitesimals, viz. the terms involving $\tau$ and $\tau^2$,

$$x = \frac{ft^2}{2},$$

the same result as (31).

223.] Some examples are added illustrative of the principles contained in the preceding articles.

Ex. 1. It is required to divide a straight line whose length is $a$ into four parts, such that a particle under the action of a finite accelerating force which acts along the line may, starting from rest, describe each part in an equal time.

Let $x_1, x_2, x_3, x_4$ be the four parts: then, by equations (38),

$$\frac{x_1}{1} = \frac{x_2}{3} = \frac{x_3}{5} = \frac{x_4}{7} = \frac{x_1 + x_2 + x_3 + x_4}{1+3+5+7}$$

$$= \frac{a}{16};$$

$$\therefore \quad x_1 = \frac{a}{16}, \qquad x_2 = \frac{3a}{16}, \qquad x_3 = \frac{5a}{16}, \qquad x_4 = \frac{7a}{16}.$$

Ex. 2. A particle moves in a straight line, under the action of an uniformly accelerating force, and describes spaces $P$ and $Q$ in the $p$th and $q$th units of time respectively; determine the velocity of projection, and the magnitude of the accelerating force.

Let $u =$ the velocity of projection, and let $f$ be the accelerating force; then

The space described in one unit of time on account of the velocity of projection is $u$: and that due to the accelerating force in the $n$th unit of time $= \frac{f}{2}(2n-1)$;

$$\therefore \quad P = u + \frac{f}{2}(2p-1), \qquad Q = u + \frac{f}{2}(2q-1);$$

$$\therefore \quad f = \frac{P-Q}{p-q}, \qquad u = \frac{Q(2p-1) - P(2q-1)}{2(p-q)}.$$

Ex. 3. A particle is projected with a given velocity $u$ in a line along which an accelerating force acts, and in a direction opposite to that of the force's action: and the time is given between its leaving a given point and its return to it: it is required to determine the velocity of projection and the whole time of motion.

Let $u =$ the velocity with which the particle leaves the origin O: and let the time between the particle's passage through A, at a distance $a$ from O, and its return to it be $2\text{T}$: let B be the extreme point which the particle reaches: then, by Art. 220,

$$\text{OB} = \frac{u^2}{2f}, \text{ and time from O to B} = \frac{u}{f};$$

$$\therefore \quad \text{the distance AB} = \frac{u^2}{2f} - a;$$

and the time due to this distance $= \text{T}$: therefore by (31),

$$\frac{u^2}{2f} - a = \frac{f\text{T}^2}{2};$$

$$\therefore \quad u = (2af + f^2\text{T}^2)^{\frac{1}{2}};$$

and therefore the whole time of motion $= \dfrac{2u}{f}$

$$= 2\left(\text{T}^2 + \frac{2a}{f}\right)^{\frac{1}{2}}.$$

## SECTION 3.—*On gravity as an uniformly accelerating force.*

224.] In Chapter VI, on attractions, it is shewn that the attraction, on an external particle $m$, of a sphere consisting of homogeneous concentric shells, the density of each one of which may be different, is the same as if the whole sphere were condensed into its centre, and therefore the attraction of such a sphere on an external particle varies as the square of the distance of the particle from the centre of the sphere. Hence if a particle moves in vacuo towards such a sphere, and under the influence of its attraction, the law of force is that of the inverse square of the distance from the centre of the sphere. But when the attracted particle is nearly on the surface of such a sphere, and moves only over distances which are small in comparison of the radius of the sphere, the variation of the attraction is so small that it may be neglected, and the force which increases its velocity may be considered constant: the same result also follows from the investigation of the attraction of a thin plate on an external particle which is given in Art. 175: and if the attracted particle lies within the surface of the sphere, the law of attraction depends on the densities of those concentric shells of which the sphere is composed, and which are within that surface on which the attracted particle is; for the resultant attraction of all the shells external to that one vanishes: see Art. 181.

Now these results are approximately applicable to the attraction of the earth on particles and on bodies; only *approximately*, I say: because the mean bounding surface of the earth is not a sphere, but an oblate spheroid, of which the equatorial diameter is 7925 miles, and the polar diameter is 7899 miles; and thus the ratio of these diameters is nearly that of the numbers 299 to 298. Now the effect of this oblateness (1) is an increase in the earth's attraction, and thus in weight and in the accelerating force of gravity, on particles at or near to the earth's surface as we pass from the equator to the poles; and the amount of this increase is in weight about the 590th part of the weight of a body at the equator: (2) is a change of the line of action of the earth's attraction. If the earth were a sphere consisting of homogeneous concentric shells, the line of action on a given particle would be the line joining the position of the particle and the earth's centre: as the case now is, the line of action is, by the principle of fluid-equilibrium, perpendicular to the surface of still water at the place: and is therefore along the normal to the spheroid: all these lines of action therefore touch the evolute-surface of the spheroid, but do not pass through the centre. Laplace has calculated the effect of the oblateness of the earth on the motion of the moon; and observation verifies his results.

Again, as the earth rotates about its polar diameter, the centrifugal force, which diminishes the weight of particles and the earth's accelerating force on particles near to the surface, is greatest at the equator, and is zero at the poles: of this cause of diminution and its measure we shall speak hereafter: I may observe, however, that at the equator the weight of a body is diminished by about the 289th of its true weight, and that the effect of centrifugal force in passing from the equator to the poles varies as the square of the cosine of the latitude.

Also the effect of the earth's accelerating force varies by reason of local causes: it is affected by near mountains both as to intensity and as to line of action: it is different on an island which is surrounded by a large mass of water, and on a continent: it even varies, as delicate observations with the time-measuring pendulum shew, with the materials of the earth at the place of observation: thus does the pendulum, as M. Poisson observes, and as we shall shew hereafter, become an indicator of geological conditions.

Gravity also manifestly varies with the altitude of a place above the level of the sea: experiments however by which its value has been determined, are supposed to be made at the level of highwater-mark.

225.] And notwithstanding all these variations of the earth's attraction, for bodies near to the surface the accelerating force due to it is nearly constant at any given place, and increases as we pass from the equator to the pole; and decreases as we remove further from the centre of the earth. The exact measure of it as an accelerating force, that is, the velocity-increment which it impresses on a given particle in one unit of time, for a given place is, of course, to be determined by experiment: now at Greenwich, if one second is the unit of time, at the level of highwater-mark, and in vacuo, the most exact pendulum experiments exhibit a velocity-increment of 386.28 inches, that is, of 32.19 feet. That is, if a particle falls in vacuo towards the earth, the excess of the velocity at the end of any second of time over that at the beginning of the second of time is 32.19 feet.

The velocity-increment is the measure of the accelerating force called gravity; and it is independent of the matter, form, and magnitude of bodies. Thus in the common experiments under the exhausted receiver of an air-pump, the heaviest metals and the lightest pith fall from rest through the same distance in the same time, and acquire equal velocities. And also the time of oscillation of a pendulum is independent of the matter of which the pendulum is made: gravity therefore as an accelerating force is independent of the particular kind of matter which it communicates velocity to.

226.] As to the experimental proof that gravity is an uniformly accelerating force: when a heavy particle or body falls freely by itself in vacuo, the velocity of it quickly becomes so great and increases so rapidly that the law of the increase cannot be observed with accuracy; hence arises the need of some contrivance which may diminish the velocity, and not change the law. There are chiefly two contrivances for this purpose: firstly, Attwood's machine; in which two unequal masses differing slightly from each other in weight, are connected by a very fine, and, as nearly so as may be, flexible and inextensible string: this is suspended over a pulley, see fig. 87, ABC; which, by means of friction-wheels and any other appliance whereby friction may be diminished, moves as easily as possible. Of course the greater mass descends; and as both the masses move with the same velocities, their expressed momentum-increment is the product of the sum of their masses, and their common expressed velocity-increment; and their impressed momentum-increment is that due to the excess of the momentum-increment of the larger mass over that of the smaller; that is, is due to the difference of their weights; see Art. 213. And these momentum-increments are equal, except that some small part of the impressed momentum is spent in producing the velocity of the pulley and of the string, which we at present neglect. Now as the difference of the weights of the two masses may be as small as it is convenient, so may the expressed velocity-increment of the masses be diminished as much as we please, and we are thereby enabled to measure the rate of increase of the velocity, and also the whole velocity which is expressed in a given time: and after very careful and numerous observations it is found that,

(1) The velocity of the descending mass varies as the time during which it has been in motion from rest.

(2) The spaces described by the descending mass vary as the squares of the times during which they are described.

(3) The spaces described in successive units of time vary as the odd numbers $1, 3, 5, \dots (2n-1)$.

And as these results are in accordance with those which have been deduced in Art. 220, when the accelerating force is constant, and as the processes by which these results were proved may be inverted, it follows that the moving force by which, in Attwood's machine, moment-increment is impressed is uniform; and therefore the earth's attraction or gravity is an uniformly accelerating force*.

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* From this Article, and from Art. 212, it appears that two principal results are established by Attwood's machine: (1) the matter of the earth is such, that the expressed momentum-increment is equal to the product of the

Secondly, the oscillating pendulum is a contrivance by which great accuracy is attained, in which the velocity-increment is easily measured, and which is actually employed for the purpose. To the lower end of a fine straight rigid rod a body is attached, the mass of which is so large in comparison of that of the rod, that the mass of the latter may approximately be neglected: the upper end of the rod is so fixed, that the whole rod and body can vibrate freely about an axis passing through it. Now if the rod is moved out of its position of rest, and turns about this axis, the rod and body will vibrate; let the vibrations be small, and let the motion take place wholly in one plane: then it is observed that the oscillations are isochronous, that is, are performed in equal times. In a future Chapter it will be shewn that such isochronism of bodies moving in small circular arcs is true, when the accelerating force is constant; and therefore we infer that the force of gravity under the action of which these isochronous oscillations are performed is a constant accelerating force.

227.] In the following examples of the action of gravity, the time-unit is taken to be one second, the space-unit one foot; and the velocity-increment is supposed to be 32.2 feet (rather greater than its correct value 32.19 feet in the latitude of Greenwich) for facility of calculation, and is symbolized by $g$; and let $m$ be the mass of the moving particle.

We will consider the case (1) of a falling body: (2) of a body projected vertically upwards with a certain velocity: and in both cases I would observe that if the time results at last with a negative sign, it expresses an epoch anterior to that at which we suppose our time to commence.

(1) The motion of a particle $m$ falling towards the earth.

Let a certain point O, ($a$) fig. 88, in the line of the particle's motion be taken as the origin: and let $x = \text{OP}$ be its distance from O at the time $t$: then, if $dx = \text{PQ}$ is the space described in $dt$, that is, in $dt$ units of time,

$$\begin{aligned}\frac{dx}{dt} &= \text{the space described in one unit of time,}\\ &= \text{the velocity at the point P, and at the time } t,\\ &= v \text{ (say);}\end{aligned}$$

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mass and the expressed velocity-increment: (2) gravity is an uniformly accelerating force.

and therefore $m\dfrac{dx}{dt} = mv$ is the momentum of the particle at that time. Hence $\dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ is the expressed velocity-increment in an unit of time, and $m\dfrac{d^2x}{dt^2}$ is the expressed momentum-increment in an unit of time. This last expression is to be equated to $mg$, which is the earth's impressed momentum-increment on $m$ due to a second of time; so that we have

$$m\frac{d^2x}{dt^2} = mg;$$

$$\therefore \quad \frac{d^2x}{dt^2} = g. \tag{41}$$

Now if $u =$ the velocity of $m$, when $t = 0$, the definite integral of (41), the superior and inferior limits on both sides corresponding to $t = t$ and to $t = 0$, is

$$\frac{dx}{dt} - u = gt; \tag{42}$$

$$\therefore \quad \frac{dx}{dt} = \text{the vel. of } m = u+gt; \tag{43}$$

that is, the velocity is equal to the sum of the initial velocity, and of that which gravity has impressed in $t''$; and if the particle is projected upwards from o in a direction contrary to that in which $x$ is measured and $g$ acts, then

$$\frac{dx}{dt} = gt-u. \tag{44}$$

Again integrating (43), and supposing $a$ to be the distance of $m$ from o when $t = 0$, let us take the definite integrals with limits corresponding to $t = t$ and to $t = 0$; and since

$$dx = (u+gt)\,dt; \qquad \therefore \quad x-a = ut+\frac{gt^2}{2};$$

$$x = a+ut+\frac{gt^2}{2}. \tag{45}$$

Also multiplying both sides of (41) by $2dx$, we have

$$d.\frac{dx^2}{dt^2} = 2g\,dx;$$

and taking the same limits of integration as before,

$$\therefore \quad \frac{dx^2}{dt^2} - u^2 = 2g(x-a);$$

$$\therefore \quad \frac{dx^2}{dt^2} = (\text{vel.})^2 = u^2+2g(x-a). \tag{46}$$

If when $t = 0$, $x = 0$ and the particle is at rest, then $a = 0$, and $u = 0$, and

$$\frac{dx}{dt} = gt, \tag{47}$$

$$x = \frac{gt^2}{2}, \tag{48}$$

$$\frac{dx^2}{dt^2} = 2gx. \tag{49}$$

Whence in a particle falling from rest,

(1) the expressed velocity varies as the time,

(2) the expressed velocity varies as the square root of the space,

(3) the space varies as the square of the time.

And generally, (43) gives the velocity in terms of the time: (45) gives the space in terms of the time: and (46) gives the velocity in terms of the space: (45) is of course identical with the equation which would result from the elimination of the velocity by means of (43) and (46).

(2) Suppose the particle $m$ to be projected vertically upwards from O, see ($\beta$), fig. 88; $\text{OP} = x$, $\text{PQ} = dx$; and let $\text{OH} = h$, where H is the highest point which $m$ reaches; and let $u =$ the velocity of projection from O: then since $g$ in this case causes both the velocity and the distance to decrease as $t$ increases,

$$m\frac{d^2x}{dt^2} = -mg;$$

$$\therefore \quad \frac{d^2x}{dt^2} = -g: \tag{50}$$

then taking definite integrals with limits the same as heretofore,

$$\frac{dx}{dt} - u = -gt,$$

$$\frac{dx}{dt} = u - gt, \tag{51}$$

$$x = ut - \frac{gt^2}{2}; \tag{52}$$

also from (50),
$$\frac{dx^2}{dt^2} - u^2 = -2gx,$$

$$\frac{dx^2}{dt^2} = u^2 - 2gx. \tag{53}$$

When $\frac{dx}{dt} = 0$, the particle comes to rest; therefore from (51) and (53),

$$\text{the time, when } m \text{ comes to rest, } = \frac{u}{g}; \tag{54}$$

$$\text{the distance from o to H } = \frac{u^2}{2g}; \tag{55}$$

after $m$ has come to rest at H, it begins to descend: and from the preceding formula it is manifest that the time of the descent from H to O is equal to that of the ascent from O to H; also that the velocity acquired in the descent is equal to that lost in the ascent. These results are also evident from first principles.

228.] Ex. 1. A particle falls from rest; determine its velocity, and the space which it has described at the end of 6″.

$$\frac{d^2x}{dt^2} = g, \qquad \frac{dx}{dt} = gt, \qquad x = \frac{gt^2}{2};$$

$$\therefore \quad \text{the velocity at the end of } 6'' = 6 \times 32.2 \text{ feet,}$$

$$\text{the space at the end of } 6'' = \frac{32.2 \times 36}{2} \text{ feet.}$$

Ex. 2. A particle is projected vertically upwards with a velocity of 100 feet in one second: find the height to which it ascends, and the time of its ascent.

$$\frac{d^2x}{dt^2} = -g,$$

$$\frac{dx}{dt} - u = -gt,$$

$$\frac{dx^2}{dt^2} - u^2 = -2gx;$$

$$\therefore \quad \text{when } \frac{dx}{dt} = 0, \qquad t = \frac{u}{g} = \frac{1000}{322},$$

$$x = \frac{u^2}{2g} = \frac{10000}{64.4}.$$

Ex. 3. A particle is projected upwards with a velocity $u$; find the time which intervenes between its leaving, and returning to, a given point in its path.

Let $a =$ the distance of the given point A from O, the point of projection: then if $t$ is the time from O to A,

$$a = ut - \frac{gt^2}{2};$$

$$\therefore \quad t = \frac{u}{g} \pm \frac{(u^2 - 2ag)^{\frac{1}{2}}}{g}; \tag{56}$$

And the time to the highest point $= \dfrac{u}{g}$.

In (56) the upper sign refers to the passage of the particle through the given point in its descent, and when it has left the highest point; and the lower sign refers to the passage of $m$ through the given point in its first ascent: therefore

$$\text{the intervening time} = \frac{2(u^2-2ag)^{\frac{1}{2}}}{g}.$$

Ex. 4. With what velocity must a particle be projected downwards, that it may in $n''$ overtake another particle which has already fallen through $a$ feet.

Let $u =$ the required velocity: therefore the space which the first particle will pass through in $n''$ is

$$un + \frac{gn^2}{2};$$

and the velocity which the second particle has, when the former starts from rest, is $(2ag)^{\frac{1}{2}}$: therefore at the end of $n''$, its distance from the origin, is $a+(2ag)^{\frac{1}{2}}n+\frac{gn^2}{2}$; which, being equated to the preceding distance, gives

$$u = \frac{a}{n} + (2ag)^{\frac{1}{2}}.$$

Ex. 5. A particle whose elasticity is $e$ falls through a given vertical distance $a$ and strikes a horizontal plane, whence it rebounds, and falls again; and so on continually: find the whole space which it passes through before it comes to rest.

By (49), the velocity of impact in the plane $= (2ag)^{\frac{1}{2}}$;

$\therefore$ by (15), Art. 216, the velocity of rebound $= e(2ag)^{\frac{1}{2}}$:

$\therefore$ the height to which the particle ascends, by (55), $= e^2a$;

similarly after the second impact, the height $= e^4a$; and so on:

$$\begin{aligned}\text{therefore the whole space} &= a+2\{e^2a+e^4a+\dots\}\\ &= a+\frac{2e^2a}{1-e^2}\\ &= a\frac{1+e^2}{1-e^2}.\end{aligned}$$

229.] Let us also investigate, and apply to certain examples, the equations of motion of two given masses connected by a fine inextensible and flexible string, which is suspended over a pulley, as in Attwood's machine, see fig. 87: we shall suppose the pulley and the string to be without inertia, so that no part of the impressed momentum is spent in giving velocity to them.

Let $m$ and $m'$ be the masses respectively at P and Q at the time $t$: of these suppose $m$ to be the greater, so that P descends: let AP $= x$, BQ $= x'$: then, since the string is inextensible,

$$x + x' = \text{a constant};$$

$$\therefore \quad \frac{dx}{dt} + \frac{dx'}{dt} = 0, \tag{57}$$

$$\frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0; \tag{58}$$

whence we conclude that the velocity and the velocity-increments of the two particles are equal, and have opposite signs.

Now the whole mass which receives and developes velocity-increment is $m+m'$, and as $\frac{d^2x}{dt^2}$ is the velocity-increment of the whole of this, the expressed momentum-increment $= (m+m')\frac{d^2x}{dt^2}$; and the impressed momentum-increment is the excess of that of $m$ over that of $m'$; that is, is $mg - m'g$; therefore

$$(m+m')\frac{d^2x}{dt^2} = mg - m'g, \tag{59}$$

$$\frac{d^2x}{dt^2} = \frac{m-m'}{m+m'}g; \tag{60}$$

from which equation all the circumstances of a motion such as we have supposed are to be deduced. The corresponding equation of motion of $m'$ is

$$\frac{d^2x'}{dt^2} = \frac{m'-m}{m'+m}g. \tag{61}$$

As to the circumstances of the initial velocity: suppose $m$ to be projected vertically downwards, so that if it were detached from the string it would have the velocity $a$; and similarly let $m'$ be projected vertically downwards, so that if it were free, it would have the velocity $a'$: and let the velocity with which, by virtue of these two separate velocities, $m$ and $m'$ move when connected by the string, be $u$: then

$$(m+m')u = ma - m'a';$$

$$\therefore \quad u = \frac{ma - m'a'}{m+m'}; \tag{62}$$

which gives the initial velocity with which $m$ begins to descend, if $ma$ is greater than $m'a'$; and with which $m'$ begins to descend if $m'a'$ is greater than $ma$.

Let the initial value of $x$, that is, when $t = 0$, be $a$: therefore from (60),

$$\frac{dx}{dt} - u = \frac{m-m'}{m+m'} gt; \tag{63}$$

$$\frac{dx^2}{dt^2} - u^2 = 2\frac{m-m'}{m+m'} g(x-a); \tag{64}$$

$$x = a + ut + \frac{m-m'}{m+m'} \frac{gt^2}{2}; \tag{65}$$

$u$ being given by (62).

And if $a'$ is the initial value of $x'$, we have

$$x' = a' - ut - \frac{m-m'}{m+m'} \frac{gt^2}{2}. \tag{66}$$

230.] Examples illustrative of the preceding formulæ.

Ex. 1. $m = 16.6$ oz.: $m' = 15.6$ oz., and they start from rest: required the space through which $m$ passes in 5″, and the velocity which it has at the end of the time. From (63) and from (65),

$$\text{vel. of } m = 5 \text{ feet,}$$

$$\text{the space} = x = 12.5 \text{ feet.}$$

Ex. 2. A mass of 10 lbs. is distributed at the ends of a cord passing over a fixed pulley, so that the heavier weight descends through $3g$ feet in 10″: it is required to find the weights at each end of the cord.

Let $w =$ one weight; therefore $10-w =$ the other: then from (65),

$$x = \frac{m-m'}{m+m'} \frac{gt^2}{2}, \qquad \therefore \quad 3g = \frac{2w-10}{10} \frac{g100}{2};$$

$$\therefore \quad w = 5.3, \qquad 10-w = 4.7.$$

Ex. 3. A mass $m$ draws another $m'$ by means of a flexible and inextensible string over a pulley: at the starting of the weights, $m$ is thrown downwards through $a$ feet, and $m'$ through $a'$ feet: it is required to determine the distance through which $m$ descends in $t''$.

Using the notation of equation (62),

$$a = (2ag)^{\frac{1}{2}}, \qquad a' = (2a'g)^{\frac{1}{2}};$$

$$\therefore \quad u = \frac{ma^{\frac{1}{2}} - m'a'^{\frac{1}{2}}}{m+m'} (2g)^{\frac{1}{2}};$$

therefore from (65),

$$x = \frac{ma^{\frac{1}{2}} - m'a'^{\frac{1}{2}}}{m+m'} (2g)^{\frac{1}{2}} t + \frac{m-m'}{m+m'} \frac{gt^2}{2}.$$

Ex. 4. It is required to determine the velocity-increment of the centre of gravity of two masses $m$ and $m'$ which are connected by a string passing over a fixed pulley.

Let $x$ and $x'$ be the vertical distances of $m$ and $m'$ at the time $t$ below the horizontal line passing through the centre of the fixed pulley: and let $\bar{x}$ be the vertical distance from the same line of their centre of gravity. Then

$$(m+m')\bar{x} = mx+m'x',$$

$$\therefore \quad (m+m')\frac{d^2\bar{x}}{dt^2} = m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2};$$

and substituting from (60) and (61),

$$(m+m')\frac{d^2\bar{x}}{dt^2} = m\frac{m-m'}{m+m'}g+m'\frac{m'-m}{m'+m}g,$$

$$\therefore \quad \frac{d^2\bar{x}}{dt^2} = \left(\frac{m-m'}{m+m'}\right)^2 g. \tag{67}$$

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## Section 4.—*Rectilinear motion of particles in vacuo under the action of varying accelerating forces.*

231.] The varying accelerating forces whose effects will be considered in this section are supposed to be explicit functions of the distance between the moving particle ($m$) and the point wherein the force resides, and whence its influence emanates; and the motion of the particle is supposed to be along this line. Thus the force is only implicitly a function of the time: that is, only so far as the passage of the particle through a certain distance requires time, and the distance may thus become a function of that time: and thus our equation of motion will be of the form

$$\frac{d^2x}{dt^2} = f(x), \tag{68}$$

and not of the form

$$\frac{d^2x}{dt^2} = f(t). \tag{69}$$

We limit our considerations to the former form chiefly, because it expresses the laws of communication of velocity which present themselves in the salient phænomena of nature: although in some problems, the latter law will also occur.

The point whence the influence of a force emanates is called the centre of the force; and according as the force attracts or repels, so is it called an *attractive* or a *repulsive* force.

Let us consider briefly the general case. Let $m =$ the mass of the moving particle, and let the centre of the force be the origin O, fig. 86: let P be the position of $m$ at the time $t$: let $\text{OP} = x$: let the force vary as the $n$th power of the distance, and let $\mu$, which is called the absolute* force, be the value of it when $x = 1$ and $m = 1$; so that the impressed momentum-increment is $\mu m x^n$, which is to be affected with a positive or negative sign according as the force is repulsive or attractive. Now the expressed momentum-increment is $m\frac{d^2x}{dt^2}$: therefore by Art. 211, if the force is repulsive,

$$m\frac{d^2x}{dt^2} = m\mu x^n, \tag{70}$$

because both $x$ and the velocity increase as $t$ increases; and thus both $\frac{dx}{dt} = v$ and $\frac{dv}{dt} = \frac{d^2x}{dt^2}$ are positive; and therefore, dividing (70) through by $m$,

$$\frac{d^2x}{dt^2} = \mu x^n. \tag{71}$$

If the force varies inversely as the $n$th power of the distance, and is repulsive, (71) becomes

$$\frac{d^2x}{dt^2} = \frac{\mu}{x^n}; \tag{72}$$

and (71) and (72) must have negative signs if the force is attractive, because in that case $x$ decreases as $t$ increases, when the force makes $m$ move towards its centre.

Instead however of deducing from these general values the circumstances of such rectilinear motion, it will be more convenient to consider the results for particular laws of force: and we shall choose such examples as will either elucidate natural phænomena or will suggest general methods for solving problems in rectilinear motion.

232.] A particle $m$ moves towards a centre of force which attracts directly as the distance: it is required to determine the circumstances of motion.

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* In the preceding investigations on attraction, Chap. VI, I have taken the mass of the attracting body to be the absolute force, so that at an unit of distance and on an unit-mass the attraction is equal to the attracting mass: and thus the attraction of one unit-mass on another unit-mass at an unit-distance apart is made the attraction-unit.

Let O the centre of force be the origin; and let P, fig. 89, be the position of $m$ at the time $t$: let $\text{OP} = x$ and $\text{OA} = a$, where A is the position of the particle when $t = 0$: let $\mu =$ the absolute force: then the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x, \tag{73}$$

$$\cdot \quad \frac{2\,dx\,d^2x}{dt^2} = -2\mu x\,dx;$$

and if the limits of integration are those values which correspond to $t = t$ and to $t = 0$, then if the particle is at rest, when $x = a$ and $t = 0$,

$$\begin{aligned} \frac{dx^2}{dt^2} &= -\mu(x^2 - a^2) \\ &= \mu(a^2 - x^2); \end{aligned} \tag{74}$$

$$\therefore \quad \frac{-dx}{(a^2 - x^2)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\,dt;$$

the negative sign of the root being taken, because, according to our configuration, $x$ decreases as $t$ increases: therefore integrating between the limits corresponding to $t = t$ and to $t = 0$,

$$\cos^{-1}\frac{x}{a} = \mu^{\frac{1}{2}} t, \tag{75}$$

$$\therefore \quad x = a\cos\mu^{\frac{1}{2}} t. \tag{76}$$

From (74) it appears that the velocity of the particle is zero when $x = a$, and when $x = -a$; and is a maximum, viz. $a\mu^{\frac{1}{2}}$, when $x = 0$; the particle therefore moves from rest at A; its velocity increases until it reaches O, where it becomes a maximum, and where the force is zero: so that the particle passes through that point, and its velocity decreases, and at A′, at a distance $= -a$, becomes zero: whence the particle under the action of the force returns, and continually oscillates over the distance $2a$, of which O is the middle point. The distance $a$ of A from O is called the *amplitude of the vibration.*

Also from (76) it appears that when $x = 0$, $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$, and when $x = a$, $t = 0$; so that the time of passing from A to $\text{O} = \frac{\pi}{2\mu^{\frac{1}{2}}}$; and the time from O to A′ is equal, so that the time of the oscillation from A to A′ is $\frac{\pi}{\mu^{\frac{1}{2}}}$. This result may also be more generally inferred by the following method. The relation between $x$ and $t$ is given by the periodic function (76), viz. $x = a\cos\mu^{\frac{1}{2}} t$. Now as the greatest value of a cosine is $+1$, and the least value is $-1$, the greatest and least values of $x$ are $+a$ and $-a$: $x$ also will have passed once through all its values when $\mu^{\frac{1}{2}}t$ is increased by $2\pi$; that is, when $t$ is increased by $\frac{2\pi}{\mu^{\frac{1}{2}}}$; this therefore is the time of a complete double oscillation; and therefore the time of one oscillation, viz. from A to A′, is $\frac{\pi}{\mu^{\frac{1}{2}}}$.

And hence we have the remarkable fact that the time of an oscillation is independent of the distance from the centre of the point from which the particle began to move, and only depends on the absolute force, and is the greater the less that is.

233.] The two following cases in nature, wherein an attraction the law of which is that of the direct distance presents itself, deserve mention.

(1) A homogeneous sphere attracts a particle within its bounding surface with a force varying directly as the distance from the centre of the sphere; see Ex. 5, Art. 176. Let us therefore consider the earth to be such a homogeneous sphere, and let us suppose a particle to move under the action of the earth's attraction within the shaft of a mine the direction of which is vertical. Thus if C is the centre of the earth, fig. 90, and P is the position of $m$ at the time $t$, the force acting on $m$ varies as CP; and thus if the shaft were continued straight through the earth, such as that represented by A′CPA in the figure, and if the particle were free at A, it would move to C, where its velocity would be a maximum, and thence on to A′ on the opposite side, where it would come to rest: and thence it would return through C to A again; and its motion would continue to be oscillatory, and the time of the oscillation would be independent of CA, the earth's radius.

(2) In the undulatory or wave theory of light, all space is supposed to be pervaded in a greater or less degree by the particles of a fluid excessively elastic and jelly-like; in the motion of these particles light is supposed to consist, and when they are at rest, there is darkness. It is also supposed that these particles exercise mutual attractions on each other: that the possible relative displacements of them are very small, and that when displacements occur elastic forces are brought into action, by virtue of which, in conjunction with their mutual attractions,

the motion of them continues: the lines of action, as well as the intensities of such elastic forces, of course vary from one medium to another; and in this variety consists the optical character of the medium. Doubtless the arrangement of the particles of a crystallised substance is different to that of one which is non-crystallised. The sun, or the flame of a candle, or the electric spark, are, together with many others, exciting causes of the motion of the particles of ether; and the displacement of each particle is very small. It seems, too, that the force which acts on a particle in its displaced position varies directly as the distance of it from its original position of rest; this force being the resultant of the elastic forces which arise from the disturbance of the medium and of the attracting forces of the particles. Now a ray of light consists in the motion of a series of ethereal molecules which when at rest are in a straight line emanating from the source of motion. The mode of propagation of the motion of the particles it is not my purpose now to inquire into: I shall consider the motion of only a single molecule of a single ray. The displacement of a molecule may be in any direction with reference to the line of propagation of the ray: it might be along that line, or it might, after its first displacement, describe any curve with reference to that line; it is, however, in the theory of light supposed, and not without evidence, that the motion of the molecule takes place in a plane which is perpendicular to the line of propagation of the ray: that is, the displacement of the particle is transversal to the line of propagation. Generally the force acting on the molecule, varying directly as the distance, will have its line of action inclined to the line joining the displaced and the original position of the molecule; and, as we shall shew hereafter, the molecule will move in an ellipse, the centre of which is the original place of rest of the molecule: but in particular constitutions of the ethereal medium, the line of action of the force may be that joining the original and the displaced positions of the molecule: in which case the molecule moves along that line, and is under the action of a force varying directly as the distance from its original position of rest: we have then the case of a particle under the action of a force such as we have supposed that in Art. 232 to be, and the results of that Article are applicable. The particle therefore has an oscillatory motion, and the amplitudes of its vibrations are equal on both sides of its original positions; and the time of the oscillation is independent of the amplitude, and depends only on the absolute force at the centre. Now the intensity of light is supposed to depend on the amplitude of the vibration, and the colour of it on the time of vibration, that is, in the value of $\mu$: it follows therefore that, with such incidents of motion as we have imagined, the intensity and the colour may vary independently of each other: the former will depend on the original exciting cause of the motion; the latter on the nature of the medium: and this independence of these properties of light is amply verified by experiment.

If the motion of all the molecules of a ray is in straight lines, and is such as that described above, and if all the lines of motion are parallel to each other, the ray is said to be *plane-polarised;* and as a beam of light consists of an infinite number of rays, if the molecules of all the rays move in lines parallel to each other, the beam is said to be *plane-polarised.* And although there has not been uniformity on the subject, yet the plane, perpendicular to which the motion takes place, is correctly called *the plane of polarisation.*

234.] If in Art. 232 $m$ is projected from A with a velocity $u$ along the line, and in the direction, OA, then (74) becomes

$$\frac{dx^2}{dt^2} - u^2 = \mu(a^2 - x^2); \tag{77}$$

$$\therefore \quad \frac{-dx}{\left(a^2 + \frac{u^2}{\mu} - x^2\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\, dt;$$

and taking the definite integrals with limits corresponding to $t = t$ and to $t = 0$, we have

$$\cos^{-1}\frac{x}{\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}}} - \cos^{-1}\frac{a}{\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\, t; \tag{78}$$

$$\therefore \quad x = a\cos\mu^{\frac{1}{2}}t - \frac{u}{\mu^{\frac{1}{2}}}\sin\mu^{\frac{1}{2}}t. \tag{79}$$

From (77) it appears that the greatest and least distances of $m$ from O are

$$\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}}, \quad \text{and} \quad -\left(a^2 + \frac{u^2}{\mu}\right)^{\frac{1}{2}};$$

and from (79) the time of an oscillation is, as before, $\frac{\pi}{\mu^{\frac{1}{2}}}$.

235.] A particle $m$ moves towards a centre of force which attracts inversely as the square of the distance; it is required to determine the circumstances of motion.

Let the centre of force be the origin; and let P, fig. 91, be the position of $m$ at the time $t$; let A be the position of $m$ at rest, when $t=0$, so that the particle is moving towards O: let $\text{OP}=x$, $\text{OA}=a$; let $\mu=$ the absolute force: and let the limits of the definite integrals correspond to $t=t$ and to $t=0$. Then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}, \tag{80}$$

$$\frac{2\,dx\,d^2x}{dt^2} = -\frac{2\mu\,dx}{x^2},$$

$$\frac{dx^2}{dt^2} = \frac{2\mu}{x} - \frac{2\mu}{a} \tag{81}$$

$$= \frac{2\mu(ax-x^2)}{ax^2}; \tag{82}$$

$$\therefore\quad \frac{-x\,dx}{(ax-x^2)^{\frac{1}{2}}} = \left(\frac{2\mu}{a}\right)^{\frac{1}{2}} dt,$$

the negative sign being taken, because $x$ decreases as the time increases, according to the arrangement of our figure. Therefore integrating again, and taking the limits corresponding to $t=t$ and to $t=0$, we have

$$(ax-x^2)^{\frac{1}{2}} - \frac{a}{2}\,\text{versin}^{-1}\frac{2x}{a} + \frac{\pi a}{2} = \left(\frac{2\mu}{a}\right)^{\frac{1}{2}} t; \tag{83}$$

$$\therefore\quad t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{(ax-x^2)^{\frac{1}{2}} + a\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right\}. \tag{84}$$

From (81) it appears that the velocity $=0$, when $x=a$; and $=\infty$, when $x=0$: thus the velocity increases as the particle approaches the centre of force, and ultimately, when it arrives at the centre, becomes infinite; and from (84) it appears that the time of passing from A to O is $\dfrac{\pi a^{\frac{3}{2}}}{(8\mu)^{\frac{1}{2}}}$.

If $m$ moves from an infinite distance towards O, then $a=\infty$, and the velocity at a distance $x$ from O $= \left(\dfrac{2\mu}{x}\right)^{\frac{1}{2}}$.

If $m$ is projected from A with a velocity $u$, then (81) becomes

$$\frac{dx^2}{dt^2} - u^2 = \frac{2\mu}{x} - \frac{2\mu}{a},$$

and the process of integration is the same as the preceding.

This problem is that of a particle moving in vacuo from a given place above the surface of the earth towards the earth's centre, the distance through which it moves being so great that the variation of the earth's attraction due to the distance must be taken account of. If however a particle falls towards the earth, and also passes from above to below its surface, as, for instance, down a mine, the law of force changes at the surface: and having varied inversely as the square of the distance, then varies (approximately) directly as the distance.

236.] Again, let the force vary inversely as the square root of the distance and be attractive; and suppose the particle to be at rest at a distance $a$ from the centre of force; it is required to determine the circumstances of motion.

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^{\frac{1}{2}}},$$

$$\frac{dx^2}{dt^2} = 4\mu\,(a^{\frac{1}{2}} - x^{\frac{1}{2}}), \tag{85}$$

$$\frac{-dx}{(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}}} = 2\mu^{\frac{1}{2}}\,dt,$$

$$\frac{4}{3}(x^{\frac{1}{2}} + 2a^{\frac{1}{2}})\,(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}} = 2\mu^{\frac{1}{2}}t;$$

$$\therefore\quad t = \frac{2}{3\mu^{\frac{1}{2}}}(x^{\frac{1}{2}} + 2a^{\frac{1}{2}})\,(a^{\frac{1}{2}} - x^{\frac{1}{2}})^{\frac{1}{2}}; \tag{86}$$

and thus the velocity at any point of the path, and time occupied in arriving at that point are known: and when the particle arrives at the centre, $x = 0$, and

$$\text{the velocity} = 2\mu^{\frac{1}{2}}a^{\frac{1}{4}},$$

$$\text{the time} = \frac{4a^{\frac{3}{4}}}{3\mu^{\frac{1}{2}}}.$$

237.] A particle $m$ is placed at rest at a given point in the line joining two centres of force, of which the absolute forces are equal, and which vary directly as the distance: it is required to determine the circumstances of motion.

Let A and A′, fig. 92, be the centres of the forces, at a distance $2a$ apart: let O, the middle point of AA′, be the origin: let $\mu$ be the absolute force of each centre: let B be the position of $m$ at rest, P its position at the time $t$: $\text{OA} = \text{OA}' = a$: $\text{OB} = b$: $\text{OP} = x$. Then the equation of motion is

$$m\frac{d^2x}{dt^2} = m\mu\,\text{AP} - m\mu\,\text{A}'\text{P};$$

$$\frac{d^2x}{dt^2} = \mu(a-x)-\mu(a+x)$$
$$= -2\mu x;$$
$$\therefore \quad \frac{dx^2}{dt^2} = 2\mu(b^2-x^2), \tag{87}$$
$$x = b\cos(2\mu)^{\frac{1}{2}}t. \tag{88}$$

From (87) it appears that the velocity of the particle is zero when $x = \pm b$: the particle therefore moves from rest at B, and comes to rest again at a point B′ on the opposite side of O, and at a distance from it equal to that of B: also the velocity is a maximum at O: and the particle returns from B′ to O, and again to B, and thus oscillates continually: and from (88) it appears that the time of an oscillation is $\frac{\pi}{(2\mu)^{\frac{1}{2}}}$.

Also again suppose two particles $m$ and $m'$, which attract each other directly as their masses, and as the distance between them, to be placed at rest at two given points, and then to be left to their mutual action: it is required to determine their positions at a given time, and the other circumstances of their motion.

Let $a$ and $a'$ be the distances of $m$ and $m'$ from a certain point O, fig. 93, in the line joining them, when they are at rest, and when $t=0$: and let $x$ and $x'$ be the distances of them from the same point O, when $t=t$: let $OA = a$, $OA' = a'$: $OP = x$, $OP' = x'$: then the equations of motion of $m$ and $m'$ respectively are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= m'(x'-x) \\ \frac{d^2x'}{dt^2} &= m(x-x') \end{aligned}\right\}; \tag{89}$$

$$\therefore \quad m\frac{d^2x}{dt^2}+m'\frac{d^2x'}{dt^2} = 0;$$

$$m\frac{dx}{dt}+m'\frac{dx'}{dt} = 0, \tag{90}$$

the initial values of $\frac{dx}{dt}$ and of $\frac{dx'}{dt}$ being simultaneously zero; and taking the definite integral, with limits corresponding to $t=t$ and to $t=0$, we have

$$m(x-a)+m'(x'-a') = 0. \tag{91}$$

If $\bar{x}$ refers to the centre of gravity of $m$ and $m'$, then

$$(m+m')\bar{x} = mx+m'x'$$
$$= ma+m'a'; \qquad (92)$$

and therefore the centre of gravity remains at rest. Again from (89),

$$\frac{d^2x'}{dt^2} - \frac{d^2x}{dt^2} = -(m'+m)(x'-x).$$

Let $x'-x = z$: and let $m'+m = \mu$; (93)

$$\frac{d^2z}{dt^2} = -\mu z, \qquad \therefore \quad \frac{dz^2}{dt^2} = \mu\{(a'-a)^2 - z^2\};$$

because when $t=0$, $\frac{dz}{dt} = \frac{dx'}{dt} - \frac{dx}{dt} = 0$: therefore

$$\frac{-dz}{\{(a'-a)^2-z^2\}^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\,dt;$$

$$\therefore \quad \cos^{-1}\frac{z}{a'-a} = \mu^{\frac{1}{2}}t,$$

because when $t=0$, $z = a'-a$. Therefore substituting

$$x'-x = (a'-a)\cos(m'+m)^{\frac{1}{2}}t. \qquad (94)$$

Therefore from (91), (92), and (94),

$$x' = \bar{x} + \frac{m(a'-a)}{m+m'}\cos(m'+m)^{\frac{1}{2}}t;$$

$$x = \bar{x} - \frac{m'(a'-a)}{m+m'}\cos(m'+m)^{\frac{1}{2}}t.$$

238.] A particle $m$ is placed at rest at a certain point in the line joining the centres of two forces, which vary inversely as the square of the distance: it is required to determine the circumstances of motion of $m$.

Let A and A′ be the centres of force, fig. 94, of which let the absolute forces be $\mu$ and $\mu'$: let the point O, which is the neutral point of attraction between them, be the origin; OA $= a$, OA′ $= a'$; let AA′ $= c$: then

$$\frac{\mu}{a^2} = \frac{\mu'}{a'^2};$$

$$\therefore \quad \frac{\mu^{\frac{1}{2}}}{a} = \frac{\mu'^{\frac{1}{2}}}{a'} = \frac{\mu^{\frac{1}{2}}+\mu'^{\frac{1}{2}}}{c}; \qquad (95)$$

whereby $a$ and $a'$ are known. Let B and P be respectively the places of $m$ when $t=0$, and when $t=t$: let OB $= b$, OP $= x$: then the equation of motion is

$$\frac{d^2x}{dt^2} = \frac{\mu}{(a-x)^2} - \frac{\mu'}{(a'+x)^2}; \qquad (96)$$

therefore multiplying by $2dx$ and integrating, and taking the limits corresponding to $t=t$ and to $t=0$,

$$\frac{dx^2}{dt^2}=\frac{2\mu}{a-x}+\frac{2\mu'}{a'+x}-\frac{2\mu}{a-b}-\frac{2\mu'}{a'+b}; \tag{97}$$

which equation involves an elliptic transcendent, and does not generally admit of further integration.

Suppose however the circumstances to be such that the particle is projected from B with a velocity $u$, and comes to rest at O: then from (96) by integration we have generally

$$\frac{dx^2}{dt^2}-u^2=\frac{2\mu}{a-x}+\frac{2\mu'}{a'+x}-\frac{2\mu}{a-b}-\frac{2\mu'}{a'+b};$$

and since $\frac{dx}{dt}=0$, when $x=0$,

$$u^2=\frac{2\mu b}{a(a-b)}-\frac{2\mu' b}{a'(a'+b)}. \tag{98}$$

If the velocity of projection is less than that thus determined, $m$ will not reach O, but will come to rest at some point short of it, and will, as the force at A is greater than that at A′, return to A: similarly if the velocity of projection is greater, the particle will pass beyond O, and will eventually fall into A′.

Now if A and A′ are the centres of two spheres, each of which is composed of concentric homogeneous shells, every particle of which attracts with a force varying directly as the mass, and inversely as the square of the distance, then each sphere will attract an external particle with a force which varies directly as its mass, and inversely as the square of the distance of the particle from its centre; see Art. 179. Now suppose A′ and A to be the centres of the earth and the moon, which are assumed to be spheres, and to be at rest: and suppose P to be the position at the time $t$ of a particle in the line joining their centres, and acted on by their attractions; then we have the circumstances of the preceding problem: and since the mass of the earth is about 75 times that of the moon*, as determined by tidal observation, and by the phænomenon of nutation, we have

$$\mu'=75\,\mu;$$

and also since the mean distance of the moon's centre from that of the earth is about 60 (actually 59.9643) of the earth's equatorial radii, or about 237000 miles, we have $a+a'=237000$ miles: therefore

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* See Herschel's Outlines of Astronomy, Art. 759, 4th edit., 1851.

$$a' = 212466 \text{ miles}, \qquad a = 24534 \text{ miles}; \tag{99}$$

such are the distances from the centres of the earth and the moon of the neutral point of attraction of the two bodies.

Suppose now a particle to be projected from the surface of the moon towards the earth, and with such a velocity as just to arrive at the neutral point, and to remain at rest there. Then since the mean radius of the moon is 1080 miles,

$$b = (24534-1080) \text{ miles} = 23454 \text{ miles}: \text{ and therefore}$$

$$a-b = 1080 \text{ miles}, \qquad a'+b = 235920 \text{ miles}.$$

Also if $g$ is gravity at the earth's surface, $g = \frac{32.2}{1760 \times 3}$ miles; and if $r$ is the earth's mean radius, $r = 4000$ miles,

$$g = \frac{\mu'}{r^2}; \qquad \therefore \quad \mu' = (4000)^2 \frac{32.2}{1760 \times 3};$$

therefore substituting in (98), and reducing, we have ultimately,

$$u = 7852 \text{ feet in } 1'';$$

and therefore if the moon were not moving, and if there is no atmosphere, so that the projectile does not meet with a resisting medium, a particle thrown from the surface with a velocity greater than 7852 feet in 1″ towards the earth, will pass beyond the neutral point of attraction, and will move towards the earth.

239.] A centre of force, whose law of variation is that of the direct distance moves with an uniform velocity along a straight line, and attracts a particle, which is projected with a given velocity from a given point in the line of motion of the centre of the force and along that line: it is required to determine the circumstances of motion of the particle.

Let O, fig. 95, the position of the centre of force when $t = 0$, be taken as the origin; let $a =$ the constant velocity with which the centre of force moves along OA, and let C be its position at the time $t$, so that $\text{OC} = at$: let A and P be respectively the positions of $m$ when $t = 0$, and when $t = t$: $\text{OA} = a$, $\text{OP} = x$: and let $m$ be projected from A along the line, and in the direction, AP with a velocity $u$. The equation of motion is

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\mu\,\text{CP} \\ &= -\mu(x-at); \end{aligned} \tag{100}$$

$$\therefore \quad \frac{d^2x}{dt^2} + \mu x = \mu a t.$$

Therefore integrating by the method of Art. 350, Vol. II,

$$x = \left(\frac{d^2}{dt^2}+\mu\right)^{-1}\mu a t$$

$$= at + c_1 \sin \mu^{\frac{1}{2}} t + c_2 \cos \mu^{\frac{1}{2}} t; \tag{101}$$

where $c_1$ and $c_2$ are arbitrary constants introduced in integration: and which are determined by the following conditions: when $t=0$, $x=a$, therefore $c_2 = a$; also when $t=0$, $\frac{dx}{dt} = u$; therefore $c_1 = \frac{u-a}{\mu^{\frac{1}{2}}}$: therefore

$$x = at + \frac{u-a}{\mu^{\frac{1}{2}}} \sin \mu^{\frac{1}{2}} t + a \cos \mu^{\frac{1}{2}} t; \tag{102}$$

whence it appears that the *mean* value of $x$ varies directly as the time: that the particle is sometimes before and sometimes behind the centre of force; and that while it has on the whole a progressive motion, it oscillates from one side to the other of the centre of force; and that the period of an oscillation is $\frac{\pi}{\mu^{\frac{1}{2}}}$.

(100) also might have been integrated by the following process.

Let $\qquad x - at = z;$

$$\therefore \quad \frac{dx}{dt} - a = \frac{dz}{dt}; \qquad \frac{d^2x}{dt^2} = \frac{d^2z}{dt^2}:$$

therefore (100) becomes

$$\frac{d^2z}{dt^2} = -\mu z;$$

whence integrating, and taking the limits corresponding to $t=t$, and to $t=0$, we have

$$\frac{dz^2}{dt^2} - (u-a)^2 = -\mu(z^2-a^2);$$

and integrating again, and taking the same limits, we find the equation (102).

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### SECTION 5.—*Rectilinear motion of particles in resisting media.*

240.] A particle is projected from a given point with a given velocity, in a medium of which the density is constant, and of which the resistance varies as the square of the velocity, and where no other force acts on the particle: it is required to determine the circumstances of motion.

Let the point from which the particle is projected be taken as the origin: and the line in which it is projected as the axis of $x$: let $u$ be the velocity of projection, and let $x$ be the distance of the particle from the origin at the time $t$: let the particle be of the mass $m$, and be of such a form as to present a plane surface $\omega$ to the medium in the direction in which it moves: then by (21), Art. 219, the equation of motion is

$$m\frac{d^2x}{dt^2} = -\rho\omega\left(\frac{dx}{dt}\right)^2;$$

and as $\rho$ is constant, let $\rho\omega = mk$;

$$\therefore \quad \frac{d^2x}{dt^2} = -k\left(\frac{dx}{dt}\right)^2; \tag{103}$$

$k$ is commonly called the coefficient of resistance. Now putting (103) in the following form,

$$\frac{d.\frac{dx}{dt}}{\frac{dx}{dt}} = -k\,dx,$$

and taking the definite integral at limits corresponding to $t = t$ and to $t = 0$; we have

$$\log\frac{dx}{dt} - \log u = -kx;$$

$$\therefore \quad \frac{dx}{dt} = ue^{-kx}; \tag{104}$$

$$e^{kx}dx = u\,dt,$$

$$e^{kx}-1 = kut. \tag{105}$$

(104) gives the relation between the velocity and the distance through which the particle has passed: and (105) gives the relation between the distance and the time. From (104) it appears that $\frac{dx}{dt} = 0$, or that the particle comes to rest, when $x = \infty$: in which case also $t = \infty$, as appears by (105).

241.] A heavy particle $m$ acted on by gravity (a constant accelerating force) moves in the air, which is supposed to be a resisting medium, whose density is uniform, and the resistance of which (according to Art. 219) varies as the square of the velocity: it is required to determine the circumstances of motion.

Firstly, let us suppose the particle to descend towards the earth and to start from rest: then if $\rho$ is the constant density of the air, and if $\omega$ is the area of the face which $m$ presents to the medium transversal to the direction of its line of motion,

$$m\frac{d^2x}{dt^2} = mg - \rho\omega\left(\frac{dx}{dt}\right)^2, \tag{106}$$

affecting the resistance with a negative sign, because it tends to diminish the velocity:

$$\therefore\quad \frac{d^2x}{dt^2} = g - \frac{\rho\omega}{m}\left(\frac{dx}{dt}\right)^2.$$

Let $\frac{\rho\omega}{m} = k$, the coefficient of resistance: so that we have

$$\frac{d^2x}{dt^2} = g - k\left(\frac{dx}{dt}\right)^2. \tag{107}$$

$$\therefore\quad \frac{d.\frac{dx}{dt}}{\frac{g}{k} - \left(\frac{dx}{dt}\right)^2} = k\,dt; \tag{108}$$

whence integrating, and taking the definite integrals corresponding to $t = t$ and to $t = 0$, we have

$$\frac{k^{\frac{1}{2}}}{2g^{\frac{1}{2}}}\log\frac{g^{\frac{1}{2}} + k^{\frac{1}{2}}\frac{dx}{dt}}{g^{\frac{1}{2}} - k^{\frac{1}{2}}\frac{dx}{dt}} = kt;$$

$$\therefore\quad \frac{dx}{dt} = \left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{e^{2(kg)^{\frac{1}{2}}t} - 1}{e^{2(kg)^{\frac{1}{2}}t} + 1}. \tag{109}$$

Also again from (107),

$$\frac{d.\left(\frac{dx}{dt}\right)^2}{\frac{g}{k} - \left(\frac{dx}{dt}\right)^2} = 2k\,dx;$$

therefore integrating, and taking the limits as before,

$$\log\frac{g - k\left(\frac{dx}{dt}\right)^2}{g} = -2kx;$$

$$\frac{dx^2}{dt^2} = \frac{g}{k}\{1 - e^{-2kx}\}. \tag{110}$$

(109) gives the velocity in terms of the time, and (110) in terms of the distance through which $m$ has passed. Also from (109),

$$k\,dx = (kg)^{\frac{1}{2}}\frac{e^{(kg)^{\frac{1}{2}}t} - e^{-(kg)^{\frac{1}{2}}t}}{e^{(kg)^{\frac{1}{2}}t} + e^{-(kg)^{\frac{1}{2}}t}}\,dt;$$

therefore integrating, and taking the same limits as before,

$$kx = \log\{e^{(kg)^{\frac{1}{2}}t} + e^{-(kg)^{\frac{1}{2}}t}\} - \log 2\,;$$

$$\therefore \quad 2e^{kx} = e^{(kg)^{\frac{1}{2}}t} + e^{-(kg)^{\frac{1}{2}}t}; \tag{111}$$

which gives the relation between the distance and the time to which it is due; and which equation might have been found by eliminating the velocity between (109) and (110).

When $t = \infty$, $x = \infty$; that is, an infinite time is required for an infinite space: but when $x = \infty$, and $t = \infty$, the velocity $= \left(\frac{g}{k}\right)^{\frac{1}{2}}$, that is, becomes uniform; in which case, as appears from (107), $\frac{d^2x}{dt^2} = 0$; and although this state is never attained to, yet it is that to which the circumstances of motion approach; also this limiting velocity is greater, the less $k$ is; but $k$ varies directly as the density of the medium, directly as the surface which the particle presents, and inversely as the mass of the particle: therefore the terminal velocity is greater, the greater the mass of the particle is, and the less the density of the medium is, and the less the area of the face is which the particle presents to it in its motion. These results are in accordance with experience. From the form of (107) it appears that the equation is satisfied if

$$\frac{dx^2}{dt^2} = \frac{g}{k}, \tag{112}$$

because in that case $\frac{d^2x}{dt^2} = 0$: this therefore is a solution of the equation: and is a singular one, because it does not arise by giving any particular values to the arbitrary constants, to which the limits of the integrals are equivalent, and which are therefore dependent on the initial circumstances of motion. It appears therefore that the general integral represents the circumstances until the velocity attains its constant value; and that then the singular solution represents the motion. Other and similar peculiar properties of singular solutions will be exhibited hereafter.

242.] Secondly let us suppose $m$ to be projected upwards (that is, in a direction contrary to that of the action of gravity) with a given velocity $u$; it is required to determine the circumstances of motion.

Let us moreover suppose $m$ to be of such a form, that it presents to the medium an equal area transversal to the line of motion, whether it falls, or whether it moves upwards: then if $x$ is measured upwards, gravity and the resistance of the medium both tend to diminish the velocity as $t$ increases: so that the equation of motion is

$$m\frac{d^2x}{dt^2} = -mg - \rho\omega\left(\frac{dx}{dt}\right)^2;$$

and if $\rho\omega = mk$, we have

$$\frac{d^2x}{dt^2} = -g - k\left(\frac{dx}{dt}\right)^2; \tag{113}$$

$$\therefore \quad \frac{d.\frac{dx}{dt}}{\frac{g}{k}+\left(\frac{dx}{dt}\right)^2} = -k\,dt;$$

therefore integrating, and taking the limits which correspond to $t = t$ and to $t = 0$,

$$\left(\frac{k}{g}\right)^{\frac{1}{2}}\left\{\tan^{-1}\left(\frac{k}{g}\right)^{\frac{1}{2}}\frac{dx}{dt} - \tan^{-1}\left(\frac{k}{g}\right)^{\frac{1}{2}}u\right\} = -kt; \tag{114}$$

$$\therefore \quad \frac{dx}{dt} = \left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{uk^{\frac{1}{2}} - g^{\frac{1}{2}}\tan(kg)^{\frac{1}{2}}t}{g^{\frac{1}{2}} + uk^{\frac{1}{2}}\tan(kg)^{\frac{1}{2}}t}, \tag{115}$$

which gives us the velocity in terms of the time.

Again, from (113), if we multiply both sides by $2\,dx$,

$$\frac{d.\left(\frac{dx}{dt}\right)^2}{\frac{g}{k}+\left(\frac{dx}{dt}\right)^2} = -2k\,dx;$$

therefore integrating, and taking limits the same as before,

$$\log\frac{g+k\left(\frac{dx}{dt}\right)^2}{g+ku^2} = -2kx;$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = u^2e^{-2kx} - \frac{g}{k}(1-e^{-2kx}), \tag{116}$$

which gives the velocity in terms of the distance.

Also, from (115),

$$dx = \left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{uk^{\frac{1}{2}}\cos(kg)^{\frac{1}{2}}t - g^{\frac{1}{2}}\sin(kg)^{\frac{1}{2}}t}{uk^{\frac{1}{2}}\sin(kg)^{\frac{1}{2}}t + g^{\frac{1}{2}}\cos(kg)^{\frac{1}{2}}t}\,dt;$$

and therefore integrating, and taking the limits the same as before,

$$x = \frac{1}{k}\log\frac{uk^{\frac{1}{2}}\sin(kg)^{\frac{1}{2}}t + g^{\frac{1}{2}}\cos(kg)^{\frac{1}{2}}t}{g^{\frac{1}{2}}}, \tag{117}$$

which gives the space described by the particle in terms of the time to which it is due.

From (115) and (116), when $\frac{dx}{dt} = 0$, that is, when $m$ has reached the highest point,

$$t = \frac{1}{(kg)^{\frac{1}{2}}} \tan^{-1} u \left(\frac{k}{g}\right)^{\frac{1}{2}}, \tag{118}$$

$$x = \frac{1}{2k} \log\left(1 + \frac{k}{g} u^2\right), \tag{119}$$

which give the distance of the highest point, and the time of reaching it. After which the particle begins to fall, and the investigations of the preceding Article are applicable.

Since $k$ is the same in this and the preceding Articles, that is, since $m$ presents an equal area $\omega$ in the ascent and the descent, then by (110) the velocity acquired by $m$ in descending to the point whence it was projected with $u$ is

$$\frac{u g^{\frac{1}{2}}}{(g + ku^2)^{\frac{1}{2}}}, \tag{120}$$

which is less than $u$: hence the velocity acquired in the descent is less than lost in the ascent, the reason being that momentum is withdrawn from $m$ both in the ascent and in the descent, and is transferred to the molecules of the medium.

Again, substituting (119) in (111), the time occupied in the descent is

$$\frac{1}{2(kg)^{\frac{1}{2}}} \log \frac{(g + ku^2)^{\frac{1}{2}} + uk^{\frac{1}{2}}}{(g + ku^2)^{\frac{1}{2}} - uk^{\frac{1}{2}}}, \tag{121}$$

and which is different to that required for the ascent, and which is given in (118).

243.] Let us also consider the motion of a particle under the action of a constant force in the line of its motion, and moving in a medium, the resistance of which varies as the velocity; and let us suppose the particle to be projected with a velocity $u$, when $t = 0$ and $x = 0$. The equation of motion is, in terms of velocity-increment,

$$\frac{d^2 x}{dt^2} = f - k\frac{dx}{dt}; \tag{122}$$

wherein $f$ expresses the constant force, and $k$ is the coefficient of resistance. Therefore integrating, and taking the limits which correspond to $t = t$ and to $t = 0$,

$$\frac{dx}{dt} - u = ft - kx;$$

$$\therefore \quad \frac{dx}{dt} + kx = u + ft; \tag{123}$$

$$\begin{aligned} \therefore \quad x &= \left(\frac{d}{dt} + k\right)^{-1}(u+ft) \\ &= e^{-kt}\int_0^t e^{kt}(u+ft)\,dt \\ &= \frac{ft}{k} + \frac{f-uk}{k^2}(e^{-kt}-1). \end{aligned} \tag{124}$$

Thus from (123) we have the velocity in terms of $x$ and $t$; and in (124) the relation is given between $x$ and $t$: hence also

$$\frac{dx}{dt} = \frac{f}{k} - \frac{f-uk}{k}e^{-kt}. \tag{125}$$

And if $t = \infty$, $x = \infty$, and $\frac{dx}{dt} = \frac{f}{k}$; that is, the velocity has this finite limiting value, which it attains only when $t = \infty$. This result also follows from the equation of motion: $\frac{dx}{dt} - \frac{f}{k} = 0$ is a singular solution of it: and thus the particular integrals (123) and (124) express the circumstances of the motion, so long as the time is finite; and when $t = \infty$, the singular solution expresses them.

244.] Lastly, let us consider the case of a particle moving in a resisting medium, where the density of the medium varies; and let us suppose the resistance to vary as the square of the velocity, and the density to vary inversely as the square of the distance from a given point; and the particle also to move under the action of an attracting force which varies inversely as the cube of the distance from the same point.

Let $a$ and $x$ be the distances of $m$ from the given point when $t = 0$ and when $t = t$. Let $u =$ the velocity of $m$ when $t = 0$, and let $\mu$ be the absolute force of the central force: then the equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{m\mu}{x^3} + \frac{k'}{x^2}\omega\left(\frac{dx}{dt}\right)^2.$$

Let $k'\omega = mk$: so that we have

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3} + \frac{k}{x^2}\left(\frac{dx}{dt}\right)^2. \tag{126}$$

Multiplying by $2\,dx$, we have

$$d.\frac{dx^2}{dt^2} = -\frac{2\mu}{x^3}dx + \frac{2k}{x^2}\left(\frac{dx}{dt}\right)^2 dx;$$

$$\therefore \quad d.\frac{dx^2}{dt^2} - \frac{2k}{x^2}\left(\frac{dx}{dt}\right)^2 dx = -\frac{2\mu}{x^3}dx;$$

a linear differential equation, of which the integrating factor is $e^{\frac{2k}{x}}$; see Vol. II, Art. 291: therefore integrating, and taking the limits which correspond to $t = t$ and to $t = 0$, we have

$$e^{\frac{2k}{x}}\left(\frac{dx}{dt}\right)^2 - e^{\frac{2k}{a}}u^2 = -2\mu\int_a^x e^{\frac{2k}{x}}\frac{dx}{x^3}$$

$$= \frac{\mu}{2k^2}\left\{\frac{2k-x}{x}e^{\frac{2k}{x}} - \frac{2k-a}{a}e^{\frac{2k}{a}}\right\},$$

which gives the velocity in terms of the distance, and does not admit of further integration.

# CHAPTER IX.

## THE RESOLUTION AND COMPOSITION OF VELOCITIES, AND THE THEORY OF CURVILINEAR MOTION.

245.] When two or more forces act simultaneously on a material particle in motion, the effects are different, and require separate consideration, according as their lines of action are in the line of motion of the particle or make angles with that line; in the former case the effect is an acceleration or retardation of the particle in the line along which it is moving: and the total effect of many such forces is the sum of their several effects; see Art. 205. And here I may remark that the effect of a finite accelerating force is of the nature of a series of impulses rapidly succeeding each other, and at infinitesimal intervals of time; and thus producing an apparent continuously increasing or decreasing (as the case may be) velocity; and the actual velocity is the aggregate of these: this part of the subject has been explained and illustrated in the preceding Chapter. In the latter case, the effect of a force acting along a line which is inclined at a given angle to the line of motion of a particle, is partly to produce a deflexion from the rectilinear path in which by the law of inertia the particle would move, and partly to produce an acceleration or a retardation along that line. Such forces therefore will generally cause a particle to describe a curvilinear path: the nature and properties of a motion of this kind will now be discussed.

246.] In the first place let it be observed that although a particle may have a certain velocity, yet that velocity is not an impediment to the independent action of another force on the particle: the material particle, whether in motion or at rest, has the same property of inertia. Hence another force will produce its own effect on it, and precisely as if the particle was not moving. And thus the expressed velocity of the particle will be the resultant of these two several velocities, and its line of motion will depend on the lines of motion and the intensities of the two component velocities, and according to a law which we will now investigate. The law of inertia however becomes extended, and we have the following proposition:

When two or more forces impress velocity on a particle, the change in velocity and line of motion of the particle due to each is the same as if the others did not act.

This proposition arises partly from the inertia of matter, and partly from the fact that an adequate and intelligible conception of force requires that it acts on matter and causes it to move in a definite line. Hereby we shall be able to obtain from the combined action of two or more forces a velocity single and definite as to intensity and line of action; and thus may a velocity or a velocity-increment be resolved into other velocities or velocity-increments along given lines according to a definite law.

Thus a material particle under the action of two or more forces, of which the intensities and the lines of action are given, will have at any assigned time a definite velocity and line of motion: in the most general case, the lines of action and the intensities of the forces may vary from one point to another, and therefore the velocity and the line of motion of the particle will also vary; hence arises the convenience of fixing certain lines drawn in certain directions, of taking them as coordinate axes, and of estimating velocities, velocity-increments, and lines of motion in reference to them. In the most general case, the path of the particle will be a curve of double curvature.

Let us first take a fixed system of rectangular coordinate axes, and let us suppose $(x, y, z)$ to be the position of the particle at the time $t$. If three relations can be found between $x, y, z$ and $t$, the position of the particle at the time $t$ will be completely determined; and if $t$ is eliminated, two equations in terms of $x, y, z$ will result, which will represent two surfaces, the line of intersection of which is the path described by the particle. The path is called the *trajectory* of the particle; we shall first consider the cinematical part of the subject with respect to expressed velocities and velocity-increments, and in the most general form: and then, by a method similar to that of Art. 211, we shall be able to equate the expressed momenta to the impressed momenta, and thus obtain equations of motion similar to (14) of that article.

247.] Let the coordinates of the particle at the time $t$ be $x, y, z$: and at the time $t+dt$, $x+dx$, $y+dy$, $z+dz$: then if

$$dx^2+dy^2+dz^2 = ds^2, \qquad (1)$$

$ds$ is the space described in the time $dt$: and therefore the measure of the expressed velocity of the particle in the line of its motion at the time $t$ is $\frac{ds}{dt}$. But since $dx, dy, dz$ are the several increments of $x, y, z$ in the time $dt$,

$$\left.\begin{array}{l} \frac{dx}{dt} \text{ is the measure of the expressed velocity along the axis of } x \\ \frac{dy}{dt} \text{ - - - - - - - - - - - - - - - - - - - - - } y \\ \frac{dz}{dt} \text{ - - - - - - - - - - - - - - - - - - - - - } z \end{array}\right\}; \quad (2)$$

the velocities being considered positive or negative according as the coordinates increase or decrease as $t$ increases.

Dividing each side of (1) by $dt^2$, we have

$$\frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}; \tag{3}$$

and therefore the square of the expressed velocity is equal to the sum of the squares of the several expressed velocities along the coordinate axes.

Also if $a, \beta, \gamma$ are the direction-angles of $ds$, that is, of the line of motion of the particle during the time $dt$, then

$$\left.\begin{array}{l} dx = ds \cos a \\ dy = ds \cos \beta \\ dz = ds \cos \gamma \end{array}\right\}; \tag{4}$$

and dividing each side by $dt$, we have

$$\left.\begin{array}{l} \frac{dx}{dt} = \frac{ds}{dt} \cos a \\ \frac{dy}{dt} = \frac{ds}{dt} \cos \beta \\ \frac{dz}{dt} = \frac{ds}{dt} \cos \gamma \end{array}\right\}; \tag{5}$$

that is, the velocities along the three coordinate axes are severally the product of the velocity along the line of motion and the direction-cosine with reference to each axis. Therefore the law of resolution of a velocity is the same as that of statical pressures and as that of the projections of lines and areas; and thus the following law for the resolution and composition of velocities is established:

If a particle moves with a given velocity along a certain line of action, its velocity with respect to any other line is equal to the product of the velocity and the cosine of the angle between the two lines.

This law is of the greatest importance in the treatment of complicated mechanical problems, and is called the parallelepipedon of velocities, and leads to the following construction for the resolution and composition of velocities:

Construct a rectangular parallelepipedon, of which the three sides parallel to the coordinate axes are $dx, dy, dz$, and of which the diagonal is $ds$, $ds$ being the distance described by the particle in the time $dt$; then $dx, dy, dz$ are proportional to the resolved parts of the velocity along the three axes.

248.] Hence it follows that if two forces whose lines of action are inclined at a given angle to each other act simultaneously on a particle and impress on it velocities, the resultant expressed velocity will be represented by the diagonal of the parallelogram, of which the two adjacent sides meeting at the position of the particle are the representatives of the separately impressed velocities. This proposition is commonly called the Second Law of Motion.

Let O, fig. 7, be the place of the particle $m$ at rest at the beginning of the time: let two impulsive forces P and Q act on it, of which the lines of action are OP and OQ; and let us suppose the force P to impress a velocity on $m$ so that it would describe the space OP uniformly in $t$ units of time: similarly let the force Q impress on $m$ a velocity such that it would describe uniformly the space OQ in $t$ units of time. Let the figure be constructed as in the diagram; where OR is the diagonal of the parallelogram of which OP and OQ are two containing and adjacent sides; where $Q''OP''$ is perpendicular to OR, and $OQ'' = OP'' = QQ' = PP'$, and where these four lines are all parallel to each other. Now by the last Article, the velocity of which OP is the line-representative may be resolved into two velocities, one of which will be represented by $OP'$ and the other by $OP''$; similarly may the velocity of which OQ is the line-representative be resolved into two, of which $OQ'$ and $OQ''$ are the line-representatives. Now $OQ''$ and $OP''$, being equal and in opposite directions, destroy each other; and $OP'$ and $OQ'$ acting along the same line and in the same direction must be added, and of their resultant the line-representative is the sum of $OP'$ and $OQ'$, that is, is OR; OR therefore is the line-representative of the velocity which the particle has, and therefore of the resultant of the two component velocities of which OP and OQ are the line-representatives.

Thus if on a particle $m$ two impulsive forces act, the lines of action of which are inclined at an angle $\gamma$, and if these acting singly would impress on $m$ velocities $u$ and $v$ along their lines of action, then, if $w$ is the velocity which one force acting would impress on $m$ and produce the same effect as the other two acting in combination,

$$w^2 = u^2+2uv\cos\gamma+v^2; \qquad (6)$$

and if $\alpha$ and $\beta$ are the angles between the lines of $v$ and $w$, and of $w$ and $u$ respectively, then

$$\frac{u}{\sin\alpha} = \frac{v}{\sin\beta} = \frac{w}{\sin\gamma}. \qquad (7)$$

Similarly if three forces, whose lines of action are mutually inclined at angles $\alpha, \beta, \gamma$, act on a material particle, and are such that each acting singly would impress on it velocities $u, v, w$ along their lines of action, then the one force which would impress on $m$ the same velocity as these three acting simultaneously is that which would impress a velocity $\sigma$, where

$$\sigma^2 = u^2+v^2+w^2+2vw\cos\alpha+2wu\cos\beta+2uv\cos\gamma; \qquad (8)$$

and its line of action would be parallel to the line whose equations are

$$\frac{x}{u} = \frac{y}{v} = \frac{z}{w}. \qquad (9)$$

If $\alpha=\beta=\gamma=90°$, (8) becomes identical with (3): thus this last investigation has enabled us to extend to oblique coordinates that which was proved to be true of rectangular coordinates.

Thus the parallelepipedon of velocities is true for constant velocities, which are the effects of impulsive forces: when however the velocity varies, the force which causes the variation may, by the infinitesimal principle, be conceived to be composed of a series of impulses, each of which produces a constant velocity during the infinitesimal element of time before the succeeding impulse takes place; and for each of these elements of time the theorem is true; it is true therefore generally for varying velocities.

249.] And the result may be illustrated by the following experiment: Let ABC, fig. 96, be the horizontal deck of a boat which is moving with a constant velocity in the direction indicated by the arrow, so that in the time $t$ the point A moves into the position A′, and all the other points on the deck describe straight lines equal and parallel to AA′; and suppose at A a particle $m$ to be placed, and from a force to receive a velocity in the direction AQ, so that if the boat is at rest, in the time $t$ it describes the line AQ: now if the boat is moving, this latter velocity will be combined with that of the boat, and the result is the effect of the two; but neither of them alters the effect of the other; and thus at the end of the time $t$ the particle is found at the point R, having described the diagonal AR, and which is therefore the line-representative of its velocity, because it is described uniformly and in the time $t$.

Experiments and observations such as the following shew the law to be true in the matter of the earth.

A small heavy particle let fall from the top of a mast of a ship sailing uniformly, falls at the foot of the mast, although the force under the action of which it falls is uniformly accelerating. Thus the particle retains the horizontal velocity which it had at the top of the mast, and this is combined with the vertical impressed velocity.

If a carriage moves evenly along a railway, and if an impulse is given to a ball in it, the effect of the impulse is the same, whatever is the direction in which it is given: the motion of the carriage does not alter the effect of the impulse, and the path and absolute velocity of the ball are of course compounded of the two velocities.

The earth revolves on its polar axis from west to east, and therefore all points on its surface move with a velocity due to this rotation. If therefore the law is not true, a body struck in a direction north or south, would deviate to the west, and this is not found to be the case. And this fact admits of most accurate examination: for suppose a pendulum to be suspended and to oscillate, the time and the extent of oscillation would be different for the different directions of the plane of oscillation: no difference however is observed, whatever is the azimuth of the plane: and the smallest difference, as Laplace has shewn in the Mécanique Céleste, cannot fail of being appreciable.

Again: of a lofty and vertical tower the top is of course farther from the centre of the earth than the bottom, and therefore as the earth rotates (from west to east), the horizontal velocity of the top is greater than that of the bottom. Let a heavy ball fall from the top: it will have the horizontal velocity of the top, and this is greater than that of the bottom: if therefore the ball falls on the west side of the tower, it will strike the tower before it reaches the earth: but if it falls on the east side of the tower, it will strike the earth at a small distance from the tower towards the east. These results have been actually observed; and therefore we infer the law of which they are the effects.

Also the phænomena of the aberration of light, and the accordance with observation of the results arrived at, yield another proof of the truth of the law of composition of velocities which we have here investigated. Suppose, see fig. 97, S to be the place of a star, and E to be the place of the earth in its path at the same time: now light travels with a velocity of 192000 miles in one second of time, and the earth moves in its elliptic path through 19 miles in a second: and let us suppose that in the time during which the light of the star has travelled from S to E, the earth has moved from E to E′, where EE′ is to SE as 19 to 192000; then the effect to us is the same as if the earth had been at rest, and light had a velocity represented by EE′ from E′ to E along EE′, in combination with its velocity along SE, so that the ray of light would come in the direction S′E, where S′E is the diagonal of the parallelogram of which SE and EE′ are two adjacent containing sides: the star therefore appears to us to be before its real place in the direction of the line of motion of the earth. See also Herschel's Outlines of Astronomy, Arts. 328—335.

And, omitting other experiments and observations, I may remark that the most conclusive evidence of the truth of the law of composition of velocities is the *a posteriori* proof arising out of the results of physical astronomy. The expressed velocities and velocity-increments of the planets are resolved and estimated according to it, and their places calculated at particular times; when these are compared with the observed places, no discrepancies are discovered; and thus one of the severest tests of the truth of such a law is applied and is satisfied.

250.] Let us return to the consideration of the velocity of a particle moving along its trajectory. In the most general case it will have a varying velocity along its path, and thus varying resolved velocities along the coordinate axes: hence during equal and successive $dt$'s, the $ds$'s, $dx$'s, $dy$'s, $dz$'s will not all be equal: therefore there will be $d^2s$, $d^2x$, $d^2y$, $d^2z$: and thus by a process of reasoning similar to that of Art. 206, if $t$ is equicrescent,

$$\left.\begin{array}{ll}
\dfrac{d^2s}{dt^2} & \text{represents the expressed velocity-increment in an unit of time along the curve} \\
\dfrac{d^2x}{dt^2} & \text{- - - - - - - - - along the axis of } x \\
\dfrac{d^2y}{dt^2} & \text{- - - - - - - - - - - - - - -} \quad y \\
\dfrac{d^2z}{dt^2} & \text{- - - - - - - - - - - - - - -} \quad z
\end{array}\right\}; \tag{10}$$

which expressions severally are, if $t$ is not equicrescent,

$$\left.\begin{array}{c}
\dfrac{d^2s\,dt - d^2t\,ds}{dt^3} \\
\dfrac{d^2x\,dt - d^2t\,dx}{dt^3} \\
\dfrac{d^2y\,dt - d^2t\,dy}{dt^3} \\
\dfrac{d^2z\,dt - d^2t\,dz}{dt^3}
\end{array}\right\}. \tag{11}$$

As $\frac{d^2s}{dt^2}$, $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$ are velocities, it is evident that the laws of resolution and composition of velocities of Art. 247 are applicable to them: hence as the particle at the time $t$ is describing the element of its path $ds$, the projections of which on the coordinate axes are $dx$, $dy$, $dz$, it follows that

$$\frac{d^2x}{dt^2}\frac{dx}{ds}, \qquad \frac{d^2y}{dt^2}\frac{dy}{ds}, \qquad \frac{d^2z}{dt^2}\frac{dz}{ds}, \tag{12}$$

are the resolved parts along $ds$ of the velocity-increments in an unit of time along the coordinate axes of $x$, $y$, $z$. But since

$$ds^2 = dx^2 + dy^2 + dz^2,$$

$$\therefore \quad ds\,d^2s = dx\,d^2x + dy\,d^2y + dz\,d^2z;$$

$$\therefore \quad \frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds} + \frac{d^2z}{dt^2}\frac{dz}{ds}; \tag{13}$$

and therefore the velocity-increment along each element of the trajectory is equal to the sum of the resolved parts along that element of the several velocity-increments along the coordinate axes.

Suppose the mass of the particle, whose motion we have considered, to be $m$, then $m\frac{ds}{dt}$ is its expressed momentum in the line of its motion; and $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, $m\frac{dz}{dt}$ are its expressed momenta along the axes of $x, y$, and $z$: momenta are evidently resolved and compounded according to the law of the parallelepipedon of velocities.

Hence also $m\frac{d^2s}{dt^2}$ is the expressed momentum-increment of $m$ in an unit of time along the line of its motion; and $m\frac{d^2x}{dt^2}$, $m\frac{d^2y}{dt^2}$, $m\frac{d^2z}{dt^2}$, are the several expressed momentum-increments of $m$ in an unit of time resolved along the coordinate axes of $x, y, z$.

Also from equation (13) it may be inferred that the expressed momentum-increment in the line of motion of a particle at a given time is the sum of the resolved parts along that line of the several expressed momentum-increments along the coordinate axes.

Hence also impressed momenta and momentum-increments and their causes, accelerating forces and moving forces respectively, are resolved and compounded according to the law of the parallelepipedon of velocities.

And as statical pressures, see Art. 213, are virtual dynamical momenta, it follows that statical pressures are resolved and compounded according to the same law: hence we have a proof of the parallelogram of forces.

251.] Suppose therefore $\mathrm{X}', \mathrm{Y}', \mathrm{Z}'$ to be the momentum-increments along the three coordinate axes, impressed on a particle $m$; then by reason of the principle contained in equation (14), Art. (211),

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \mathrm{X}' \\ m\frac{d^2y}{dt^2} &= \mathrm{Y}' \\ m\frac{d^2z}{dt^2} &= \mathrm{Z}' \end{aligned}\right\}; \qquad (14)$$

and if $\mathrm{X}', \mathrm{Y}', \mathrm{Z}'$ are proportional to the mass of $m$*, so that

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* The impressed velocity-increments are not always proportional to $m$: instances of the non-proportionality have already occurred in the equations of motion in Articles 241 and 242.

$$\text{X}' = m\text{X}, \qquad \text{Y}' = m\text{Y}, \qquad \text{Z}' = m\text{Z},$$

then

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \text{X} \\ \frac{d^2y}{dt^2} &= \text{Y} \\ \frac{d^2z}{dt^2} &= \text{Z} \end{aligned}\right\}; \tag{15}$$

in which equations X, Y, Z are the impressed velocity-increments which are the effects of the accelerating forces.

The three preceding equations are the general equations of motion of a particle moving under the action of finite accelerating forces, and are applicable to the solution of every problem involving such a motion and such forces. If the values of $x, y, z$ are given in terms of $t$, as for instance, if

$$x = f_1(t), \qquad y = f_2(t), \qquad z = f_3(t), \tag{16}$$

the components of the velocity along the three coordinate axes may be found by differentiation, and thence the magnitude and the line of motion of the resultant velocity may be determined: and the components of the expressed velocity-increments may be found from the components of the velocity by differentiation; and as these are equal to the impressed velocity-increments, the laws according to which velocity must be impressed on the moving particle may also be determined: thus if the equations to the trajectory are given in terms of the time by means of three equations such as (16), the laws of the impressed velocity-increments may be found by differentiation.

Again, if the laws of the impressed velocity-increments are given, and if the problem is the deduction from them of the equations of the trajectory, (15) must be integrated, whereby three relations will be given between $x, y, z$ and $t$; whence $t$ must be eliminated, and two equations will result in terms of $x, y, z$, which will represent two surfaces, the line of intersection of which will be the trajectory. In the course of integration, twelve limiting values will be required, viz., the six components of the velocities corresponding to $t = t$ and to $t = 0$; and the six coordinates of $m$ corresponding to the same values of $t$: of these, six, viz., those corresponding to $t = t$, will be left in the general equations in their general form: the other six, which correspond to $t = 0$, will enter into the final equations as arbitrary constants, because the beginning of the time $t$ is arbitrary.

252.] If the moving forces are impulsive, and if $\text{v}'_x, \text{v}'_y, \text{v}'_z$ are the momenta which are impressed on the particle $m$ along the three coordinate axes, then

$$\left.\begin{aligned} m\frac{dx}{dt} &= \text{v}'_x \\ m\frac{dy}{dt} &= \text{v}'_y \\ m\frac{dz}{dt} &= \text{v}'_z \end{aligned}\right\}; \tag{17}$$

and if the components of the impressed momenta along the coordinate axes are proportional to the mass of $m$, if (say)

$$\text{v}'_x = m\text{v}_x, \qquad \text{v}'_y = m\text{v}_y, \qquad \text{v}'_z = m\text{v}_z,$$

then (17) become

$$\left.\begin{aligned} \frac{dx}{dt} &= \text{v}_x \\ \frac{dy}{dt} &= \text{v}_y \\ \frac{dz}{dt} &= \text{v}_z \end{aligned}\right\}; \tag{18}$$

and here, it is to be remembered, the whole expressed velocity is the measure of the action of the force.

253.] If the particle $m$ is not acted on by any forces, so that in (16), $\text{X} = \text{Y} = \text{Z} = 0$, then

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = 0. \tag{19}$$

Let $(a, b, c)$ be the place of the particle when $t = 0$, and $(x, y, z)$ when $t = t$: also let $\alpha, \beta, \gamma$ be the components of its velocity when $t = 0$; then integrating (19) between the limits corresponding to these values of $t$,

$$\frac{dx}{dt} - \alpha = 0, \qquad \frac{dy}{dt} - \beta = 0, \qquad \frac{dz}{dt} - \gamma = 0,$$

$$x - a - \alpha t = 0, \qquad y - b - \beta t = 0, \qquad z - c - \gamma t = 0;$$

$$\therefore \quad \frac{x-a}{\alpha} = \frac{y-b}{\beta} = \frac{z-c}{\gamma} = t; \tag{20}$$

which are the equations to a straight line, whose direction-cosines are proportional to the components of the velocity when $t = 0$, and which passes through the point $(a, b, c)$. If there is no initial velocity, $\alpha = \beta = \gamma = 0$; in which case $x = a$, $y = b$, $z = c$; that is, the particle remains at rest and in its original position. This result is of course in accordance with the law of inertia.

254.] The method of resolving velocities and momenta along three rectangular axes chosen arbitrarily is artificial, and has not been deduced from considerations of the actual motion of the particle; but it is convenient, and adapts itself to the Cartesian system of algebraical geometry, and to the ordinary equations of curves. Let us however consider whether the actual motion of $m$ does not lead us to another and more natural method.

$\frac{d^2s}{dt^2}$ is the expressed velocity-increment in the line of motion of $m$ at any time $t$; and therefore if the moving forces impress a momentum-increment only in this line, the path will be rectilinear: generally however the particle moves in a curvilinear path, and there is therefore some other component of the impressed momenta, which causes the deflexion from the rectilinear path: the question is, What is the mathematical representative of this impressed momentum-increment, as expressed in the motion of $m$? In fig. 98, let PQ ($= ds$) be the element of the curvilinear path described in the time $dt$; let $t$ be equicrescent; then if no force acts, whereby the particle is deflected from its rectilinear path, it will in the next $dt$ describe QR; but suppose QT ($= ds + d^2s$) to be the element of the curve succeeding PQ, and to be the path taken by the particle in the second $dt$; then at the point Q and along the line QS some force has acted, whereby the particle has received a velocity with which it moves over QS, in the time $dt$, so that at the end of $dt$ $m$ is at the point T; our object is to determine the value of the force which acts along QS.

P, Q, and T being three consecutive points in the curve, the plane which contains them is the osculating plane of the trajectory at the point P, and the angle RQT is the angle of contingence; see Art. 323, Vol. I. Let $\rho$ be the radius of absolute curvature of the path at P; that is, $\rho$ is the radius of the circle passing through P, Q, and T: and therefore from the geometry

$$\text{QS} = \frac{\text{QT}^2}{2\rho} = \frac{(ds + d^2s)^2}{2\rho} = \frac{ds^2}{2\rho}. \qquad (21)$$

Now whatever is the law of force under the action of which the particle $m$ describes QS in the time $dt$, we may consider it to be constant for that infinitesimal element of time; and therefore if $f$ is the impressed velocity-increment due to that force, by equation (31), Art. 220,

$$\text{QS} = \frac{f}{2}\, dt^2; \tag{22}$$

therefore from (21) and (22),

$$f = \frac{1}{\rho}\frac{ds^2}{dt^2}$$
$$= \frac{v^2}{\rho}, \tag{23}$$

if $v$ is the velocity of $m$ at P: and the line of action of it is along the radius of absolute curvature, that is, is the principal normal.

Hence at any point P of the trajectory, if the expressed velocity-increment is resolved along the tangent to the curve at P, and along a line perpendicular to the tangent, the latter line is the principal normal; and the expressed velocity-increments in these two lines are respectively

$$\frac{d^2s}{dt^2} \text{ and } \frac{v^2}{\rho},$$

and there is no expressed velocity-increment at right-angles to the osculating plane. The impressed velocity-increments, that is, do not produce any torsion, and torsion is due to their variations only.

Therefore if T and N are the impressed velocity-increments along the curvilinear path, and perpendicular to the path in the osculating plane,

$$\text{T} = \frac{d^2s}{dt^2}, \tag{24}$$

$$\text{N} = \frac{v^2}{\rho}. \tag{25}$$

T and N are called respectively the tangential and normal components of the impressed velocity-increments; and similarly $\frac{d^2s}{dt^2}$ and $\frac{v^2}{\rho}$ are respectively the tangential and normal components of the expressed velocity-increments.

255.] These results may also be deduced from the equations of motion (15). Let R be the resultant of the impressed velocity-increments; therefore

$$\begin{aligned}
\mathrm{R}^2 &= \mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2 \\
&= \frac{1}{dt^4}\{(d^2x)^2+(d^2y)^2+(d^2z)^2\} \\
&= \frac{1}{dt^4}\{(d^2x)^2+(d^2y)^2+(d^2z)^2-(d^2s)^2+(d^2s)^2\} \\
&= \frac{1}{dt^4}\left\{\frac{ds^4}{\rho^2}+(d^2s)^2\right\},
\end{aligned}$$

if $\rho =$ the radius of absolute curvature; see (20), Art. 325, Vol. I;

$$\therefore \quad \mathrm{R}^2 = \left(\frac{v^2}{\rho}\right)^2+\left(\frac{d^2s}{dt^2}\right)^2; \tag{26}$$

the form of which equation (the right-hand member being the sum of two squares) shews that the resultant of the impressed velocity-increments, when expressed, consists of two parts, which have lines of action perpendicular to each other: one of which, viz. $\frac{d^2s}{dt^2}$, is the expressed velocity-increment along the tangent to the curve: and the line of action of the other, viz. $\frac{v^2}{\rho}$, is perpendicular to the tangent: and if $\lambda, \mu, \nu$ are the direction-angles of its line of action, they may be determined by the following process. Since $\frac{d^2s}{dt^2}$ and $\frac{v^2}{\rho}$ are the components of R, the lines of action of the three are in one plane: let the direction-angles of the normal of that plane be proportional to A, B, C; then, because this plane contains (1) $\frac{d^2s}{dt^2}$, (2) $\frac{v^2}{\rho}$, (3) R, we have

$$\left.\begin{array}{r}
\mathrm{A}\,dx + \mathrm{B}\,dy + \mathrm{C}\,dz = 0 \\
\mathrm{A}\cos\lambda+\mathrm{B}\cos\mu+\mathrm{C}\cos\nu = 0 \\
\mathrm{A}\,d^2x + \mathrm{B}\,d^2y + \mathrm{C}\,d^2z = 0
\end{array}\right\}; \tag{27}$$

$$\begin{aligned}
\therefore \quad \cos\lambda\,(d^2y\,dz-d^2z\,dy)+\cos\mu\,(d^2z\,dx-d^2x\,dz)& \\
+\cos\nu\,(d^2x\,dy-d^2y\,dx) &= 0. \qquad (28)
\end{aligned}$$

Also, since the line of action of $\frac{v^2}{\rho}$ is perpendicular to the tangent of the path of $m$, we have

$$dx\cos\lambda+dy\cos\mu+dz\cos\nu = 0; \tag{29}$$

therefore from (28) and (29),

$$\frac{\cos\lambda}{dy\,\{d^2x\,dy-d^2y\,dx\}-dz\,\{d^2z\,dx-d^2x\,dz\}} = \ldots = \ldots;$$

and by a reduction similar to that in Art. 294, Vol. I, we have

$$\frac{\cos\lambda}{d.\dfrac{dx}{ds}} = \frac{\cos\mu}{d.\dfrac{dy}{ds}} = \frac{\cos\nu}{d.\dfrac{dz}{ds}}; \tag{30}$$

the denominators of which fractions are proportional to the direction-cosines of the principal normal; therefore the line of action of the normal component is the principal normal.

As N is the normal component of the impressed velocity-increment it is the effect of the force which deflects $m$ from its rectilinear path and draws it towards the centre of absolute curvature; and thus is the measure of that force, and, as such, is called *the centripetal force;* and since $m$ would, if left to itself, by the law of inertia proceed in a straight line with the velocity which it has at P, but is in its actual motion deflected out of that course, its expressed velocity in the normal line is, in reference to $m$, called *the centrifugal force*:* hence

$$\text{the centrifugal force of } m = \frac{mv^2}{\rho}. \tag{31}$$

Centripetal and centrifugal forces are therefore the same quantity under different aspects. Centripetal force is the force which impresses the momentum, and is spoken of in reference to that force: centrifugal force is spoken of with reference to the particle $m$, and is that force of which $m\dfrac{v^2}{\rho}$ is the expressed momentum. The inquiry into other general properties and principles which are deducible from the equations (15) are deferred until applications of them have been made to particular examples in the three following Chapters.

256.] Let the lines of action of all the moving forces which act on $m$ be in one plane, and which we will take to be the plane of $xy$; and if $m$ has any initial velocity we shall suppose that to be also in the same plane.

And first let us resolve along two rectangular axes in that plane; and let $m$X, $m$Y be the resolved impressed momentum-increments along the axes of $x$ and $y$, and acting to increase $x$ and $y$ as $t$ increases; let the place of the particle $m$ at the time $t$ be $(x, y)$: so that the equations of motion, in terms of velocity-increments, are

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* This term is inaccurate; $\dfrac{v^2}{\rho}$ is the expressed effect of an accelerating force.

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \text{X} \\ \frac{d^2y}{dt^2} &= \text{Y} \end{aligned}\right\}. \tag{32}$$

Secondly, let the motion of the particle be referred to a system of polar coordinates, $r$ and $\theta$; and let P and Q be the impressed velocity-increments along, and perpendicular to, the radius vector; P having a positive sign when it acts to increase $r$ as $t$ increases, and Q having a positive or negative sign according as it acts to increase or diminish $\theta$ as $t$ increases. P and Q are called respectively the *radial* and the *transversal* impressed velocity-increments.

Let X and Y be the impressed velocity-increments on $m$, see fig. 99, parallel to the axes of $x$ and $y$: let O be the origin, or the pole: $\text{OM} = x$, $\text{MP} = y$, $\text{OP} = r$, $\text{PO}x = \theta$.

Then since $\text{X} = \frac{d^2x}{dt^2}$, $\text{Y} = \frac{d^2y}{dt^2}$, if we resolve these along and perpendicular to OP, we have

$$\left.\begin{aligned} \text{P} &= \text{Y} \sin\theta + \text{X} \cos\theta \\ \text{Q} &= \text{Y} \cos\theta - \text{X} \sin\theta \end{aligned}\right\}; \tag{33}$$

and replacing X and Y by their values,

$$\left.\begin{aligned} \text{P} &= \frac{d^2y}{dt^2} \sin\theta + \frac{d^2x}{dt^2} \cos\theta \\ \text{Q} &= \frac{d^2y}{dt^2} \cos\theta - \frac{d^2x}{dt^2} \sin\theta \end{aligned}\right\}. \tag{34}$$

Now bearing in mind that $t$ is equicrescent, and that therefore generally neither $r$ nor $\theta$ can be simultaneously so, we have

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}, \qquad \therefore \quad \left.\begin{aligned} dx &= \cos\theta\, dr - r\sin\theta\, d\theta \\ dy &= \sin\theta\, dr + r\cos\theta\, d\theta \end{aligned}\right\}; \tag{35}$$

$$\therefore \left.\begin{aligned} d^2x &= \cos\theta\, d^2r - 2\sin\theta\, dr\, d\theta - r\cos\theta\, d\theta^2 - r\sin\theta\, d^2\theta \\ d^2y &= \sin\theta\, d^2r + 2\cos\theta\, dr\, d\theta - r\sin\theta\, d\theta^2 + r\cos\theta\, d^2\theta \end{aligned}\right\}; \tag{36}$$

and substituting these in (34), we have

$$\text{P} = \frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2}; \tag{37}$$

$$\begin{aligned} \text{Q} &= 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} \\ &= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right). \end{aligned} \tag{38}$$

The right-hand members of (37) and (38) are respectively called the radial and the transversal expressed velocity-increments.

$\frac{d\theta}{dt}$ is the angle generated by the revolving radius vector in one unit of time; it is therefore the measure of the expressed angular velocity; and $\frac{d^2\theta}{dt^2}$ is in like manner the measure of the expressed angular velocity-increment due to an unit of time. Similarly $\frac{dr}{dt}$ and $\frac{d^2r}{dt^2}$ are the measures of the expressed velocity and velocity-increment of the radius vector.

Two particular forms of (37) and (38) deserve notice. If the source of all the impressed velocity is in O, so that $Q = 0$, in which case the force is said to be central, then

$$r^2 \frac{d\theta}{dt} = \text{a constant}. \tag{39}$$

And if the angular velocity is constant, so that $\frac{d\theta}{dt} = $ a constant $= \omega$ (say), then

$$\left.\begin{aligned} P &= \frac{d^2r}{dt^2} - \omega^2 r \\ Q &= 2\omega \frac{dr}{dt} \end{aligned}\right\}. \tag{40}$$

Thirdly, let the velocity-increments on $m$ be resolved into normal and tangential components, see fig. 100, and let N and T be respectively the normal and the tangential impressed velocity-increments.

Let $OM = x$, $MP = y$; then $\tan PTM = \frac{dy}{dx}$: and resolving,

$$T = X\frac{dx}{ds} + Y\frac{dy}{ds} = \frac{dx\, d^2x + dy\, d^2y}{ds\, dt^2}, \tag{41}$$

$$N = Y\frac{dx}{ds} - X\frac{dy}{ds} = \frac{dx\, d^2y - dy\, d^2x}{ds\, dt^2}. \tag{42}$$

Also since
$$dx^2 + dy^2 = ds^2,$$
$$dx\, d^2x + dy\, d^2y = ds\, d^2s;$$

and if $\rho$ is the radius of curvature at P,

$$\begin{aligned} ds &= -\rho\, d.\tan^{-1}\frac{dx}{dy} \\ &= \rho\frac{dx\, d^2y - dy\, d^2x}{ds^2}; \end{aligned}$$

$$\left.\begin{aligned} \therefore \quad \text{T} &= \frac{d^2 s}{dt^2}, \\ \text{N} &= \frac{ds^2}{\rho\, dt^2} = \frac{v^2}{\rho}, \end{aligned}\right\} \tag{43}$$

if $v = \frac{ds}{dt}$ = the velocity at the point P. These values may be deduced also from the resolution of P and Q in (37) and (38).

257.] The normal component, which is also the centrifugal force, and of which the mathematical expression is $\frac{v^2}{\rho}$, requires further consideration: for on an adequate understanding of it in a great measure depends our knowledge of curvilinear motion.

Into the preceding investigations no differentials or derived functions of the equation of the curvilinear path above the second order have entered; and therefore only *three* consecutive points on the curve have come under consideration: the preceding properties therefore are equally true of the trajectory and of its circle of curvature at the point P.

Let the circle PQRA in fig. 101 be the circle of curvature at the point P; let P be the place of *m* at the time *t*, and let PQ, QR be the two successive elements of the trajectory described in the equal succeeding elements *dt*.

Now *m*, having described PQ in the time *dt*, would, during the following *dt*, have described another element (not necessarily equal to PQ) QT in the same straight line, if no force acted whereby it is deflected from its rectilinear path: but since at the end of the second *dt*, *m* is at R, a force has deflected it, the expressed effect of which is the motion of the particle over the distance RT or MQ: let *f* represent this force; and since *f* acts only for the infinitesimal time *dt*, we may, whatever is its law of change, consider it to be constant during that infinitesimal time. Now the space through which *m* moves under the action of *f* is RT or QM; therefore by (31), Art. 220,

$$\text{QM} = \frac{f.dt^2}{2};$$

but by the geometry MQ : QR = QR : QA; and AQ is the diameter of the circle of curvature, and therefore is equal to $2\rho$: and QR = *ds*,

$$\therefore \quad \text{QM} = \frac{ds^2}{2\rho};$$

$$\therefore \quad f = \frac{ds^2}{\rho\, dt^2} = \frac{v^2}{\rho},$$

if $v =$ the velocity of $m$ at the point P. And therefore

$$\text{the centrifugal force of } m = \frac{mv^2}{\rho}. \tag{44}$$

Let us consider and apply this expression in the case of $m$ moving in a circle. Let the radius of the circle $= a$; then if T is the time during which $m$ moves round the circle, and (as we suppose) with a constant velocity V,

$$2\pi a = \text{VT}; \tag{45}$$

$$\text{therefore the centrifugal force of } m = m\frac{4\pi^2 a}{\text{T}^2}; \tag{46}$$

thus the centrifugal force in a circle varies directly as the radius of the circle, and inversely as the square of the periodic time.

Again, if $m$ moves in the circle with a constant angular velocity $\omega$, then since

$$s = a\theta, \qquad \frac{ds}{dt} = a\frac{d\theta}{dt},$$

$$\therefore \quad v = a\omega;$$

$$\therefore \quad \text{the centrifugal force of } m = m\omega^2 a; \tag{47}$$

and therefore varies directly as the radius of the circle.

258.] When a solid body rotates about an axis all its particles describe in equal times circles, the planes of which are perpendicular to the axis of rotation, the centres of which are in this axis, and the radii of which are the perpendiculars from each point on the axis: therefore from (47), as $\omega$ is the same for all the points, the centrifugal forces vary as these perpendiculars. Now as the earth revolves about its polar axis, the centrifugal forces of particles on its surface vary as the perpendicular distances from the particle on the polar axis; and therefore the centrifugal force of each particle varies as the radius of the parallel of latitude which the particle describes: and the line of action, being the radius of the circular path in which the particle moves, is perpendicular to the polar axis. As the radius of the circle of the parallel of latitude decreases from the equator to the pole, so does the centrifugal force which varies as this radius by reason of (47); the centrifugal force therefore is greatest at the equator and least at the poles, where it vanishes.

Let us consider the earth to be a perfect sphere, and to be revolving with an angular velocity such that the time of the revolution is 24 hours; and let us consider a place on it of which the latitude is $\lambda$; then, if $r$ is the radius of the earth, the radius of the circle of the parallel of latitude is $r \cos \lambda$; and therefore if $\omega$ is the angular velocity, the centrifugal force of $m$ in this line, which is perpendicular to the polar axis, is $m\omega^2 r \cos \lambda$; and if this is resolved into two parts, one of which is horizontal, and the other is vertical, at the place, the vertical component of the centrifugal force is $m\omega^2 r (\cos \lambda)^2$: by this quantity therefore will the attraction of the earth on a particle $m$ on its surface at the latitude $\lambda$ be diminished: so that if $mg$ is the weight of $m$ when diminished by the centrifugal force, and $m\text{G}$ were its weight if the earth were at rest, then

$$mg = m\text{G} - m\omega^2 r (\cos \lambda)^2, \tag{48}$$

$$\therefore \quad g = \text{G} - \omega^2 r (\cos \lambda)^2; \tag{49}$$

therefore $\omega^2 r (\cos \lambda)^2$ is the diminution of the earth's gravity due to the centrifugal force.

Let us calculate the value of the diminution of gravity when $\lambda = 0$, that is, at the equator; in which case

$$\begin{aligned} g &= \text{G} \left\{ 1 - \frac{\omega^2 r}{\text{G}} \right\} \\ &= \text{G} \left\{ 1 - \frac{\omega^2 r}{g} \right\}, \end{aligned} \tag{50}$$

since the difference between $g$ and G is very small. Let $\text{T} =$ the time of rotation of the earth about its axis; therefore $\text{T} = 24 \times 60 \times 60$ seconds. Also $2\pi = \omega\text{T}$; and $g = 32.2$ feet, $\pi = 3.14159$; therefore $r = 4000 \times 1760 \times 3$ feet,

$$\begin{aligned} \frac{\omega^2 r}{g} &= \frac{4\pi^2 r}{g\text{T}^2} \\ &= \frac{1}{289} \text{ nearly}; \end{aligned}$$

$$\therefore \quad g = \text{G} \left\{ 1 - \frac{1}{289} \right\}; \tag{51}$$

that is, the diminution of gravity at the earth's equator due to the centrifugal force is the 289th part of that which the earth's attraction at the equator would be, if the earth did not rotate.

Thus also the weight of a body $m$ at the equator is diminished by its 289th part, and the diminution of its weight at the latitude $\lambda$, is

$$\frac{mg(\cos\lambda)^2}{289}.$$

The preceding calculation is made on the hypothesis that the earth is a perfect sphere, whereas it is an oblate spheroid: and the attraction of the earth on particles at its surface decreases as we pass from the poles to the equator: this diminution takes place according to the law known as Clairault's theorem, and which is expressed by the equation

$$g = \text{G}\{1+k(\sin\lambda)^2\};$$

where G is the equatorial gravity, $k$ is a constant depending on the eccentricity of the generating ellipse, and on the equatorial centrifugal force, and where $\lambda$ is the latitude of the place to which $g$ corresponds. Thus gravity becomes diminished at the equator from both causes, and the whole diminution is nearly a 200th part of the original gravity. Hence also the weight of a body at the poles is one 200th more than its weight at the equator.

Since 289 is the square of 17, it follows that if the earth completed a revolution about its polar axis in the 17th part of a day, the centrifugal force at the equator would be equal to, and would neutralize, the earth's attraction; in which case matter at the equator would have no weight.

259.] The motion of the moving particle $m$, which we suppose at the time $t$ to be at the point $(x, y, z)$, has been in the preceding investigations referred to a fixed origin and to a fixed system of coordinate axes: and its velocity and other incidents of motion are thus absolutely determined: in many cases however, and especially in astronomical inquiries, it is convenient to consider the motion of a particle relatively to another particle, the place of which we suppose to be the origin of a system of axes, the position and direction of which continually change under the action of given forces: and the motion of $m$ thus referred is called relative motion. For the sake of simplicity I shall consider the system of moveable axes to have only motion of translation, so that while the origin describes a certain path the directions of the coordinate axes are unaltered.

Let $(x, y, z)$ $(a, b, c)$ be the positions of $m$ and of the moving origin at the time $t$, with reference to a fixed origin and axes;

and let $x', y', z'$ be the coordinates of $m$, with reference to the moving system: then

$$x = a+x', \qquad y = b+y', \qquad z = c+z';$$

$$\therefore \frac{d^2x}{dt^2} = \frac{d^2a}{dt^2}+\frac{d^2x'}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2b}{dt^2}+\frac{d^2y'}{dt^2}, \quad \frac{d^2z}{dt^2} = \frac{d^2c}{dt^2}+\frac{d^2z'}{dt^2};$$

$$\therefore \frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2}-\frac{d^2a}{dt^2}, \ \ldots\ldots$$

Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the components of the impressed velocity-increment acting on $m$ at P; and let $\mathrm{A}, \mathrm{B}, \mathrm{C}$ be the components of the impressed velocity-increments which act on the particle at the moving origin: then we have

$$\frac{d^2x'}{dt^2} = \mathrm{X}-\mathrm{A}, \qquad \frac{d^2y'}{dt^2} = \mathrm{Y}-\mathrm{B}, \qquad \frac{d^2z'}{dt^2} = \mathrm{Z}-\mathrm{C}; \qquad (52)$$

so that the relative expressed velocity-increments of $m$ are the excess of the absolute velocity-increments impressed on it over those impressed on the particle at the moving origin.

The same result may also be found by the following process. Let O be the moving origin and P the place of the moving particle $m$; and let A, B, C, X, Y, Z be employed in the same significations as above: then if forces are applied to O and to $m$, producing equal velocities in both in each direction, the relative motion of the two, by reason of their other forces, will be the same as if these other forces did not act. Suppose forces to be applied which will impress velocity-increments in each axis equal to those impressed on the particle at O, and in opposite directions: then O necessarily is at rest; and the components of the impressed velocity-increments on $m$ at P are

$$\mathrm{X}-\mathrm{A}, \qquad \mathrm{Y}-\mathrm{B}, \qquad \mathrm{Z}-\mathrm{C};$$

and the motion of $m$ under the action of these forces is absolute, since O is at rest: and thus

$$\left.\begin{aligned} \frac{d^2x'}{dt^2} &= \mathrm{X}-\mathrm{A} \\ \frac{d^2y'}{dt^2} &= \mathrm{Y}-\mathrm{B} \\ \frac{d^2z'}{dt^2} &= \mathrm{Z}-\mathrm{C} \end{aligned}\right\}. \qquad (53)$$

# CHAPTER X.

## APPLICATION OF THE EQUATIONS OF THE PRECEDING CHAPTER TO PARTICLES MOVING FREELY IN SPACE.

### SECTION 1.—*Oblique impact and collision of particles and of smooth spherical balls.*

260.] The laws of resolution of velocities and momenta, and the principles of the theory of curvilinear motion have been investigated in the preceding Chapter; our object now is further to elucidate these, and to apply them to the special case of particles moving freely in space, reserving to subsequent Chapters the cases of particles moving on curves or on surfaces or in tubes, cases, that is, of constrained motion.

The most simple case of resolution of momenta is that of a material particle or of a smooth homogeneous spherical ball (which is supposed to have motion of translation only) which moves in a rectilinear path with a constant velocity, and impinges at an oblique angle on a given plane. As the velocity of the ball is constant before impact, so will the velocity be also constant after impact: and thus the inquiry is limited to the circumstances of change during the collision, and to the determination of the line of motion and of the velocity of the ball after impact. The elasticity of the ball is supposed to be the same as that which was explained in Art. 214, that is, the momentum acquired during the restitution of the figure of the ball is supposed to bear a constant ratio to that lost during the compression of the figure: the limiting values of the ratio being 1 and 0, according as the ball and plane are perfectly elastic or are perfectly inelastic.

One observation however is to be made on the circumstances of oblique impact, which was not applicable in that of direct impact. In oblique impact we assume that the mutual action of the balls during collision is along the line joining their centres at the instant when compression is a maximum, and along that line only; that is, in other words, we assume the balls to be perfectly smooth. For suppose a ball of mass $m$, see fig. 102, to move uniformly along the line $a$A and to impinge on the ball of mass $m'$, which is moving at an uniform velocity along the line $b$B: and suppose the line OAB to be that which passes through their centres A, B at the instant at which compression is a maximum: we assume the action of the two balls on each other for the time during which the collision takes place to be wholly along this line; along this line has momentum been lost during the compression, along this line will momentum be acquired during the restitution: the momenta in a line perpendicular to AB have not been altered by the collision: thus by virtue of the statements made in Art. 214, if P represents the momentum along the line AB lost during the compression; $e$P represents that acquired during the restitution along the same line. Thus although for the time during which the balls are in collision, they, by reason of their velocity which is perpendicular to AB, slide on each other, and thereby the momentum in that line would be changed if the balls are not perfectly smooth, yet for the sake of simplicity we assume the roughness of the balls to be so far diminished, that it is not necessary to take account of the change of momentum along the line perpendicular to AB.

Hence if a smooth ball impinges obliquely on a smooth plane, the line of reaction of the plane will be perpendicular to its surface, and the momentum of the impinging ball will be affected along that line only, and not along the plane.

261.] The results of the direct impact of a ball on a plane are given in equations (15), (16), Art. 216, so that if $v$ is the velocity of impact, $-ev$ is the velocity of rebound.

But suppose a perfectly smooth and spherical ball, whose mass is $m$, and whose elasticity is $e$, to move with an uniform velocity $v$, and to impinge at P on a smooth plane in the line $a$A, making an angle $\alpha$ with the normal to the plane at the point P, fig. 103, so that $a\text{AN} = \alpha$; let $u$ be the velocity of $m$ when the compression is a maximum: at which instant the motion of the ball is wholly along the plane; and suppose V to be the velocity of rebound, and $\beta = \text{NA}b$ to be the angle which the line of motion of the ball after rebound at P, makes with the normal: let P be the momentum in the line AN which is lost by the ball during the compression, and let $e$P be that recovered during the restitution, the line of action of both these momenta being the normal AN. Let us resolve the momenta along and perpendicular to the plane: then at the instant when the compression is a maximum, we have (1) along the plane,

$mv \sin a$ = the impressed momentum of $m$,

$mu$ = the expressed momentum of $m$;

(2) perpendicular to the plane,

$mv \cos a$ = the impressed momentum of $m$,

P = momentum lost by $m$ during compression;

$$\therefore \quad mv \sin a = mu; \tag{1}$$

$$mv \cos a = \mathrm{P}: \tag{2}$$

by a similar process when restitution ends, we have (1) along the plane,

$mu$ = the impressed momentum of $m$,

$m\mathrm{v} \sin \beta$ = the expressed momentum of $m$;

(2) perpendicular to the plane,

$e\mathrm{P}$ = the impressed momentum of $m$,

$m\mathrm{v} \cos \beta$ = the expressed momentum of $m$;

$$\therefore \quad mu = m\mathrm{v} \sin \beta, \tag{3}$$

$$e\mathrm{P} = m\mathrm{v} \cos \beta; \tag{4}$$

$$\therefore \quad v \sin a = \mathrm{v} \sin \beta,$$

$$e \tan \beta = \tan a;$$

$$\therefore \quad \tan \beta = \frac{\tan a}{e}; \tag{5}$$

$$\mathrm{v} = \frac{\sin a}{\sin \beta} v; \tag{6}$$

whereby $\beta$ and v are known in terms of given quantities; $a$ and $\beta$ are called respectively the angles of *incidence* and *reflexion*. Generally if $e$ is less than unity, $\beta$ is greater than $a$. If the ball is perfectly elastic, $e = 1$: in which case $a = \beta$, that is, the angles of incidence and reflexion are equal; and $\mathrm{v} = v$, that is, the velocities of incidence and reflexion are equal to each other.

If the ball is perfectly inelastic, $e = 0$; in which case $\beta = 90°$, and $\mathrm{v} = v \sin a$; the ball, that is, after impact moves along the plane with the component in that line of its velocity of impact.

These results are equally true, when the ball impinges on a curved surface, the plane of impact being in this case the tangent plane to the surface at the point of impact.

262.] Examples in illustration of the preceding principles.

Ex. 1. To find the line along which a ball of given elasticity $e$ must be projected from a given point A, so that after reflexion at a given plane it may strike another given ball at B.

Let CD, fig. 104, be the given plane, A and B the given positions of the balls. From A and B draw AC and BD at right angles to the plane CD; produce AC to A′, making CA′ = $e$.CA. Join BA′ cutting CD in P, and join AP; APB is the required path. At P draw the line PN normal to the plane: then APN is the angle of incidence, and NPB is the angle of reflexion: and since

$$\tan \text{APN} = \cot \text{APC} = \frac{\text{CP}}{\text{AC}};$$

and
$$\tan \text{BPN} = \cot \text{BPD} = \cot \text{A}'\text{PC} = \frac{\text{CP}}{\text{A}'\text{C}} = \frac{\text{CP}}{e.\text{AC}};$$

$$\therefore \quad \tan \text{BPN} = \frac{1}{e} \tan \text{APN};$$

and therefore the path APB satisfies the condition (5), and therefore is such that a ball projected from A along AP will strike another ball at B.

Ex. 2. To determine the path which a ball of elasticity $e$ must take with reference to two given inclined planes, so that when projected from a given point A it may after reflexion successively at the two planes strike another ball at B, A and B being in the plane which is perpendicular to the line of intersection of the two planes of reflexion.

Let the plane of the paper, fig. 105, be that in which the two points A and B are, and thus the planes of reflexion are perpendicular to the plane of the paper.

From A draw ADA′ perpendicular to the plane DO, and make A′D = $e$.AD: also from B draw BCB′ perpendicular to OC, and such that BC = $e$.B′C: draw A′B′ cutting DO in P and OC in Q, join AP and BQ; APQB shall be the path required.

It is evident by the construction in the former example that the lines AP and PQ satisfy at P the condition required in (5): and also that PQ and QB satisfy the same condition at Q: therefore APQB is the required path.

Also by a similar process may the path be determined, by which a ball of given elasticity may after projection from a given point and reflexion at given planes strike a ball placed at another given point.

Ex. 3. A ball of given elasticity $e$ is projected from a given point in the circumference of a circle, and after two reflexions from the circle returns to the same point: determine the angle at which it was projected.

Let A, fig. 106, be the point in the circle whence the ball is projected, and let AP, PQ, QA be the paths which it successively describes: let CAP $= \theta =$ CPA, CPQ $= \phi =$ CQP, CQA $= \psi =$ CAQ. Then by (5),

$$\tan\phi = \frac{1}{e}\tan\theta, \qquad \tan\psi = \frac{1}{e}\tan\phi = \frac{1}{e^2}\tan\theta;$$

$$\text{also} \qquad \theta+\phi+\psi = 90^\circ;$$

$$\therefore \quad \tan\theta = \cot(\phi+\psi) = \frac{1-\tan\phi\tan\psi}{\tan\phi+\tan\psi}$$

$$= \frac{e^3-(\tan\theta)^2}{(e^2+e)\tan\theta};$$

$$\therefore \quad \tan\theta = \left(\frac{e^3}{1+e+e^2}\right)^{\frac{1}{2}}.$$

263.] Two smooth spheres of given elasticity and of masses $m$ and $m'$, moving in given lines, and with given velocities, and with their centres in the same plane, impinge on each other; it is required to determine their velocities and lines of motion after impact.

Let us suppose the two spheres to be moving in the directions indicated by the arrows, fig. 102, and along the lines $a$A and $b$B, and let OAB$x$ be the line passing through their centres at the instant when the compression is a maximum: and let $m$, whose centre is A, impinge on $m'$, whose centre is B. Now by Art. 259, the elastic action of the two spheres takes place along the line OAB only: let $e =$ the elasticity, and P $=$ the momentum lost during compression, so that $e$P is that acquired during the restitution of the figure of the balls. The momenta will be resolved along, and at right angles to, the line OAB.

Let $v, v'$ be the velocities of $m, m'$ before collision begins,
$u, u'$ - - - - - - - when compression is a maximum,
$\mathrm{v}, \mathrm{v}'$ - - - - - - - when collision ends.
$\alpha, \alpha'$ be the angles between OAB, and the lines of motion of $m, m'$ when collision begins,
$\theta, \theta'$ - - - - - - - - - - when compression is a maximum,
$\beta, \beta'$ - - - - - - - - - - when collision ends.

Then at the instant when the compression is a maximum, the equations of resolved momenta are,

$$\text{for } m \begin{cases} mv\cos\alpha = mu\cos\theta + \text{P}, & (7) \\ mv\sin\alpha = mu\sin\theta, & (8) \end{cases}$$

$$\text{for } m' \begin{cases} m'v'\cos\alpha' = m'u'\cos\theta' - \text{P}, & (9) \\ m'v'\sin\alpha' = m'u'\sin\theta'; & (10) \end{cases}$$

and at the instant when collision ends,

$$\text{for } m \begin{cases} mu\cos\theta = m\text{V}\cos\beta + e\text{P}, & (11) \\ mu\sin\theta = m\text{V}\sin\beta, & (12) \end{cases}$$

$$\text{for } m' \begin{cases} m'u'\cos\theta' = m'\text{V}'\cos\beta' - e\text{P}, & (13) \\ m'u'\sin\theta' = m'\text{V}'\sin\beta'. & (14) \end{cases}$$

But when the compression is a maximum, both balls are moving with the same velocity along OAB: therefore

$$u\cos\theta = u'\cos\theta'. \tag{15}$$

From these nine equations, $\text{V}$, $\text{V}'$, $\beta$, $\beta'$ are to be detrmined.

From (7), (9) and (15),

$$u\cos\theta = u'\cos\theta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'}; \tag{16}$$

from which and from (7) and (11), we have

$$\text{V}\cos\beta = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} - \frac{em'}{m+m'}(v\cos\alpha - v'\cos\alpha'). \tag{17}$$

Similarly,

$$\text{V}'\cos\beta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} + \frac{em}{m+m'}(v\cos\alpha - v'\cos\alpha'). \tag{18}$$

Also from (8) and (12), and from (10) and (14),

$$\text{V}\sin\beta = v\sin\alpha, \tag{19}$$

$$\text{V}'\sin\beta' = v'\sin\alpha'; \tag{20}$$

so that $\text{V}$, $\text{V}'$, $\beta$, $\beta'$ are completely determined.

Also from the preceding we have

$$\left.\begin{aligned} m\text{V}\cos\beta + m'\text{V}'\cos\beta' &= mv\cos\alpha + m'v'\cos\alpha' \\ m\text{V}\sin\beta + m'\text{V}'\sin\beta' &= mv\sin\alpha + m'v'\sin\alpha' \end{aligned}\right\}, \tag{21}$$

that is, the sum of the momenta both along, and at right angles to, OAB is the same before and after impact.

**264.]** Let the balls be perfectly elastic, that is, let $e = 1$: then

$$\text{V}\cos\beta = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} - \frac{m'(v\cos\alpha - v'\cos\alpha')}{m+m'}, \tag{22}$$

$$\text{V}'\cos\beta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'} + \frac{m(v\cos\alpha - v'\cos\alpha')}{m+m'}, \tag{23}$$

$$\mathrm{v} \sin\beta = v \sin\alpha, \tag{24}$$

$$\mathrm{v}' \sin\beta' = v' \sin\alpha'; \tag{25}$$

$$\therefore \quad m\mathrm{v}^2 + m'\mathrm{v}'^2 = mv^2 + m'v'^2, \tag{26}$$

that is, the sum of the vires vivæ is the same before and after impact.

Again, let the balls be perfectly inelastic, that is, let $e = 0$; then

$$\mathrm{v} \cos\beta = \mathrm{v}' \cos\beta' = \frac{mv\cos\alpha + m'v'\cos\alpha'}{m+m'}, \tag{27}$$

$$\mathrm{v} \sin\beta = v \sin\alpha, \tag{28}$$

$$\mathrm{v}' \sin\beta' = v' \sin\alpha', \tag{29}$$

that is, the balls after impact have the same velocity along the line OAB, but unequal velocities at right angles to that line; also

$$\frac{\tan\beta}{\tan\beta'} = \frac{v\sin\alpha}{v'\sin\alpha'}. \tag{30}$$

265.] The velocity and the line of motion of the centre of gravity of two smooth balls which impinge on each other is the same before and after impact.

Let $e$ be the elasticity of the balls: and let us take a line parallel to that which joins the centres of the balls at the instant when the compression is a maximum to be the axis of $x$; and a line perpendicular to it to be that of $y$: let $(x, y)$, $(x', y')$ be the places of the centres of the balls at the time $t$, either before or after impact; and let $(\bar{x}, \bar{y})$ be the place of the centre of gravity at the same time: then

$$\left.\begin{aligned}(m+m')\,\bar{x} &= mx + m'x' \\ (m+m')\,\bar{y} &= my + m'y'\end{aligned}\right\}, \tag{31}$$

$$\therefore \quad \left.\begin{aligned}(m+m')\frac{d\bar{x}}{dt} &= m\frac{dx}{dt} + m'\frac{dx'}{dt} \\ (m+m')\frac{d\bar{y}}{dt} &= m\frac{dy}{dt} + m'\frac{dy'}{dt}\end{aligned}\right\}, \tag{32}$$

but $\frac{dx}{dt}$ and $\frac{dx'}{dt}$ are before impact respectively $v\cos\alpha$ and $v'\cos\alpha'$; and are after impact respectively $\mathrm{v}\cos\beta$ and $\mathrm{v}'\cos\beta'$: and $\frac{dy}{dt}$ and $\frac{dy'}{dt}$ are before impact respectively $v\sin\alpha$ and $v'\sin\alpha'$, and are after impact respectively $\mathrm{v}\sin\beta$ and $\mathrm{v}'\sin\beta'$: therefore by virtue of equations (21), the right-hand members of (32) are the same before and after impact, therefore also the left-hand members are: and therefore $\frac{d\bar{x}}{dt}$ and $\frac{d\bar{y}}{dt}$ are the same before and after impact: and therefore the velocity and the line of motion of the centre of gravity of $m$ and $m'$ are the same before and after impact.

266.] Examples illustrative of the preceding equations.

Ex. 1. A smooth and homogeneous sphere of mass $m$ and elasticity $e$ moves with a velocity $v$ and impinges directly on another of the mass $m'$, which is moving with a velocity $v'$ in a line of motion at right angles to that of $m$; it is required to find the velocities and the lines of motion of both balls after the collision.

In this case $a = 0$, $a' = 90°$; therefore from (17), &c.,

$$\begin{aligned} \mathrm{v} \cos \beta &= v \frac{m - em'}{m + m'}, \\ \mathrm{v} \sin \beta &= 0, \\ \mathrm{v}' \cos \beta' &= (1+e) \frac{mv}{m+m'}, \\ \mathrm{v}' \sin \beta' &= v'; \end{aligned}$$

$$\therefore \quad \beta = 0, \quad \tan \beta' = \frac{v'}{v} \frac{m+m'}{(1+e)m},$$

$$\mathrm{v} = v \frac{m - em'}{m+m'}, \qquad \mathrm{v}'^2 = v'^2 + \left\{ \frac{(1+e)mv}{m+m'} \right\}^2.$$

Hence the ball $m$ will continue to move in the same line of motion, but with a velocity diminished in the ratio of $m - em'$ to $m + m'$ to its former velocity; the velocity of $m'$ will be increased, and its line of motion will make an angle $\beta'$ with that of $m'$s motion.

Ex. 2. Two balls $m$ and $2m$, whose elasticity is $\frac{2}{3}$, move with velocities $2v$ and $v$, and impinge, so that the line of motion of each one makes an angle of 60° with the line joining their centres at the instant when compression is a maximum; it is required to determine their motion after collision.

Here $m' = 2m$, $v' = \frac{v}{2}$, $e = \frac{2}{3}$, $a = a' = 60°$; therefore from (17), &c.,

$$\mathrm{v} \cos \beta = \frac{2v}{9}, \qquad \mathrm{v}' \cos \beta' = \frac{7v}{18},$$

$$\mathrm{v} \sin \beta = \frac{(3)^{\frac{1}{2}} v}{2}, \qquad \mathrm{v}' \sin \beta' = \frac{3^{\frac{1}{2}} v}{4},$$

whence v, v′, $\beta$, and $\beta'$ are easily determined.

267.] In Article 219 was estimated the loss of momentum of a body in its passage through a resisting medium, when the body presented to the medium a plane surface of area $\omega$, which is perpendicular to the line of motion of the body. The investigation can now be extended to the case in which the surface which acts on the medium is inclined at any angle to the line of motion. The velocity of the body will be resolved into two components, of which one is perpendicular to the plane of the surface, and the other is along the surface: the former alone causes the resistance by reason of momentum being transferred to the elements of the resisting medium; the latter produces only a friction along the surface, the calculation of which does not belong to the present part of our work.

Let there then be a plane surface, of which the area is $\omega$, perpendicular to the plane of the paper, and of which the section by the plane of the paper is the line OP, fig. 107; and suppose it to be moving in the line MO, and its normal to be inclined to MO at the angle $i$; let $v$ be the velocity of the body along the line of motion, so that $v \cos i$ is the velocity of OP in the line of its normal; and therefore by a process similar to that of Art. 219 it follows, that the momentum which is impressed by $\omega$ on the particles of the resisting medium during the time $dt$, and which has therefore been withdrawn from the moving body, is

$$\rho \omega v^2 (\cos i)^2 \, dt;$$

but the line of action of this resistance is in the normal to OP; therefore its component in the line of motion of the moving body is

$$\rho \omega v^2 (\cos i)^3 \, dt; \qquad (33)$$

and therefore if $m$ is the mass of the moving body, and $dv$ its loss of velocity in the line of its motion during the time $dt$ owing to the resistance of the medium,

$$-m \, dv = \rho \omega v^2 (\cos i)^3 \, dt;$$

$$\therefore \quad m \frac{dv}{dt} = -\rho \omega v^2 (\cos i)^3. \qquad (34)$$

Hence it appears that the resistance of a plane rudder passing through the water varies as the cube of the sine of the angle at which it is inclined to the keel of the vessel.

A few examples illustrative of (33) are subjoined.

Ex. 1. An isosceles triangular wedge, of which the vertical angle is $2\alpha$, the depth is $b$, and the altitude is $a$, moves in a resisting medium, firstly with its edge forward, secondly with its top forward: compare the resistances in the two cases.

Let $R_1$ and $R_2$ be the resistances in the first and second cases respectively, then

$$R_1 = 2\rho v^2 (\sin a)^3 ab \sec a,$$

$$R_2 = 2\rho v^2 ab \tan a;$$

$$\frac{R_1}{R_2} = (\sin a)^2.$$

Ex. 2. A semicircular lamina of given thickness $\tau$ moves in a fluid, firstly with its convex edge forwards, secondly with its base forwards; compare the resistances in the two cases.

Let the resistances be $R_1$ and $R_2$: let $a =$ the radius of the semicircle: then, fig. 108, if $PCA = \theta$, $QCP = d\theta$, $PQ = a\,d\theta$,

$$\therefore \quad R_1 = 2a\tau\rho v^2 \int_0^{\frac{\pi}{2}} (\cos\theta)^3 \,d\theta$$

$$= \frac{4a\tau\rho v^2}{3},$$

$$R_2 = 2a\tau\rho v^2;$$

$$\therefore \quad \frac{R_1}{R_2} = \frac{2}{3}.$$

268.] By means of (33) also can be determined the resistance which a solid of revolution meets with in its passage through a resisting medium, such as water or air.

Let OPB, fig. 109, be the generating curve of the bounding surface of the solid, and let its equation be, $y = f(x)$; $OM = x$, $MP = y$; and let PG be the normal to the curve at the point P, so that

$$\cos PGO = \frac{dy}{ds}.$$

Let an element of the curve at $P = ds$; so that of a surface-element at P, $ds$ is the section by the plane of the paper: also let the surface-element subtend at an angle $d\theta$ at M; and thus, if $\omega =$ the surface-element,

$$\omega = ds\, y\, d\theta;$$

and therefore the loss of momentum corresponding to $\omega$ in the line AO and in the time $dt$ is

$$\rho v^2 y \left(\frac{dy}{ds}\right)^3 ds\, d\theta\, dt;$$

and as the loss of momentum corresponding to every equal element of the ring generated by the revolution of $ds$ about AO is the same, therefore the loss of momentum due to the ring is

$$2\pi\rho v^2 y \left(\frac{dy}{ds}\right)^3 ds\, dt;$$

and therefore the loss of momentum of the whole surface in the time $dt$ is

$$2\pi\rho v^2 dt \int y \left(\frac{dy}{ds}\right)^3 ds; \tag{35}$$

and therefore the loss of momentum to the moving body in an unit of time, or the resistance of the medium, as it is called, is

$$2\pi\rho v^2 \int y \left(\frac{dy}{ds}\right)^3 ds, \tag{36}$$

the limits of the integral being quantities assigned by the conditions of the problem.

Ex. 1. Let the surface COB be a hemisphere; it is required to compare the resistance of the hemisphere with the resistance of the base.

Let $a$ be the radius; then if the line of motion is the axis of $x$, and O is the origin,

$$y^2 = 2ax - x^2;$$

$$\therefore \quad \frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a};$$

$$\therefore \quad \text{the resistance} = 2\pi\rho v^2 \int_0^a \frac{(a-x)^3}{a^3} dx$$

$$= \frac{\pi\rho v^2 a^2}{2};$$

and the resistance of hemisphere moving with the base forwards

$$= \pi\rho v^2 a^2.$$

Therefore the resistance of a hemisphere moving with its convex surface forwards is one-half of its resistance when it moves with its base forwards.

Ex. 2. A right cone passes through a resisting medium, firstly with its vertex forwards, secondly with its base forwards: it is required to compare the resistances in the two cases.

Let the resistances be $R_1$ and $R_2$: let $a =$ the altitude of the cone, $b =$ the radius of its base; so that in the first case

$$\frac{y}{b} = \frac{x}{a}, \qquad \therefore \quad \frac{dy}{b} = \frac{dx}{a} = \frac{ds}{(a^2+b^2)^{\frac{1}{2}}};$$

$$\therefore \quad R_1 = \frac{2\pi\rho v^2 b^4}{a^2(a^2+b^2)} \int_0^a x\, dx$$

$$= \frac{\pi\rho v^2 b^4}{a^2+b^2};$$

also $$\text{R}_2 = \pi\rho v^2 b^2;$$

$$\therefore \quad \frac{\text{R}_1}{\text{R}_2} = \frac{b^2}{a^2+b^2}.$$

In these investigations no account has been taken of the action of the particles of the fluid on each other, nor of the friction of the particles against the surface of the moving body: also as the body moves forward it leaves a space behind it, which the particles of the resisting medium rush into and occupy: doubtless some momentum is imparted by these to the moving body: it is not therefore to be expected that the preceding results will be entirely accordant with experiment; and they are not; and in fact it appears that the law of the resistance is not, cæteris paribus, that of the square of the velocity. It is however worth while even to approximate to a solution of a problem of such difficulty, and therefore I have inserted the preceding theory of resistance, springing as it does out of that of impact and collision. There is also one other problem in the subject which deserves insertion, requiring as it does the calculus of variations, and of which the solution was first given by sir Isaac Newton.

269.] To determine the form of a surface of revolution COB, so that the resistance of a fluid, through which it moves in the line of its axis, may be the least.

Let $u$ represent the resistance: then

$$u = 2\pi\rho v^2 \int_0^1 y\frac{dy^3}{ds^2};$$

and taking the variation, and equating it to zero, we have

$$0 = \int_0^1 \left\{ \frac{dy^3}{ds^2}\delta y + y\frac{3\,dy^2\,ds\,\delta.dy - 2\,dy^3\,\delta.ds}{ds^3} \right\};$$

but since $$ds^2 = dx^2+dy^2,$$

$$\therefore \quad \delta.ds = \frac{dx}{ds}\delta.dx + \frac{dy}{ds}\delta.dy;$$

$$\therefore \quad 0 = \int_0^1 \left\{ \frac{dy^3}{ds^2}\delta y + \frac{3y\,dy^2}{ds^2}d.\delta y - \frac{2y\,dy^4}{ds^4}d.\delta y - \frac{2y\,dy^3\,dx}{ds^4}d.\delta x \right\};$$

$$\therefore \quad 0 = \left[\left(3y\frac{dy^2}{ds^2} - 2y\frac{dy^4}{ds^4}\right)\delta y - \frac{2y\,dy^3\,dx}{ds^4}\delta x\right]_0^1$$
$$+\int_0^1 \left\{ \left(\frac{dy^3}{ds^2} - d.\frac{3y\,dy^2}{ds^2} + d.\frac{2y\,dy^4}{ds^4}\right)\delta y + d.\frac{2y\,dy^3\,dx}{ds^4}\delta x \right\}. \quad (37)$$

To determine the function which represents the curve, we have, by the Calculus of Variations,

$$d.\frac{2y\,dy^3\,dx}{ds^4} = 0,$$

$$\frac{dy^3}{ds^2} - d.\frac{3y\,dy^2}{ds^2} + d.\frac{2y\,dy^4}{ds^4} = 0; \tag{38}$$

therefore from the former,

$$\frac{2y\,dy^3\,dx}{ds^4} = \text{a constant} = c\text{ (say)};$$

$$\therefore\quad c(dy^2+dx^2)^2 = 2y\,dy^3\,dx,$$

$$\text{or,}\qquad c\left(1+\frac{dy^2}{dx^2}\right)^2 = 2y\,\frac{dy^3}{dx^3}; \tag{39}$$

and replacing $y$ in (38) by its value from (39), we have

$$2\,\frac{dy^3}{ds^2} - c\,d.\frac{dy}{dx} - 3\,c\,d.\frac{dx}{dy} = 0,$$

$$2\,dx - c\,d.\frac{dy}{dx}\left(\frac{dx}{dy}+\frac{dx^3}{dy^3}\right) - 3c\,\frac{dx}{dy}\left(1+\frac{dx^2}{dy^2}\right)d.\frac{dx}{dy} = 0,$$

$$\therefore\quad 2x - c\log\frac{dy}{dx} + \frac{c}{2}\,\frac{dx^2}{dy^2} - \frac{3c}{2}\,\frac{dx^2}{dy^2} - \frac{3c}{4}\,\frac{dx^4}{dy^4} = c'; \tag{40}$$

where $c'$ is also an arbitrary constant. And thus (40) becomes

$$2x - c\log\frac{dy}{dx} - c\,\frac{dx^2}{dy^2} - \frac{3c}{4}\,\frac{dx^4}{dy^4} = c', \tag{41}$$

(39) and (41) are together the equations to the required curve: and if $\frac{dy}{dx}$ could be eliminated, the resulting equation would be that required in terms of $x$ and $y$. The properties of the curve at the limits would be given by the integrated part of (37).

## SECTION 2.—*Motion of particles on smooth inclined planes, under the action of the constant accelerating force of gravity.*

270.] As the problem of particles moving on smooth inclined planes under the action of a constant force, and which, to fix our thoughts, I will take to be the resolved force of gravity, is the most simple in which a constant force is resolved, it is convenient to treat of it in this part of our work: yet as it properly belongs to the theory of constrained motion, we are unable to give a complete solution of it, until the principles of such motion have been explained in a future Chapter.

Let the smooth plane be inclined to the horizon at the angle $a$: and let OA, AB, fig. 110, be the sections of the inclined and horizontal planes made by the plane of the paper, which is supposed to be vertical and perpendicular to the line of intersection of the two planes.

Let P be the place of the particle $m$ at the time $t$, and suppose $m$ to be under the action of gravity: let $g$, as in Section 3, Chap. VIII, represent the velocity-increment impressed by the earth in one second of time, so that $mg$ represents the earth's impressed momentum on $m$ due to a second of time in its own vertical line of action: therefore the component of it along the plane OA is $mg \sin a$: let $\text{OP} = x$, and suppose $m$ to be moving down the plane, then the expressed momentum-increment of $m$ along the plane in an unit of time is $m\dfrac{d^2x}{dt^2}$; and as the plane and $m$ are smooth, there is no friction, and the impressed momentum-increment along the plane is equal to the expressed momentum-increment: therefore

$$m\frac{d^2x}{dt^2} = mg \sin a, \tag{42}$$

$$\frac{d^2x}{dt^2} = g \sin a; \tag{43}$$

$g \sin a$ being positive, because both $x$ and the velocity of $m$ increase as $t$ increases. Let the velocity of $m$ be $u$ when $t = 0$, therefore

$$\frac{dx}{dt} - u = gt \sin a,$$

$$\frac{dx}{dt} = u + gt \sin a; \tag{44}$$

whereby the velocity due to the time $t$ is known.

Also let $x = a$ when $t = 0$, therefore

$$x - a = ut + \frac{gt^2 \sin a}{2}; \tag{45}$$

$$x = a + ut + \frac{gt^2 \sin a}{2}; \tag{46}$$

whereby the distance due to the time $t$ is given.

If $m$ moves from rest when $t = 0$, and from O where $x = 0$, then (44) and (46) become,

$$\left.\begin{aligned} \frac{dx}{dt} &= gt \sin a \\ x &= \frac{gt^2 \sin a}{2} \end{aligned}\right\}. \tag{47}$$

Again, multiplying both sides of (43) by $2\,dx$, and integrating for the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{2\,dx\,d^2x}{dt^2} = 2g\,dx \sin a;$$

$$\therefore \quad \frac{dx^2}{dt^2} - u^2 = 2g(x-a)\sin a; \qquad (48)$$

and thus the velocity is given in terms of the space described.

If $m$ is at rest when $t = 0$, and also at O, which is the origin of distance, then

$$\frac{dx^2}{dt^2} = 2gx \sin a. \qquad (49)$$

Thus if OA, the length of the plane, is equal to $l$, and OB, the vertical projection of $l$, $= h$, then

$$\begin{aligned}(\text{velocity due to the plane})^2 &= 2gl \sin a \\ &= 2gh; \qquad (50)\end{aligned}$$

but $2gh$, see Art. 227, equation (49), is equal to the square of the velocity acquired by $m$ in falling down the altitude OB; therefore the velocity acquired by $m$ in falling down the plane, depends only on the vertical projection of the length of the plane, and not separately on its length or its angle of inclination; that is, depends only on the distance through which the force has acted in its own line of action. Therefore the velocity acquired by $m$ in falling down a plane is the same for all planes, the vertical heights of which are equal.

This is a particular case of the general law of vis viva, which will hereafter be shewn to rule generally in cases of dynamical action.

If $m$ is projected up the plane, and $x$ is measured from the bottom of the plane, and thus in the direction contrary to that in which the resolved part of gravity acts, so that, in fig. 110, AP $= x$, then

$$\frac{d^2x}{dt^2} = -g \sin a;$$

and if $t = 0$, when $m$ is at A, and if the velocity of projection $= u$, then

$$\frac{dx}{dt} = u - gt \sin a;$$

$$\frac{dx^2}{dt^2} = u^2 - 2gx \sin a;$$

$$x = ut - \frac{gt^2 \sin a}{2};$$

so that $m$ ascends until $\frac{dx}{dt} = 0$, in which case,

$$t = \frac{u}{g \sin a}, \qquad x = \frac{u^2}{2g \sin a}.$$

271.] If a circle is placed in a vertical plane, the times of descent down all chords drawn from the highest point are the same.

Let O, fig. 111, be the highest point of the circle OQA, which is supposed to be in a vertical plane: let $a =$ the radius, $\text{AOQ} = \theta$, therefore $\text{OQ} = 2a \cos \theta$; $\text{OP} = r$: then

$$\frac{d^2 r}{dt^2} = g \cos \theta, \qquad \frac{dr}{dt} = gt \cos \theta;$$

$$\therefore \quad r = 2a \cos \theta = \frac{gt^2 \cos \theta}{2}; \qquad \therefore \quad t = 2\left(\frac{a}{g}\right)^{\frac{1}{2}};$$

which is independent of $\theta$, and is therefore the same, whatever is the inclination of OQ to the vertical line OCA. Therefore the times of descent down all chords drawn from O the highest point are the same.

By reason of this property the circle is called the synchronous curve of all straight lines in a vertical plane passing through O.

Similarly it may be shewn that the times of descent down all chords drawn to A, the lowest point, are equal; that is, the time down QA is equal to that down OA.

If the plane of the circle is inclined to the horizon at an angle $i$, a similar property is true; for the resolved part of gravity along the diameter OA becomes $g \sin i$, of which the resolved part along OQ is $g \sin i \cos \theta$. Therefore using the same notation as in the preceding problem,

$$\frac{d^2 r}{dt^2} = g \sin i \cos \theta;$$

$$\therefore \quad r = 2a \cos \theta = \frac{gt^2}{2} \sin i \cos \theta;$$

$$\therefore \quad t = 2\left(\frac{a}{g \sin i}\right)^{\frac{1}{2}};$$

which is independent of $\theta$, and is therefore the same for all chords drawn from O, the highest point of the circle to the circumference.

Similarly it may be shewn that the times down all chords from any point Q on the circle to the lowest point A are equal: the circle therefore is the synchronous curves for a pencil of lines drawn, (1) from a given point O, (2) to a given point A, on an inclined plane.

272.] By help of the preceding property of the circle, whether in a vertical, or on an inclined plane, may many problems be solved, which involve the determination of planes drawn from given points and lines to other points and lines, and which are such that the times of descent down them may be maxima or minima. Some examples are subjoined, and the principle contained in them is equally applicable to all similar problems.

Ex. 1. To determine the plane of quickest descent from, (1) a given point to a given straight line, (2) a given straight line to a given point.

(1) Let A, fig. 112, be the given point, and BC the given straight line: the solution of the problem depends on the construction of a circle which passes through A, the diameter of which is vertical, and which touches the given straight line.

Through A draw the horizontal line AB: bisect the angle ABC by BO, which intersects in O the vertical line drawn through A: from O draw OP perpendicular to BC: then OP is manifestly equal to OA, and therefore the circle described from the centre O, and with the radius OA or OP will touch the line BC at P; join AP: AP is the required line of quickest descent.

For since the time is the same down all chords of the circle drawn from A, it is manifest that the time down any line other than AP from A to the line BC is longer than that down AP.

(2) Let A be the given point, fig. 113, and BC the given straight line: through A draw the vertical line AO, and the horizontal line AB; bisect the angle ABC by BO, meeting AO in O; from O draw OP at right angles to BC, and describe a circle from O as a centre with the radius equal to either OA or OP, which are evidently equal to each other: join PA: AP is manifestly the line of quickest descent from any point in BC to the point A.

Ex. 2. To determine the line of quickest descent, (1) from a point within a circle to the circle: (2) from a circle to a point without it.

(1) Let BPD, fig. 114, be the given circle, C its centre, and A the given point within it; through A draw the vertical line AO, and draw the vertical diameter BCD: join BA, and produce it to meet the circle in P: join CP, which intersects AO in O: then

OA is manifestly equal to OP, and therefore the circle described from O as a centre with the radius OA or OP will touch the given circle at P: and AP is manifestly the line of quickest descent.

(2) Let BPD, fig. 115, be the given circle, C its centre, and A the point without it: draw the vertical diameter BCD of the circle; and join BA cutting the circle in the point P: through A draw the vertical line AO, and draw the line CPO. From the geometry it is plain that OP = OA; and therefore the circle described from O as a centre, with the radius OP or OA, will touch the given circle in the point P: and thus PA is manifestly the straight line of quickest descent.

Ex. 3. To find the straight line of longest descent from a circle to a point without it, and which lies below the circle.

Let BPD be the circle, C its centre, fig. 116, D the lowest point of its vertical diameter BCD, and A the given point; join ADP, PC; and produce PC so as to intersect a vertical line through A in the point O: then the circle described from the centre O with the radius OA or OP manifestly touches the given circle at P, and the line AP is evidently that of the longest descent.

273.] Illustrative examples of the motion of a particle on an inclined plane.

Ex. 1. Of a parabola, whose axis is vertical and vertex downwards, to find that focal radius vector the time of descent down which is a minimum.

Let $4a =$ the latus rectum: and let $\theta$ be the angle between $r$ and the shortest focal distance: so that

$$r = \frac{2a}{1+\cos\theta}.$$

Now by (47),
$$r = \frac{gt^2\cos\theta}{2};$$

$$\therefore \quad \frac{gt^2}{4a} = \frac{1}{\cos\theta\,(1+\cos\theta)};$$

$$\therefore \quad \frac{gt}{2a}\,dt = \frac{\sin\theta\,(1+2\cos\theta)}{\{\cos\theta\,(1+\cos\theta)\}^2}\,d\theta = 0;$$

therefore $\sin\theta = 0$, and the sign of $\frac{dt}{d\theta}$ changes from $-$ to $+$; therefore $t$ is a minimum: so that the line from the focus to the vertex is that of quickest descent: also $\frac{dt}{d\theta} = 0$ when $\cos\theta = -\frac{1}{2}$, that is, when $\theta = 120°$: the radius vector corresponding to which is the line of quickest descent from the parabola to the vertex.

With reference to these and similar problems it may be observed, that we have here determined the position of that plane down which, of all drawn from a given point or line to another given line or point, the time of descent is the least or greatest. It must not however be hence inferred that a *straight* line is that for which of all lines straight or curved, joining two given points or two given curved lines, the time of descent is the least: we shall hereafter shew that the cycloid is the curve which, in vacuo and under the action of gravity, possesses this property of Brachistochronism (as it is called); and that the cycloid required cuts each of the given curves at right angles.

Ex. 2. To determine the inclination to the horizon of a smooth inclined plane, so that the time of descent of a particle $m$ down the length may be $n$ times that down the height of the plane.

Let $\theta$ = the inclination of the plane to the horizon,
$c$ = the length of the plane,
$b$ = the height of the upper end of the plane.

Therefore $b = c \sin\theta$. Now from (47),

$$\text{the time down the length of the plane} = \left(\frac{2c}{g\sin\theta}\right)^{\frac{1}{2}};$$

$$\text{and the time down the height} = \left(\frac{2b}{g}\right)^{\frac{1}{2}} = \left(\frac{2c\sin\theta}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \left(\frac{2c}{g\sin\theta}\right)^{\frac{1}{2}} = n\left(\frac{2c\sin\theta}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \sin\theta = \frac{1}{n}.$$

Ex. 3. It is required to shew that the times of descent down all the radii of curvature of the cycloid, fig. 105, Vol. I, are equal; that is, the time down PП is equal to that down BC.

Employing our usual notation,

$$x = a\,\text{versin}^{-1}\frac{y}{a} - (2ay - y^2)^{\frac{1}{2}};$$

$$\frac{dx}{y^{\frac{1}{2}}} = \frac{dy}{(2a-y)^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}},$$

$$\rho = \text{P}\Pi = 2(2ay)^{\frac{1}{2}};$$

$$\sin \text{PGO} = \frac{dx}{ds} = \left(\frac{y}{2a}\right)^{\frac{1}{2}};$$

$$\therefore \quad (\text{time down } \mathrm{P}\Pi)^2 = \frac{4(2ay)^{\frac{1}{2}}}{g(y)^{\frac{1}{2}}}(2a)^{\frac{1}{2}}$$
$$= \frac{8a}{g}$$
$$= (\text{time down BC})^2.$$

Ex. 4. To determine the form of a surface so that the times of descent to any point in it from two given points in the same vertical line may be equal.

It is evident that the surface is one of revolution about the given vertical line; we may therefore determine the curve by the revolution of which the surface is generated: and let us suppose the curve to be in the plane of $xz$: let the given vertical line be the axis of $z$; and let the two given points A and A′, fig. 117, on it be at a distance $2a$ apart: let O, the middle point of AA′, be the origin, and P be any point so that the time down AP is equal to that down A′P: then

$$\mathrm{OA} = \mathrm{OA'} = a; \qquad \mathrm{OM} = z, \qquad \mathrm{MP} = x,$$

$$(\text{time down AP})^2 = \frac{2.\mathrm{AP}}{g \sin \mathrm{APM}} = \frac{2.\mathrm{AP}^2}{g.\mathrm{AM}};$$

$$(\text{time down A'P})^2 = \frac{2.\mathrm{A'P}}{g.\sin \mathrm{A'PM}} = \frac{2.\mathrm{A'P}^2}{g.\mathrm{A'M}};$$

therefore by the conditions of the problem,

$$\frac{\mathrm{AP}^2}{\mathrm{AM}} = \frac{\mathrm{A'P}^2}{\mathrm{A'M}};$$

$$\therefore \quad \{x^2+(z+a)^2\}(z-a) = \{x^2+(z-a)^2\}(z+a);$$

$$\therefore \quad z^2-x^2 = a^2;$$

which is the equation to the equilateral hyperbola. And therefore the surface required is that which is generated by the revolution of an equilateral hyperbola about its transverse axis. The lower sheet is that to any point on which all straight lines drawn from A and A′ are lines down which the times of descent are equal: and the upper sheet is that from any point in which the lines drawn to A and A′ are those down which the times of descent are equal.

274.] Two smooth inclined planes, the inclinations of which to the horizon are respectively $\alpha$ and $\alpha'$, have a common vertex: on these are placed two smooth particles $m$ and $m'$, connected by a perfectly flexible and inextensible string, which passes over a small pulley placed at the common vertex of the planes: it is required to determine the motion of $m$ and $m'$.

Let the section of the two planes by the plane of the paper, which is supposed to be vertical, and to pass through the pulley and to be perpendicular to the line of intersection of the two planes, be represented in fig. 118: let us suppose the pulley at C to be so small that we may consider it to be (approximately) a point, and so that the strings CP and CP′ are parallel to the respective planes. Let $\text{CAA}' = a$, $\text{CA}'\text{A} = a'$, $\text{CP} = x$, $\text{CP}' = x'$, P and P′ being the places of $m$ and $m'$ at the time $t$: and to fix our thoughts let us suppose $m$ to be descending. Because the string is inextensible,

$$x + x' = \text{a constant},$$

$$\therefore \quad \frac{dx}{dt} + \frac{dx'}{dt} = 0, \qquad \frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} = 0;$$

that is, $\frac{d^2x'}{dt^2} = -\frac{d^2x}{dt^2}$: which result is also manifest by general reasoning. Now $m+m'$ is the whole mass moved: and $\frac{d^2x}{dt^2}$ is the velocity-increment expressed in the motion of each:

$$\therefore \quad (m+m')\frac{d^2x}{dt^2} = \text{the momentum-increment expressed.}$$

And $mg\sin a$, and $m'g\sin a'$ are the respective impressed momentum-increments along the planes: but as these act in opposite directions,

$$mg\sin a - m'g\sin a' = \text{the momentum-increment impressed;}$$

therefore

$$(m+m')\frac{d^2x}{dt^2} = (m\sin a - m'\sin a')\,g; \tag{51}$$

$$\frac{d^2x}{dt^2} = \frac{m\sin a - m'\sin a'}{m+m'}\,g. \tag{52}$$

Similarly for the equation of motion of $m'$, we have

$$\frac{d^2x'}{dt^2} = \frac{m'\sin a' - m\sin a}{m'+m}\,g. \tag{53}$$

If when $t = 0$, $m$ and $m'$ are at rest,

$$\frac{dx}{dt} = \frac{m\sin a - m'\sin a'}{m+m'}\,gt; \tag{54}$$

whereby the velocity acquired during the time $t$ is known.

Also multiplying both sides of (52) by $2dx$, and supposing the limits of the integral to be such that the velocity $= 0$, when $x = a$,

$$\frac{dx^2}{dt^2} = \frac{m \sin a - m' \sin a'}{m+m'} 2g(x-a). \tag{55}$$

And integrating (54) again, and taking the limits of integration such that $x = a$, when $t = 0$, we have

$$x - a = \frac{m \sin a - m' \sin a'}{m+m'} \frac{gt^2}{2}. \tag{56}$$

If the velocity, with which $m$ and $m'$ begin to move, is $u$, then if $x = a$, when $t = 0$,

$$\frac{dx}{dt} - u = \frac{m \sin a - m' \sin a'}{m+m'} gt, \tag{57}$$

$$x - a = ut + \frac{m \sin a - m' \sin a'}{m+m'} \frac{gt^2}{2}, \tag{58}$$

$$\frac{dx^2}{dt^2} - u^2 = \frac{m \sin a - m' \sin a'}{m+m'} 2gx. \tag{59}$$

As to the initial velocity $u$; suppose $m$ and $m'$ respectively to have the velocities $v$ and $v'$ down the corresponding planes; then if $u$ is the common velocity with which the two particles by reason of their connexion by means of the string begin to move, we have from the equality of the expressed and the impressed momenta,

$$(m+m')u = mv - m'v',$$

$$\therefore \quad u = \frac{mv - m'v'}{m+m'}. \tag{60}$$

The preceding formulæ are also applicable, whatever are the inclinations of the planes. Thus suppose the plane CA′ to be horizontal, then $a' = 0$, and

$$\frac{d^2x}{dt^2} = \frac{mg \sin a}{m+m'},$$

that is, $m'$ has no impressed momentum-increment; and if $a = 90°$, $m$ is then moving vertically downwards: this case is that of $m$ hanging by a string over the edge of a horizontal table, and drawing another body $m'$ which is on the table at the other end of the string.

If $a = a' = 90°$, we have the same formulæ as those which were investigated in Art. 229.

275.] Examples in illustration.

Ex. 1. A small ball $m$ descending vertically draws an equal ball 25 feet in 2.5 seconds up a plane inclined at 30° to the horizon, by means of a string passing over a pulley at the top: it is required to determine the force of gravity.

Here $a = 90°$, $a' = 30°$, $m = m'$;

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{g}{4}, \qquad \therefore \quad \frac{dx}{dt} = \frac{g}{4}t,$$

$$x = \frac{g}{8}t^2, \qquad 25 = \frac{g}{8}(2.5)^2, \qquad g = 32 \text{ feet}.$$

Ex. 2. Two smooth inclined planes are placed as in fig. 119: BC is inclined at 30°, and CA at 60° to the horizontal line OA: it is required to determine the distance CP through which $m$ moves in $t''$, $m'$ being equal to $m$.

The equation of motion becomes

$$2m\frac{d^2x}{dt^2} = mg(\sin 30° + \sin 60°);$$

$$\therefore \quad x = \frac{3^{\frac{1}{2}}+1}{8}gt^2.$$

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## SECTION 3.—*The determination of the law of accelerating forces, that a particle moving in vacuo may describe given curves.*

276.] In the present section our object is to determine the laws of accelerating force acting in assigned lines, as, for instance, parallel to the axes of $x$ or $y$, or subject to certain conditions, so that a particle may describe a given curve: that is, the trajectory being given under certain conditions, to determine the law of force. The principle on which such problems are to be treated have been so fully unfolded in Art. 251 that it is necessary for us now only to apply them to some examples.

Ex. 1. A particle moves in an elliptic orbit under the action of an accelerating force which is parallel to the minor axis; it is required to determine the law of that force.

Let the position of $m$ at the time $t$ be $(x, y)$; and let the equation to the elliptic path be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{61}$$

As no accelerating force acts along the axis of $x$, we have

$$\frac{d^2x}{dt^2} = 0; \qquad \therefore \quad \frac{dx}{dt} = a; \tag{62}$$

that is, the velocity of $m$ parallel to the axis of $x$ is always the same; and if we assume the initial circumstances to be such that the particle is at the extremity of the minor axis when $t = 0$, then from (62),

$$x = at. \tag{63}$$

Now our object is to determine the law of the impressed velocity-increment which is parallel to the axis of $y$, and which it is convenient to represent by Y. And

$$\text{Y} = \frac{d^2y}{dt^2};$$

From (61)

$$dy = -\frac{b^2x}{a^2y}\,dx,$$

$$\therefore\quad \frac{dy}{dt} = -\frac{b^2x}{a^2y}\,\frac{dx}{dt}$$

$$= -\frac{\alpha b^2}{a^2}\,\frac{x}{y},$$

$$\frac{d^2y}{dt^2} = -\frac{b^2\alpha}{a^2}\,\frac{y\dfrac{dx}{dt} - x\dfrac{dy}{dt}}{y^2};$$

$$\therefore\quad \text{Y} = -\frac{b^4\alpha^2}{a^2y^3}. \tag{64}$$

Therefore the accelerating force parallel to the axis of $y$ varies inversely as the cube of the ordinate of the ellipse, and is attractive, as is shewn by the negative sign.

Now when the particle has passed from the extremity of the minor axis to the extremity of the major axis, $x$ has increased from 0 to $a$; therefore the time to which this passage is due $= \dfrac{a}{\alpha}$; therefore the time in which $m$ passes through the whole ellipse is $\dfrac{4a}{\alpha}$.

If the path which $m$ describes is a circle, $b = a$; and if the acting accelerating force is, as in the preceding investigation, parallel to the axis of $y$,

$$\text{Y} = -\frac{a^2\alpha^2}{y^3}; \tag{65}$$

that is, varies inversely as the cube of the ordinate, and is attractive.

Ex. 2. Suppose $m$ to move in the same elliptic orbit, and under the action of an accelerating force which is parallel to the axis of $x$; it is required to determine the law of the force.

Since no accelerating force is parallel to the axis of $y$, $\dfrac{d^2y}{dt^2} = 0$; therefore

$$\frac{dy}{dt} = \beta,$$

$\beta$ being the constant velocity of $m$ parallel to the axis of $y$. And if $m$ is at the extremity of the major axis when $t=0$, then

$$y = \beta t.$$

Also from (61), $$dx = -\frac{a^2 y}{b^2 x} dy,$$

$$\therefore \quad \frac{dx}{dt} = -\frac{a^2\beta}{b^2}\frac{y}{x},$$

$$\frac{d^2x}{dt^2} = -\frac{a^2\beta}{b^2}\frac{x\frac{dy}{dt} - y\frac{dx}{dt}}{x^2};$$

and therefore if $\text{x}$ is the impressed velocity-increment parallel to the axis of $x$,

$$\text{x} = -\frac{a^4\beta^2}{b^2x^3}, \tag{66}$$

that is, the force parallel to the axis of $x$ is attractive and varies inversely as the cube of the abscissa.

In this case the periodic time in the orbit is $\frac{4b}{\beta}$.

In the circle, $$\text{x} = -\frac{a^2\beta^2}{x^3}. \tag{67}$$

Ex. 3. A particle $m$ moves in a parabola under the action of a force parallel to the principal axis of the parabola; it is required to determine its law.

Let the equation to the parabola be

$$y^2 = 4ax; \tag{68}$$

then since no accelerating force acts parallel to the axis of $y$, $\frac{d^2y}{dt^2} = 0$,

$$\therefore \quad \frac{dy}{dt} = \beta; \tag{69}$$

that is, the velocity of $m$ parallel to the axis of $y$ is constant: and if $m$ is at the vertex when $t=0$,

$$y = \beta t.$$

Let $\text{x}$ represent the accelerating force acting on $m$ parallel to the axis of $x$: then since from (68)

$$dx = \frac{y}{2a}dy, \qquad \therefore \quad \frac{dx}{dt} = \frac{\beta}{2a}y,$$

$$\frac{d^2x}{dt^2} = \frac{\beta}{2a}\frac{dy}{dt}$$

$$= \frac{\beta^2}{2a}; \tag{70}$$

$\therefore \quad \mathrm{X} = \frac{\beta^2}{2a}$; therefore X is constant, and also repulsive, because it carries a positive sign: the accelerating force therefore parallel to the axis of $x$ repels $m$ with a constant force from the line which touches the parabola at its vertex.

Hence also conversely, if a particle $m$ is projected with a velocity $\beta$ in a horizontal line, and is attracted towards the earth by the constant force of gravity in a vertical line, $m$ will move in a parabolic path.

By a similar process, if Y represents the accelerating force acting parallel to the axis of $y$, when no force parallel to the axis of $x$ acts; and if X represents the force acting parallel to the axis of $x$, when no force parallel to the axis of $y$ acts; and if $\alpha$ and $\beta$ are the constant velocities parallel to the axes of $x$ and $y$ respectively in each case; then, if a particle $m$ describes

(1) A Hyperbola, $xy = k^2$, $\quad \mathrm{X} = \frac{2\beta^2}{k^4} x^3$,

$\mathrm{Y} = \frac{2\alpha^2}{k^4} y^3$;

(2) A Parabola, $y^2 = 4ax$, $\quad \mathrm{Y} = -\frac{4a^2\alpha^2}{y^3}$;

(3) A Hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\quad \mathrm{X} = \frac{a^4\beta^2}{b^2x^3}$,

$\mathrm{Y} = -\frac{b^4\alpha^2}{a^2y^3}$;

(4) The Logarithmic Curve $y = a^x$, $\quad \mathrm{X} = -\frac{\beta^2}{\log a \times a^{2x}}$,

$\mathrm{Y} = \alpha^2 (\log a)^2 y$;

(5) The Cycloid, starting point being origin, $\mathrm{X} = \frac{\beta^2 ay}{(2ay - y^2)^{\frac{3}{2}}}$,

$\mathrm{Y} = -\frac{\alpha^2 a}{y^2}$;

(6) The Catenary, $y = \frac{a}{2}\{e^{\frac{x}{a}} + e^{-\frac{x}{a}}\}$ $\quad \mathrm{X} = -\frac{\beta^2 ay}{(y^2 - a^2)^{\frac{3}{2}}}$,

$\mathrm{Y} = \frac{\alpha^2}{a^2} y$.

277.] To determine the laws of force parallel to the axes of $x$ and $y$, so that a particle $m$ may describe the parabola with a constant velocity.

Let the equation to the parabola be

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}};$$

$$\therefore \quad \frac{dx}{x^{\frac{1}{2}}} = \frac{-dy}{y^{\frac{1}{2}}} = \frac{ds}{(x+y)^{\frac{1}{2}}}.$$

And let the constant velocity along the curve $= c$; so that

$$ds = c\,dt;$$

$$\therefore \quad \frac{dx^2}{dt^2} = \frac{c^2 x}{x+y}, \qquad \frac{dy^2}{dt^2} = \frac{c^2 y}{x+y},$$

and differentiating,

$$\frac{d^2x}{dt^2} = \frac{c^2 (ay)^{\frac{1}{2}}}{2(x+y)^2}, \qquad \frac{d^2y}{dt^2} = \frac{c^2 (ax)^{\frac{1}{2}}}{2(x+y)^2}. \tag{71}$$

If however we take the system of tangential and normal resolution, then as the velocity along the curve is constant, $= c$, say, so the accelerating force along the curve is zero: and therefore

$$\frac{d^2s}{dt^2} = \text{T} = 0.$$

Also, $$\text{N} = \frac{v^2}{\rho} = \frac{c^2}{\rho};$$

in the parabola whose equation is given above,

$$\rho = \frac{2(x+y)^{\frac{3}{2}}}{a^{\frac{1}{2}}};$$

$$\therefore \quad \text{N} = \frac{c^2 a^{\frac{1}{2}}}{2(x+y)^{\frac{3}{2}}}, \tag{72}$$

and which is also the same as $\left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right\}^{\frac{1}{2}}$, if we substitute for these latter quantities their values given in (71).

278.] From the vertex of a parabola a particle $m$ is projected with a velocity $u$ at right angles to the principal axis, and is acted on by an attractive force which is perpendicular to that axis and varies directly as the distance of the particle from it. It is required to determine the law of force acting parallel to the axis of $x$ so that the particle may move in the parabola, and the other circumstances of motion.

Let the vertex be the origin, and the principal axis and the tangent at the vertex be the axes of $x$ and $y$; and let P, $(x, y)$, fig. 120, be the position of $m$ at the time $t$, so that the equation to the parabola is

$$y^2 = 4ax;$$

$$\therefore \quad \frac{dy}{dx} = \frac{2a}{y} = \left(\frac{a}{x}\right)^{\frac{1}{2}}.$$

By the conditions of the problem,

$$\frac{d^2x}{dt^2} = \text{X}, \text{ which is to be determined,}$$

$$\frac{d^2y}{dt^2} = -\mu y; \tag{73}$$

therefore multiplying (73) by $2dy$, integrating, and taking the limits corresponding to $t = t$ and to $t = 0$,

$$\frac{dy^2}{dt^2} - u^2 = -\mu y^2,$$

$$\therefore \quad \frac{dy^2}{dt^2} = u^2 - \mu y^2 \tag{74}$$

$$= u^2 - 4\mu a x;$$

$$\therefore \quad \frac{dx^2}{dt^2} = \frac{u^2 x - 4\mu a x^2}{a}; \tag{75}$$

$$\therefore \quad \frac{d^2x}{dt^2} = \text{X} = \frac{u^2}{2a} - 4\mu x; \tag{76}$$

therefore the required force which is parallel to the axis of $x$ partly is constant and repulsive, and partly varies as the abscissa and is attractive towards the axis of $x$.

Also from (74) and (75) $\frac{dy}{dt}$ and $\frac{dx}{dt}$ both vanish, when

$$y = \pm \frac{u}{\mu^{\frac{1}{2}}}, \quad \text{and} \quad x = \frac{u^2}{4\mu a}; \tag{77}$$

so that at this point, say B in the figure, $m$ comes to rest; and afterwards under the action of the forces returns to the vertex of the parabola, through which it passes with the original velocity $u$, and comes to rest again at B′, which is equidistant with B from the vertex: thus the motion is oscillatory. Also from (74),

$$\frac{dy}{(u^2 - \mu y^2)^{\frac{1}{2}}} = dt,$$

taking the positive sign, as we will consider the original motion from O to B. Therefore integrating, and taking the limits corresponding to $t = t$ and to $t = 0$,

$$\sin^{-1} \frac{y(\mu)^{\frac{1}{2}}}{u} = \mu^{\frac{1}{2}} t;$$

$$\therefore \quad y = \frac{u}{\mu^{\frac{1}{2}}} \sin \mu^{\frac{1}{2}} t; \tag{78}$$

therefore the time from O to B is $\frac{\pi}{2\mu^{\frac{1}{2}}}$, and therefore the time of an oscillation, viz. from B to B′, $= \frac{\pi}{\mu^{\frac{1}{2}}}$, which is independent of the velocity of projection from O, and depends on only the absolute force parallel to the axis of $y$.

I have, for the sake of simplicity, taken the parabola for the example whereby the process may be illustrated, but the method is the same in all cases. Thus in the ellipse, if $m$ is projected from the extremity of the major axis with the velocity $u$,

$$\text{X} = \frac{a^2 \mu y^2 (a^2+x^2)}{b^2 x^3} - \frac{a^4 u^2}{b^2 x^3},$$

and the coordinates of the point B to which $m$ passes are

$$y = \frac{u}{\mu^{\frac{1}{2}}}, \qquad x = a - \frac{a}{b}\left(b^2 - \frac{u^2}{\mu}\right)^{\frac{1}{2}};$$

therefore $m$ comes to rest at the extremity of the minor axis, if $u = \mu^{\frac{1}{2}} b$.

279.] A particle $m$ describes a helix with a constant velocity; it is required to determine the laws of the accelerating forces which act on it parallel to the three coordinate axes.

Let the equations to the curve be

$$\left.\begin{aligned} x &= a\cos\phi, \\ y &= a\sin\phi, \\ z &= ka\phi, \end{aligned}\right\} \qquad \therefore \quad \begin{aligned} dx &= -a\sin\phi\, d\phi, \\ dy &= a\cos\phi\, d\phi, \\ dz &= ka\, d\phi; \end{aligned}$$

$$\therefore \quad ds^2 = a^2(1+k^2)\, d\phi^2.$$

But since the velocity along the curve is constant, $ds = c\,dt$;

$$\therefore \quad \frac{d\phi}{dt} = \frac{c}{a(1+k^2)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{dx}{dt} = -\frac{c\sin\phi}{(1+k^2)^{\frac{1}{2}}}, \quad \frac{dy}{dt} = \frac{c\cos\phi}{(1+k^2)^{\frac{1}{2}}}, \quad \frac{dz}{dt} = \frac{kc}{(1+k^2)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{c^2 x}{a^2(1+k^2)}, \quad \frac{d^2y}{dt^2} = -\frac{c^2 y}{a^2(1+k^2)}, \quad \frac{d^2z}{dt^2} = 0.$$

Hence we infer that the accelerating forces parallel to the axes of $x$ and $y$ vary directly as $x$ and $y$ respectively, and are attractive, and have the same absolute force; that is, the resultant force of these two forces will have a line of action always passing through the axis of $z$, and will be constant. Also the accelerating force parallel to the axis of $z$ vanishes, and the velocity of $m$ parallel to that axis is constant.

## SECTION 4.—*The determination of the curvilinear paths described by particles moving in vacuo under the action of given accelerating forces.*

280.] In the preceding section laws of force have been found when the equation to the trajectories, and some other condition, are given: it is our object now to investigate the equation to the trajectory when the law of force is given: and I shall take first the simple case of a particle moving in vacuo under the action of gravity which is a constant accelerating force, and the line of action of which is always vertical. The projected body or particle is called a projectile, and the problem is in this case that of the motion of a projectile in vacuo.

Let $m$ be the mass of the projectile; and let us prove, in the first place, that the particle during the motion is always in one and the same plane.

Let the horizontal plane passing through the point, whence the particle is projected, be the plane of $xy$, so that the axis of $z$ is parallel to the line of action of gravity: also let the velocity of projection of the particle be $u$; and let it be projected in a line which makes an angle $\beta$ with the axis of $z$, and of which the projection on the plane of $xy$ makes an angle $a$ with the axis of $x$; so that the three components of $u$ along the three rectangular axes are

$$u \sin\beta \cos a, \qquad u \sin\beta \sin a, \qquad u \cos\beta.$$

Now the components of the impressed accelerating force give the following equations:

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = 0, \qquad \frac{d^2z}{dt^2} = -g; \tag{79}$$

therefore integrating the first two, and taking for the limits of integration the values which correspond to $t = t$ and to $t = 0$, we have from (79),

$$\frac{dx}{dt} - u \sin\beta \cos a = 0, \qquad \frac{dy}{dt} - u \sin\beta \sin a = 0,$$

$$\therefore \quad x = ut \sin\beta \cos a, \qquad y = ut \sin\beta \sin a;$$

$$\therefore \quad \frac{x}{\cos a} = \frac{y}{\sin a};$$

which is the equation to a plane perpendicular to the plane of $xy$, and containing the axis of $z$; therefore the moving particle is always in the vertical plane, which is inclined at the angle $a$ to the plane of $xz$.

281.] Let us take the plane in which the particle moves to be that of $xy$: let the point of projection, fig. 121, be the origin: let the axis of $x$ be horizontal, that of $y$ vertical: let the velocity of projection $= u$, and let the line of projection be inclined at an angle $a$ to the axis of $x$, so that $u \cos a$ and $u \sin a$ are the resolved parts of the velocity of projection along the coordinate axes of $x$ and $y$.

Let P be the position of $m$ at the time $t$, OM $= x$, MP $= y$; $g =$ the accelerating force of gravity which acts parallel to the axis of $y$; therefore $mg$ is the impressed momentum-increment; so that the equations of motion are,

$$m\frac{d^2x}{dt^2} = 0, \qquad m\frac{d^2y}{dt^2} = -mg; \tag{80}$$

the latter being affected with a negative sign, because the tendency of gravity is to make the velocity increase, and $y$ to decrease, as $t$ increases. Therefore

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g. \tag{81}$$

Now integrating these, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{dx}{dt} - u\cos a = 0; \tag{82}$$

$$x = ut\cos a; \tag{83}$$

$$\frac{dy}{dt} - u\sin a = -gt; \tag{84}$$

$$y = ut\sin a - \frac{gt^2}{2}. \tag{85}$$

From (83),
$$t = \frac{x}{u\cos a}, \tag{86}$$

and substituting this value of $t$ in (85), we have

$$y = x\tan a - \frac{gx^2}{2u^2(\cos a)^2}. \tag{87}$$

And this equation is that of a parabola; whence it follows that a parabola is the trajectory of the particle. And (87) may be put into the form

$$\left(x - \frac{u^2\cos a\sin a}{g}\right)^2 = \frac{2u^2(\cos a)^2}{g}\left\{\frac{u^2(\sin a)^2}{2g} - y\right\}; \tag{88}$$

so that we have

(1) the abscissa to the vertex $= \dfrac{u^2 \sin a \cos a}{g}$; (89)

(2) the ordinate to the vertex $= \dfrac{u^2 (\sin a)^2}{2g}$; (90)

(3) the latus rectum $= \dfrac{2u^2 (\cos a)^2}{g}$. (91)

Also the form of the equation (88) shews that the parabola is placed with its axis vertical as in the figure, and that the vertex is the highest point of the curve.

The distance OB between the point of projection and the point where the projectile strikes the horizontal plane is called *the Range* on the horizontal plane, and is the value of $x$ when $y = 0$; that is, putting $y = 0$ in (87),

$$\text{the range} = \text{OB} = \frac{u^2 \sin 2a}{g}; \tag{92}$$

also, as is geometrically manifest, OB = 2 OC, that is, the range is equal to twice the abscissa to the vertex.

From (92) it appears that for a given velocity of projection, the range is the greatest when $a = 45°$, in which case the range $= \dfrac{u^2}{g}$; and the focus of the parabola in this case lies in the horizontal line drawn through the point of projection.

Also from the value of the range (92) it appears, that if $u$ is the same, the range is unaltered when $a$ is replaced by its complementary angle: so that the range is the same for two different angles which are complementary of each other, if the velocity of projection is the same: hence if $a = 45°$, these two angles become identical, and the range is a maximum.

CA is called *the altitude* or the greatest height of the projectile, and is the value of $y$ when $\dfrac{dy}{dx} = 0$; therefore from (87),

$$\text{the greatest height} = \frac{u^2 (\sin a)^2}{2g}. \tag{93}$$

Also from (83), $x = ut \cos a$; (94)

that is, the abscissa uniformly increases along OM. Hence if we substitute the range for $x$, we shall have an expression for the time which a particle takes in passing from O to B, and which is called *the time of flight*: and thus

$$\text{the time of flight} = \frac{2u \sin a}{g}.$$

282.] Again multiplying (81) respectively by $2dx$ and $2dy$, and integrating, and taking the limits corresponding to $t=t$ and to $t=0$, we have

$$\frac{dx^2}{dt^2} = u^2 (\cos a)^2; \tag{95}$$

$$\frac{dy^2}{dt^2} = u^2 (\sin a)^2 - 2gy; \tag{96}$$

$$\therefore \quad \frac{ds^2}{dt^2} = (\text{the velocity})^2 = u^2 - 2gy. \tag{97}$$

Now this result deserves notice. Let the directrix of the parabolic trajectory be drawn as in the figure: then

$$\text{AD} = \frac{1}{4}\text{th of the latus rectum} = \frac{u^2 (\cos a)^2}{2g};$$

and since by (90), $\text{CA} = \frac{u^2 (\sin a)^2}{2g}$, therefore $\text{CD} = \frac{u^2}{2g}$; that is, see equation (49), Art. 227, CD is the vertical height through which a particle falling in vacuo will acquire the velocity with which the particle $m$ moves at its projection from O.

Let $\text{CD} = h$; therefore $u^2 = 2gh$; and substituting in (97),

$$\frac{ds^2}{dt^2} = 2g(h-y); \tag{98}$$

that is, the velocity at any point P on the curve is that which would be acquired by a heavy particle falling freely in vacuo down a vertical height equal to $h-y$, that is, to SP. Hence it follows that the velocity of $m$ at any point P in its path is that which would be acquired by a particle falling freely from the directrix to the curve. The directrix of the parabola is therefore determined by the velocity of projection, and is at a vertical distance above the point of projection equal to that down which a particle falling would have the velocity of projection. Hence also the vis viva of the projectile is at every point of the path the same as that of an equal particle acquired in falling from the directrix to the point in the curve.

283.] The equation to the path of the projectile may also be found by the following process: and as the result of simultaneous velocities taking place in combination is well exemplified by it, I do not hesitate to insert it.

Let the particle $m$ be projected from O, fig. 121, with a velocity $u$, in the line OQ making an angle $a$ with the horizontal line; then if no force acted to impress velocity on $m$, it would in the time $t$ describe a space OQ $= ut$, and its coordinates at the time $t$ would be,

$$x = ut \cos a, \qquad y = ut \sin a. \tag{99}$$

But as gravity is a constant force, and acts in a line parallel to the axis of $y$, and tends to diminish $y$ according to the arrangement which we have assumed in the figure, $y$ will by it be diminished in the time $t$ by a quantity equal to $\frac{gt^2}{2}$, see equation (48), Art. 227, so that at the end of the time $t$ we have,

$$x = ut \cos a, \qquad y = ut \sin a - \frac{gt^2}{2}; \tag{100}$$

which values are the same as (83) and (85); and therefore it appears, that if P is the place of $m$ in the parabolic path at the time $t$, QP $= \frac{gt^2}{2}$.

284.] A particle $m$ is projected from a given point on an inclined plane in a given line; it is required to determine the point at which it will strike the plane.

Let the angle of inclination of the plane to the horizon be $i$: let $a =$ the angle between the line of projection and the horizontal line: $u =$ the velocity of projection: then the equations to the inclined plane and to the path of the projectile respectively are,

$$y = x \tan i,$$

$$y = x \tan a - \frac{gx^2}{2u^2 (\cos a)^2};$$

whence eliminating $y$, we have

$$x = \frac{2u^2}{g} (\cos a)^2 (\tan a - \tan i)$$

$$= \frac{2u^2 \cos a \sin (a-i)}{g \cos i};$$

$$\therefore \quad y = \frac{2u^2 \cos a \tan i \sin (a-i)}{g \cos i};$$

which give the point on the plane at which the projectile strikes it: and the range on the plane is equal to $x \sec i$, that is,

$$\text{the range} = \frac{2u^2 \cos a \sin (a-i)}{g (\cos i)^2}. \tag{101}$$

Also the range is the greatest when

$$a = \frac{1}{2}\left\{\frac{\pi}{2} + i\right\},$$

$$a - i = \frac{1}{2}\left\{\frac{\pi}{2} - i\right\}; \qquad (102)$$

which latter value gives the angle between the plane and the line of projection for which the range is the greatest: and in this case

$$\text{the greatest range} = \frac{u^2}{g(1+\sin i)}. \qquad (103)$$

Hence it follows that if from a given point a system of straight lines is drawn in the same vertical plane, and particles are projected with a given velocity $u$, and in such lines that the ranges on the planes are the greatest, the locus of the extreme points of these ranges is given by the equation (103); and therefore if $u$ is the velocity of projection, $r =$ the range, $\theta =$ the angle between the plane and the vertical line through the point of projection, then from (103),

$$r = \frac{u^2}{g(1+\cos\theta)}; \qquad (104)$$

and if $h$ is the vertical distance to which $u$ is due

$$r = \frac{2h}{1+\cos\theta}; \qquad (105)$$

which is the equation to a parabola, the focus of which is at the origin of coordinates, whose axis is vertical, and of which $4h$ is the latus rectum.

286.] It is required to determine the angle of projection for a given velocity so that a projectile may pass through a given point.

Let the given point be $(x_1, y_1)$: then these coordinates satisfy the equation of the path of the projectile, and we have

$$y_1 = x_1 \tan a - \frac{g x_1^2}{2u^2}\{1 + (\tan a)^2\};$$

$$\therefore \tan a = \frac{u^2}{g x_1} \pm \frac{1}{g x_1}\{u^4 - 2u^2 g y_1 - g^2 x_1^2\}^{\frac{1}{2}}; \qquad (106)$$

therefore two different values of $a$ satisfy the condition, if

$$u^4 \text{ is greater than } 2u^2 g y_1 + g^2 x_1^2;$$

and only one value of $a$ satisfies it, if

$$u^4 = 2u^2 g y_1 + g^2 x_1^2;$$

that is, if
$$x_1^2 = \frac{2u^2}{g}\left\{\frac{u^2}{2g} - y_1\right\}; \qquad (107)$$

and the projectile cannot reach the point, if

$$u^4 \text{ is less than } 2u^2gy_1+g^2x_1^2.$$

Now (107) is the equation to a parabola of which o, fig. 121, is the focus, $\frac{2u^2}{g}$ is the latus rectum, and $\frac{u^2}{2g}$ or $h$, see Art. 282, is the distance to the highest point; all points therefore on this parabola are the farthest which the projectile can reach; all points without it are beyond the reach; and all points within it may be reached by two different angles of projection. The same result may also evidently be proved by the following process.

It is required to find the envelope of all parabolas described by particles projected with a given velocity $u$ from a given point in the same vertical plane.

The equation to the path of one is

$$y = x\tan\alpha - \frac{gx^2}{2u^2}\{1+(\tan\alpha)^2\}; \tag{108}$$

therefore differentiating by making $\tan\alpha$ to vary, we have

$$0 = \left(x - \frac{gx^2}{u^2}\tan\alpha\right)d.\tan\alpha;$$

$$\therefore \quad \tan\alpha = \frac{u^2}{gx};$$

so that (108) becomes

$$x^2 = \frac{2u^2}{g}\left\{\frac{u^2}{2g} - y\right\};$$

which result is of course the same as (107).

286.] Problems in illustration of the preceding equations.

Ex. 1. To determine the angle of projection so that the area contained between the path of the projectile and the horizontal line may be a maximum.

Since the area of a parabola is two-thirds of that of the circumscribed rectangle, see Ex. 1, Art. 118, Vol. II, therefore if $\mathrm{A}$ represents the required area,

$$\begin{aligned}\mathrm{A} &= \frac{2}{3}\,\text{range} \times \text{greatest altitude}\\ &= \frac{2u^4}{3g^2}(\sin\alpha)^3\cos\alpha;\end{aligned}$$

$$\therefore \quad \frac{d\mathrm{A}}{d\alpha} = \frac{2u^4}{3g^2}(\sin\alpha)^2\{3(\cos\alpha)^2-(\sin\alpha)^2\} = 0,$$

if $\tan\alpha = 3^{\frac{1}{2}}$, and changes sign from + to −: therefore the area is a maximum and $= \frac{u^4 3^{\frac{1}{2}}}{8g^2}$, if $\alpha = 60°$.

Ex. 2. It is required to compare the areas of the two parabolas described by projectiles, of which the horizontal ranges are the same, and the angles of projection are therefore complementary to each other.

Let $A_1$ and $A_2$ be the areas: then as the ranges are equal, these are to each other as the greatest altitudes: therefore

$$\frac{A_1}{A_2} = \frac{(\sin\alpha)^2}{(\cos\alpha)^2} = (\tan\alpha)^2.$$

Ex. 3. From the top of a tower two particles are projected at angles $\alpha$ and $\beta$ to the horizon with the same velocity $u$, and both strike the horizontal plane passing through the bottom of the tower at the same point; it is required to find the height of the tower.

Let $h =$ the height of the tower; $u =$ the velocity of projection: then if the particles are projected from the edge of the top of the tower, and $x$ is the distance from the bottom of the tower to the point where they strike the horizontal plane,

$$-h = x\tan\alpha - \frac{gx^2}{2u^2}\{1 + (\tan\alpha)^2\}, \tag{109}$$

$$-h = x\tan\beta - \frac{gx^2}{2u^2}\{1 + (\tan\beta)^2\}; \tag{110}$$

therefore by subtraction,

$$x = \frac{2u^2}{g(\tan\alpha + \tan\beta)} = \frac{2u^2\cos\alpha\cos\beta}{g\sin(\alpha+\beta)};$$

substituting which in either (109) or (110), we have

$$h = \frac{2u^2\cos\alpha\cos\beta\cos(\alpha+\beta)}{g\{\sin(\alpha+\beta)\}^2}.$$

Ex. 4. Particles are projected with a given velocity in all lines in a vertical plane from the point O: it is required to find the locus of them at a given time $t$.

From (83) and (85) we have,

$$x = ut\cos\alpha, \qquad y = ut\sin\alpha - \frac{gt^2}{2};$$

$$\therefore \quad \cos\alpha = \frac{x}{ut}, \qquad \sin\alpha = \frac{y + \frac{gt^2}{2}}{ut};$$

therefore squaring and adding, we have

$$x^2 + \left(y + \frac{gt^2}{2}\right)^2 = u^2t^2;$$

the equation to a circle of which the radius is $ut$, and the centre is on the axis of $y$ at a distance $\frac{gt^2}{2}$ below the origin.

Ex. 5. Particles are projected from o with a given velocity in all lines in a vertical plane: it is required to find the locus of their highest points.

Let $x$ and $y$ be the coordinates to the highest point: then from (89) and (90),

$$x = \frac{u^2 \sin a \cos a}{g}, \qquad y = \frac{u^2 (\sin a)^2}{2g};$$

$$\therefore \quad (\sin a)^2 = \frac{2gy}{u^2}, \qquad (\cos a)^2 = \frac{gx^2}{2u^2y};$$

therefore adding,

$$4y^2 + x^2 = \frac{2u^2y}{g};$$

which is the equation to an ellipse, of which the major axis $= \frac{u^2}{g}$, and the minor axis $= \frac{u^2}{2g}$; and the origin is at the extremity of the minor axis.

The preceding investigations into the motion of projectiles would explain the theory of gunnery, if it were allowable to neglect the resistance of the air; but as the velocity with which balls and shells traverse their paths is very great, so is much of their momentum lost by the resistance of the medium; and the ratio of the vertical and horizontal velocities is so much altered, that the form of the trajectory is very different from that of a parabolic path. In the last Section of the present Chapter we shall investigate, as far as it is possible, the path, and shall take account of the loss of momentum which is due to the resistance of the medium.

We proceed now to other cases of curvilinear motion in vacuo; and I would observe, once for all, that if a particle is projected with a given velocity in a plane, and if the lines of action of the forces, which act on the particle, are in that plane, the particle is during its motion in that plane: this is evident by the principle of sufficient reason.

287.] From a given point in the axis of $y$ a particle is projected with a velocity $u$ in a line parallel to the (rectangular) axis of $x$, and is attracted by a force the intensity of which varies directly as the distance, and which has its source in the origin of coordinates: it is required to find the equation of the path of the particle, and to define the circumstances of motion.

Let $b$ be the distance on the axis of $y$ from the origin of the point whence the particle is projected with the velocity $u$: let P, fig. 122, be the place of $m$ at the time $t$, B its place when $t = 0$; $\text{OM} = x$, $\text{MP} = y$, $\text{OP} = r$, $\text{OB} = b$; and let $\mu$ be the absolute force of the attraction at O. Then the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\mu r\frac{x}{r} = -\mu x \\ \frac{d^2y}{dt^2} &= -\mu r\frac{y}{r} = -\mu y \end{aligned}\right\}. \tag{111}$$

Now multiplying these equations respectively by $2dx$ and $2dy$, and integrating, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{dx^2}{dt^2} - u^2 = -\mu x^2; \tag{112}$$

$$\frac{dy^2}{dt^2} = -\mu(y^2 - b^2); \tag{113}$$

therefore adding,

$$\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} = \frac{ds^2}{dt^2} = u^2 - \mu(x^2 + y^2 - b^2); \tag{114}$$

which gives the velocity of $m$ at any point of its path. Also from (112) and (113),

$$\frac{dx}{\left(\frac{u^2}{\mu} - x^2\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}dt = \frac{-dy}{(b^2 - y^2)^{\frac{1}{2}}}; \tag{115}$$

$$\therefore \quad x = \frac{u}{\mu^{\frac{1}{2}}}\sin \mu^{\frac{1}{2}}t; \tag{116}$$

$$y = b\cos \mu^{\frac{1}{2}}t; \tag{117}$$

whence we have

$$\frac{\mu x^2}{u^2} + \frac{y^2}{b^2} = 1, \tag{118}$$

which is the equation to an ellipse whose $x$-axis is $\frac{2u}{\mu^{\frac{1}{2}}}$, and whose $y$-axis is $2b$, and whose centre is at the origin, that is, at the source of the force.

From the preceding values we have

$$\frac{ds^2}{dt^2} = (\text{velocity})^2 = u^2 (\cos \mu^{\frac{1}{2}} t)^2 + \mu b^2 (\sin \mu^{\frac{1}{2}} t)^2 ; \qquad (119)$$

and from (116) and (117), the time from B to A $= \dfrac{\pi}{2\mu^{\frac{1}{2}}}$; therefore the whole periodic time $= \dfrac{2\pi}{\mu^{\frac{1}{2}}}$, and is therefore independent of the velocity and distance of projection, and involves only the absolute force of the impressed force.

If the force at O had been repulsive, then the sign of $\mu$ would be changed, and the equations of motion would be

$$\frac{d^2x}{dt^2} = \mu x, \qquad \frac{d^2y}{dt^2} = \mu y;$$

$$\therefore \quad \frac{dx^2}{dt^2} - u^2 = \mu x^2, \qquad \frac{dy^2}{dt^2} = \mu (y^2 - b^2);$$

$$\frac{dx}{\left(\dfrac{u^2}{\mu} + x^2\right)^{\frac{1}{2}}} = \mu^{\frac{1}{2}} dt = \frac{dy}{(y^2 - b^2)^{\frac{1}{2}}};$$

$$\therefore \quad \log \frac{x\mu^{\frac{1}{2}} + (u^2 + \mu x^2)^{\frac{1}{2}}}{u} = \mu^{\frac{1}{2}} t = \log \frac{y + (y^2 - b^2)^{\frac{1}{2}}}{b};$$

$$\therefore \quad \left.\begin{aligned} \frac{2x\mu^{\frac{1}{2}}}{u} &= e^{\mu^{\frac{1}{2}} t} - e^{-\mu^{\frac{1}{2}} t}, \\ \frac{2y}{b} &= e^{\mu^{\frac{1}{2}} t} + e^{-\mu^{\frac{1}{2}} t}, \end{aligned}\right\}$$

whence squaring, and subtracting the former from the latter, we have

$$\frac{y^2}{b^2} - \frac{\mu x^2}{u^2} = 1, \qquad (120)$$

which is the equation to a hyperbola with its centre at the origin; and which might have been deduced from (118) by affecting $\mu$ with a negative sign.

If, in the case of the force being attractive, the velocity and distance of projection are such that $u = b\mu^{\frac{1}{2}}$, the path which $m$ describes is a circle, and the velocity in it is constant and equal to that of projection.

288.] A particle $m$ is projected from a given point with a given velocity, and is acted on by a force which varies inversely as the square of the distance from a given point which is its source; it is required to determine the path of the particle, and the other circumstances of motion.

Let $u$ be the velocity of projection, $\alpha =$ the angle between the line of projection and the axis of $x$, so that the components of the velocity of projection along the axes are $u\cos\alpha$ and $u\sin\alpha$; let $(a, b)$ be the point of projection, $\mu =$ the absolute force, and let $a^2+b^2=c^2$: let $r$ be the distance of $m$ at the time $t$ from the centre of force, which we will take to be the origin, and let $(x, y)$ be the place of $m$ at the time $t$; then the equations of motion are,

$$\frac{d^2x}{dt^2} = -\frac{\mu x}{r^3}, \qquad \frac{d^2y}{dt^2} = -\frac{\mu y}{r^3}, \tag{121}$$

which are simultaneous differential equations, and from which the solution of the problem is to be obtained. Multiply (121) respectively by $y$ and $x$, and subtract; then

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0,$$

and adding and subtracting $\frac{dx}{dt}\,\frac{dy}{dt}$, and integrating, and taking definite integrals with limits corresponding to $t=t$ and to $t=0$, we have

$$x\frac{dy}{dt} - y\frac{dx}{dt} = u\,(a\sin\alpha - b\cos\alpha) \tag{122}$$

$$= h \text{ (say)}. \tag{123}$$

Again, from (123) and the first of (121), we have

$$\begin{aligned} h\frac{d^2x}{dt^2} &= -\frac{\mu x}{r^3}\,\frac{x\,dy - y\,dx}{dt} \\ &= \frac{-\mu}{r^3}\,\frac{x^2\,dy - xy\,dx}{dt}; \end{aligned}$$

but $x^2+y^2=r^2$; therefore $x\,dx+y\,dy=r\,dr$;

$$\begin{aligned} \therefore\quad h\frac{d^2x}{dt^2} &= -\frac{\mu}{r^3}\,\frac{x^2\,dy - y\,(r\,dr - y\,dy)}{dt} \\ &= -\frac{\mu}{r^3}\,\frac{r^2\,dy - ry\,dr}{dt} \\ &= -\mu\frac{d}{dt}\frac{y}{r}; \end{aligned}$$

$$\therefore\quad h\frac{dx}{dt} - hu\cos\alpha = -\frac{\mu y}{r} + \frac{\mu b}{c}; \tag{124}$$

similarly, from the second of (121),

$$h\frac{dy}{dt} - hu\sin\alpha = \frac{\mu x}{r} - \frac{\mu a}{c}. \tag{125}$$

Multiplying (125) by $x$, and (124) by $y$, and subtracting, we have

$$h\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) - hu\,(x\sin\alpha - y\cos\alpha) = \mu r - \frac{\mu}{c}(ax+by);$$

therefore by (123)

$$h^2 - hu(x\sin\alpha - y\cos\alpha) = \mu r - \frac{\mu}{c}(ax+by), \tag{126}$$

which is of the form, $r = \text{A}x + \text{B}y + \text{C}$; (127)

and as $r$ is a rational function of the coordinates $x$ and $y$, the equation is that of a conic of which the focus is the origin. A conic therefore is the trajectory, with the source of the impressed force in the focus; and the constants A, B, C are given in terms of the velocity, the direction of the line of action, and the coordinates of the point, of projection.

Also from (121), multiplying respectively by $2\,dx$ and by $2\,dy$, and adding and integrating with the limits assigned above,

$$\frac{ds^2}{dt^2} - u^2 = \frac{2\mu}{r} - \frac{2\mu}{c}, \tag{128}$$

whereby the velocity is given at any point of the curve.

This problem and that contained in the last Article are cases where the impressed forces are central; and I have selected them for the purpose of illustrating the method of coordinate resolution. As central forces however have such large application in nature, it is necessary to consider their properties at greater length, and the following Chapter will contain the development of them: we need not therefore pursue at present the inquiry which the preceding problems suggest.

289.] From a given point in the axis of $y$ a particle is projected with a given velocity in a line parallel to the axis of $x$, and is acted on by an attractive force parallel to the axis of $y$ and which varies as the distance from the axis of $x$; it is required to determine the circumstances of motion.

Let $b$ = the distance from the origin of the point of projection,
$u$ = the velocity of projection;

so that the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -\mu y,$$

$$\therefore \quad \frac{dx}{dt} = u, \qquad \frac{dy^2}{dt^2} = \mu(b^2-y^2),$$

$$x = ut, \qquad \frac{-dy}{(b^2-y^2)^{\frac{1}{2}}} = \mu^{\frac{1}{2}}\,dt;$$

$$\therefore \quad \cos^{-1}\frac{y}{b} = \mu^{\frac{1}{2}} t,$$

$$y = b\cos\mu^{\frac{1}{2}} t$$

$$= b\cos\frac{\mu^{\frac{1}{2}}x}{u}; \tag{129}$$

the limits of integration being the values of the several quantities which correspond to $t = t$ and to $t = 0$, when the position of $m$ is $(0, b)$.

From (129) it appears that the particle moves in the curve of cosines, and therefore describes an undulating path, the greatest distances of which from the axis of $x$ are $+b$ and $-b$: and the velocity parallel to the axis of $x$ is constant.

290.] For a problem wherein it is convenient to adopt the method of radial and transversal resolution explained in Article 256, we will take the following:

A particle $m$ moves about a fixed point with a constant angular velocity, and is acted on by a transversal force only: it is required to find the law of this force and the equation to the orbit.

In accordance with the symbolism of Art. 256, in this case

$$\mathrm{P} = 0, \qquad \frac{d\theta}{dt} = \text{a constant} = \omega \text{ (say)}.$$

Let the prime radius be the line passing through the fixed point and the initial place of the particle, that is, when $t = 0$, where we will suppose it to be at rest; and let $a$ be the distance of the particle from the fixed point at the beginning of motion; and let, fig. 123, $\mathrm{SP} = r$, $\mathrm{SA} = a$, $\mathrm{PSX} = \theta$; so that the equations of motion are,

$$\frac{d^2 r}{dt^2} - \omega^2 r = 0, \tag{130}$$

$$\mathrm{Q} = 2\omega \frac{dr}{dt}. \tag{131}$$

Now (130) is a linear differential equation of which the coefficient is constant, and therefore might be integrated by the methods of Vol. II. But the following process is easier. Let us multiply through by $2\,dr$, and taking the limits corresponding to $t = t$ and to $t = 0$, we have by integration,

$$\frac{dr^2}{dt^2} - \omega^2 (r^2 - a^2) = 0;$$

$$\therefore \quad \frac{dr}{(r^2 - a^2)^{\frac{1}{2}}} = \omega\, dt, \qquad \therefore \quad \log \frac{r + (r^2 - a^2)^{\frac{1}{2}}}{a} = \omega t,$$

$$\therefore \quad r = \frac{a}{2} \{e^{\omega t} + e^{-\omega t}\}. \tag{132}$$

Also as $\dfrac{d\theta}{dt} = \omega$, therefore $\theta = \omega t$, because $m$ is at $\mathrm{A}$ when $t = 0$, in which case $\theta = 0$; therefore (132) becomes

$$r = \frac{a}{2} \{e \;\; + e^{-\theta}\}, \tag{133}$$

which is the equation to the path described by $m$; and from (132),

$$\frac{dr}{dt} = \frac{a\omega}{2}\{e^{\omega t} - e^{-\omega t}\};$$

$$\begin{aligned} \therefore \quad \mathrm{Q} &= a\omega^2\{e^{\omega t} - e^{-\omega t}\} \\ &= a\omega^2\{e^{\theta} - e^{-\theta}\} \\ &= 2\omega^2(r^2 - a^2)^{\frac{1}{2}}; \end{aligned}$$

whereby Q is known.

291.] In illustration of the process of tangential and normal resolution we will consider the simple case of the motion of a projectile in vacuo under the action of gravity.

A particle of mass $m$ is projected from a given point, in a given line, and with a given velocity; and moves subject to the action of gravity; it is required to determine the curvilinear path.

Let the point of projection be taken as the origin of coordinates; and let a vertical and horizontal line drawn through it be the axes of $y$ and $x$ respectively: let $u$ be the velocity of projection, and let $a$ be the angle between the line of projection and the axis of $x$: and let $(x, y)$ be the position of $m$ at the time $t$: then because the vertical line, in which gravity acts, makes with the tangent to the curve at the point $(x, y)$ $\tan^{-1}\frac{dx}{dy}$, we have the following equations of motion from (43) Art. 256,

$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds}, \tag{134}$$

$$\frac{v^2}{\rho} = g\frac{dx}{ds}. \tag{135}$$

From (134) by definite integration we have,

$$\frac{2\,ds\,d^2s}{dt^2} = -2g\,dy,$$

$$\therefore \quad \frac{ds^2}{dt^2} - u^2 = -2gy,$$

$$\frac{ds^2}{dt^2} = u^2 - 2gy, \tag{136}$$

which result is the same as (97) Art. 282. Also from (135), since

$$\rho = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{-\frac{d^2y}{dx^2}},$$

$$-\frac{d^2y}{dx^2}(u^2-2gy) = g\frac{dx}{ds}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}$$

$$= g\left(1+\frac{dy^2}{dx^2}\right);$$

$$\therefore \quad \frac{d.\frac{dy^2}{dx^2}}{1+\frac{dy^2}{dx^2}} = \frac{-2g\,dy}{u^2-2gy},$$

$$\log\frac{1+\frac{dy^2}{dx^2}}{(\sec a)^2} = \log\frac{u^2-2gy}{u^2},$$

$$\frac{dy}{\{u^2(\sin a)^2-2gy\}^{\frac{1}{2}}} = \frac{dx}{u\cos a},$$

$$-\frac{1}{g}\{u^2(\sin a)^2-2gy\}^{\frac{1}{2}}+\frac{u\sin a}{g} = \frac{x}{u\cos a};$$

$$\therefore \quad y = x\tan a-\frac{gx^2}{2u^2(\cos a)^2}, \tag{137}$$

which result is the same as that obtained by the method of coordinate resolution in Art. 281.

292.] In some cases oblique coordinates may be used with advantage. Thus suppose, as in Art. 287, a particle $m$ to be projected with a given velocity $u$ in a given line from a given point, and to be attracted by a force the intensity of which varies directly as the distance, and which has its source in a given point; it is required to determine the path which it describes.

Let the given source of the force be the origin; and let the line passing through it and the point of projection be the axis of $y$: and let the axis of $x$ be drawn parallel to the line of projection; let the distance from the origin to the point of projection be $b_1$: then the equations of motion are

$$\frac{d^2x}{dt^2} = -\mu x, \qquad \frac{d^2y}{dt^2} = -\mu y;$$

and by a process parallel to that of Art. 287, we shall have

$$\frac{\mu x^2}{u^2}+\frac{y^2}{b_1^2} = 1,$$

which is the equation to an ellipse, referred to oblique coordinates, whose centre is at the origin, and of which the angle of ordination is (say) $\omega$, where $\omega$ is the angle between the line of projection and the line joining the point of projection and the centre of force. If $a$ and $b$ are the principal semi-axes, then by the properties of such axes we have,

$$a^2+b^2 = \frac{u^2}{\mu}+b_1^2,$$

$$ab = \frac{u\,b_1}{\mu^{\frac{1}{2}}}\sin\omega;$$

$$\therefore\ a = \frac{1}{2}\left\{\left(\frac{u^2}{\mu}+b_1^2+\frac{2u\,b_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}}+\left(\frac{u^2}{\mu}+b_1^2-\frac{2u\,b_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right\},$$

$$b = \frac{1}{2}\left\{\left(\frac{u^2}{\mu}+b_1^2+\frac{2u\,b_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}}-\left(\frac{u^2}{\mu}+b_1^2-\frac{2u\,b_1\sin\omega}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{2}}\right\}.$$

**293.]** Two material particles $m$ and $m'$ attract each other with a force varying directly as their masses and inversely as the square of the distance; it is required to determine the motion of $m$ relatively to $m'$.

Let, in reference to $m'$ placed at the origin o (which is moveable), $x, y, z$ be the coordinates to $m$ at the time $t$, and let $r$ be the distance of $m$ from o; then since the impressed velocity-increment which acts on $m'$ and attracts it towards $m$ along the line $r$ is $\frac{m}{r^2}$, therefore the components of the impressed velocity-increments on $m'$ at the origin are

$$\frac{mx}{r^3},\quad \frac{my}{r^3},\quad \frac{mz}{r^3};$$

and the components of the impressed velocity-increments on $m$ are

$$-\frac{m'x}{r^3},\quad -\frac{m'y}{r^3},\quad -\frac{m'z}{r^3}.$$

Therefore by reason of (53) in Art. 259,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+\frac{(m+m')\,x}{r^3} &= 0\\ \frac{d^2y}{dt^2}+\frac{(m+m')\,y}{r^3} &= 0\\ \frac{d^2z}{dt^2}+\frac{(m+m')\,z}{r^3} &= 0\end{aligned}\right\}; \qquad (138)$$

and the equations of motion of $m'$ relatively to $m$ will be similar in form: and therefore $m$ will describe relatively to $m'$ a curve similar to that which $m'$ will describe relatively to $m$.

And to determine the path of $m$ relatively to $m'$: multiplying the second of (138) by $z$, and the third by $y$, and subtracting, we have

$$z\frac{d^2y}{dt^2} - y\frac{d^2z}{dt^2} = 0;$$

therefore integrating,

$$\left.\begin{aligned} z\frac{dy}{dt} - y\frac{dz}{dt} &= h_1 \\ \text{similarly,}\qquad x\frac{dz}{dt} - z\frac{dx}{dt} &= h_2 \\ y\frac{dx}{dt} - x\frac{dy}{dt} &= h_3 \end{aligned}\right\}; \tag{139}$$

where $h_1, h_2, h_3$ are arbitrary constants: now multiplying these severally by $x, y, z$, and adding,

$$h_1x + h_2y + h_3z = 0; \tag{140}$$

which is the equation to a plane passing through the origin, that is, through $m'$, and therefore the motion of the particles is wholly in one plane.

Again from (138) and (139), and writing $\mu$ for $m+m'$, we have

$$h_2\frac{d^2z}{dt^2} - h_3\frac{d^2y}{dt^2} = -\left(x\frac{dz}{dt} - z\frac{dx}{dt}\right)\frac{\mu z}{r^3} + \left(y\frac{dx}{dt} - x\frac{dy}{dt}\right)\frac{\mu y}{r^3};$$

$$\begin{aligned} d\left(h_2\frac{dz}{dt} - h_3\frac{dy}{dt}\right) &= \frac{\mu}{r^3}\{(x^2+y^2+z^2)\,dx - x\,(x\,dx+y\,dy+z\,dz)\} \\ &= \mu\frac{r\,dx - x\,dr}{r^2} \\ &= \mu\, d.\frac{x}{r}, \end{aligned}$$

since $x^2+y^2+z^2 = r^2$, and $x\,dx+y\,dy+z\,dz = r\,dr$;

$$\left.\begin{aligned} \therefore\qquad h_2\frac{dz}{dt} - h_3\frac{dy}{dt} &= \frac{\mu x}{r} + f_1 \\ \text{similarly,}\qquad h_3\frac{dx}{dt} - h_1\frac{dz}{dt} &= \frac{\mu y}{r} + f_2 \\ h_1\frac{dy}{dt} - h_2\frac{dx}{dt} &= \frac{\mu z}{r} + f_3 \end{aligned}\right\}; \tag{141}$$

where $f_1, f_2, f_3$ are three undetermined constants: now multiplying these severally by $x, y, z$, and adding, and observing (139), we have

$$\mu r + f_1x + f_2y + f_3z = h_1^2 + h_2^2 + h_3^2; \tag{142}$$

and if $r$ is replaced by $(x^2+y^2+z^2)^{\frac{1}{2}}$, and the equation is cleared of radical quantities, it is that to a surface of the second order; and as the intersection of it by the plane (140) is a conic, it follows that the path of $m$ relatively to $m'$ is a conic: and similarly $m'$ describes a conic of the same species relatively to $m$.

In reference to (142) I would observe, that if $(x, y, z)$ is a point in the orbit of $m$, $f_1x+f_2y+f_3z-(h_1^2+h_2^2+h_3^2)$ is proportional to the perpendicular from $(x, y, z)$ to the plane

$$f_1x+f_2y+f_3z-(h_1^2+h_2^2+h_3^2) = 0; \tag{143}$$

and $r$ is the distance from the origin to the same point: therefore from (142) it appears that the distance from the origin to any point on the surface has a constant ratio to the perpendicular distance from the point on a fixed plane: the surface therefore is one of revolution of a conic about its axis, which is perpendicular to the given plane (143), the origin being the focus of the conic, and the given plane being generated by the revolution of the directrix. Hence also the direction-cosines of the axis of the surface are proportional to $f_1, f_2, f_3$: and since by reason of (141),

$$h_1f_1+h_2f_2+h_3f_3 = 0, \tag{144}$$

it appears that the plane (140) passes through the axis: therefore the conic in which $m$ moves is a principal section of the surface (142); and $m'$ is placed in the focus: $m$ therefore describes relatively to $m'$ a conic with $m'$ in the focus: and similarly $m'$ describes relatively to $m$ a conic about $m$ in its focus.

Also multiplying the equations (138) severally by $2\,dx$, $2\,dy$, and $2\,dz$, and adding and integrating, we have

$$\frac{ds^2}{dt^2} = \frac{2\mu}{r}+c; \tag{145}$$

where $c$ is another undetermined constant; but it is to be observed that all the undetermined constants may be found in terms of the initial velocity, the direction angles of its line of motion, and the coordinates of the point of projection.

The eccentricity of the conic may thus be found. Since the ratio of the focal radius of a conic to the perpendicular from a point on it to the directrix is that of $e$ to 1: and since from (143) the perpendicular from the point $(x, y, z)$ on the directrix is

$$\pm\frac{f_1x+f_2y+f_3z-(h_1^2+h_2^2+h_3^2)}{(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}} = p \text{ (say)};$$

$$\begin{aligned}
\therefore\ \frac{r}{p} &= \pm\frac{r\,(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}{f_1x+f_2y+f_3z-(h_1^2+h_2^2+h_3^2)};\\
&= \frac{r\,(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}{\mu r}\\
&= \frac{(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}{\mu};
\end{aligned}$$

$$\therefore \quad e = \frac{(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}{\mu}. \tag{146}$$

If the path of $m$ is an ellipse, the equations of the major axis are

$$\frac{x}{f_1} = \frac{y}{f_2} = \frac{z}{f_3} = \frac{\pm r}{(f_1^2+f_2^2+f_3^2)^{\frac{1}{2}}}; \tag{147}$$

and substituting for these values in (142), and taking $r_1$ and $r_2$ to be the greatest and least values of $r$, and substituting $h^2 = h_1^2+h_2^2+h_3^2$, $f^2 = f_1^2+f_2^2+f_3^2$, we have

$$r_1 = \frac{h^2}{\mu - f}, \qquad r_2 = \frac{h^2}{\mu+f};$$

$$\therefore \quad r_1 + r_2 = 2a = \frac{2h^2\mu}{\mu^2-f^2};$$

$$\therefore \quad a = \frac{h^2\mu}{\mu^2-f^2}$$

$$= \frac{\mu(h_1^2+h_2^2+h_3^2)}{\mu^2-(f_1^2+f_2^2+f_3^2)}; \tag{148}$$

and if $i$ is the inclination to the plane of $xy$ of the plane of motion, then from (140),

$$\cos i = \frac{h_3}{(h_1^2+h_2^2+h_3^2)^{\frac{1}{2}}}; \tag{149}$$

and if $\Omega$ is the angle between the axis of $x$ and the line of intersection of the plane of $m$'s motion with the plane of $xy$, then

$$\tan \Omega = -\frac{h_2}{h_1}; \tag{150}$$

and thus the plane of the motion is completely determined.

And thus (149) and (150) give the position of the plane in which $m$ moves: (147) give the direction-angles of the major axis of the ellipse, and therefore assign the position of the ellipse; and (146) and (148) give the dimensions of the elliptic path.

This problem is manifestly the astronomical one of two bodies $m$ and $m'$ moving relatively to each other, and under the action of their mutual attractions, and on this account I have considered it at greater length than would otherwise be necessary. The determination of the other incidents of the motion requires data which it would be out of place here to enter upon.

294.] If two particles $m$ and $m'$ move subject to their mutual attractions, the centre of gravity of them either remains at rest or moves in a straight line.

Let the positions of the particles $m$ and $m'$ at the time $t$ be respectively $(x, y, z)$, and $(x', y', z')$: and let $r$ be the distance between them, and to fix our thoughts, let us suppose $m$ farther from the origin than $m'$; then the equations of motion of the two relatively to the fixed origin are,

$$\frac{d^2x}{dt^2} = \frac{-m'(x-x')}{r^3}, \quad \frac{d^2y}{dt^2} = \frac{-m'(y-y')}{r^3}, \quad \frac{d^2z}{dt^2} = \frac{-m'(z-z')}{r^3}; \tag{151}$$

$$\frac{d^2x'}{dt^2} = \frac{m(x-x')}{r^3}, \quad \frac{d^2y'}{dt^2} = \frac{m(y-y')}{r^3}, \quad \frac{d^2z'}{dt^2} = \frac{m(z-z')}{r^3}; \tag{152}$$

$$m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} = 0, \; m\frac{d^2y}{dt^2} + m'\frac{d^2y'}{dt^2} = 0, \; m\frac{d^2z}{dt^2} + m'\frac{d^2z'}{dt^2} = 0. \tag{153}$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be the position of the centre of gravity of $m$ and $m'$ at the time $t$: therefore

$$\left.\begin{aligned} (m+m')\,\bar{x} &= mx+m'x' \\ (m+m')\,\bar{y} &= my+m'y' \\ (m+m')\,\bar{z} &= mz+m'z' \end{aligned}\right\}; \tag{154}$$

and differentiating these twice we have, by reason of (153),

$$(m+m')\frac{d^2\bar{x}}{dt^2} = (m+m')\frac{d^2\bar{y}}{dt^2} = (m+m')\frac{d^2\bar{z}}{dt^2} = 0; \tag{155}$$

$$\therefore \quad \frac{d^2\bar{x}}{dt^2} = \frac{d^2\bar{y}}{dt^2} = \frac{d^2\bar{z}}{dt^2} = 0. \tag{156}$$

Suppose $\alpha, \beta, \gamma$ to be the components of the velocity of the centre of gravity, when $t=0$; and let $(a, b, c)$ be its position: then integrating (156),

$$\frac{d\bar{x}}{dt} = \alpha, \qquad \frac{d\bar{y}}{dt} = \beta, \qquad \frac{d\bar{z}}{dt} = \gamma;$$

$$\bar{x}-a = \alpha t, \qquad \bar{y}-b = \beta t, \qquad \bar{z}-c = \gamma t; \tag{157}$$

$$\therefore \quad \frac{\bar{x}-a}{\alpha} = \frac{\bar{y}-b}{\beta} = \frac{\bar{z}-c}{\gamma}; \tag{158}$$

which are the equations to the rectilinear path of the centre of gravity of $m$ and $m'$; if $\alpha = \beta = \gamma = 0$, so that the centre of gravity is at rest when $t=0$, then for all values of $t$ we have

$$\bar{x} = a, \qquad \bar{y} = b, \qquad \bar{z} = c; \tag{159}$$

so that the centre of gravity remains in the same position; of this property an example has already occurred in Art. 237.

The equations of motion of $m$ and $m'$ relatively to their centre of gravity may be calculated as follows.

Let $(\xi, \eta, \zeta)$ $(\xi', \eta', \zeta')$ be the positions of $m$ and $m'$ relatively to the centre of gravity as origin; let $r$ be the distance between $m$ and $m'$; $\rho$ and $\rho'$ the distances of $m$ and $m'$ from their centre of gravity, so that

$$\frac{m+m'}{r} = \frac{m'}{\rho} = \frac{m}{\rho'}.$$

Now the $x$-component of the expressed velocity-increment of $m$ is as follows:

$$\begin{aligned} \frac{d^2\xi}{dt^2} &= -\left(\frac{m'}{r^2} + \frac{m}{r^2}\right)\frac{\xi}{\rho} \\ &= -\frac{m'^2}{m+m'}\,\frac{\xi}{\rho^3}; \end{aligned}$$

similarly $$\frac{d^2\eta}{dt^2} = -\frac{m'^2}{m+m'}\,\frac{\eta}{\rho^3}; \qquad \frac{d^2\zeta}{dt^2} = -\frac{m'^2}{m+m'}\,\frac{\zeta}{\rho^3}.$$

By a similar process we find,

$$\frac{d^2\xi'}{dt^2} = -\frac{m^2}{m+m'}\,\frac{\xi'}{\rho'^3}, \quad \frac{d^2\eta'}{dt^2} = -\frac{m^2}{m+m'}\,\frac{\eta'}{\rho'^3}, \quad \frac{d^2\zeta'}{dt^2} = -\frac{m^2}{m+m'}\,\frac{\zeta'}{\rho'^3}.$$

The identity of form of these equations shews that the paths which $m$ and $m'$ describe about their centre of gravity are similar; and as the form is the same as that of the equations (138), it follows that the paths are conics, of which the foci are in the centre of gravity of the particles.

295.] There is another important problem of the same kind, the differential equations of which it is desirable to insert.

It is required to calculate the motion of $m$ relatively to M, when M and $m$ are acted on by another particle $m'$, the law of attraction of all three being that of gravitation.

Let $(x, y, z)$ $(x', y', z')$ be the positions of $m$ and of $m'$ relatively to M at the time $t$; and let $r$ and $r'$ be the distances of $m$ and of $m'$ from M at the same time: and to fix our thoughts let us suppose $m'$ to be farther from M than $m$: then by virtue of the principle of Article 259, the velocity-increment of M due to the attraction of $m$ and of $m'$ is to be impressed on M in a direction the opposite of that along which the attractions of $m$ and $m'$ act; therefore, for the $x$-component of the velocity-increment of $m$, we have

$$\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{mx}{r^3} - \frac{m'x'}{r'^3} - \frac{\mathrm{M}x}{r^3} + \frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}} \\ &= -\frac{(m+\mathrm{M})x}{r^3} - \frac{m'x'}{r'^3} + \frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}; \end{aligned}$$

similarly

$$\frac{d^2y}{dt^2} = -\frac{(m+\text{M})\,y}{r^3} - \frac{m'y'}{r'^3} + \frac{m'(y'-y)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}},$$

$$\frac{d^2z}{dt^2} = -\frac{(m+\text{M})\,z}{r^3} - \frac{m'z'}{r'^3} + \frac{m'(z'-z)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}};$$

Let

$$\frac{m'(xx'+yy'+zz')}{r'^3} - \frac{m'}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{1}{2}}} = \text{R}; \quad (160)$$

$$\therefore \quad \left(\frac{d\text{R}}{dx}\right) = \frac{m'x'}{r'^3} - \frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}; \quad (161)$$

and with similar values for $\left(\frac{d\text{R}}{dy}\right)$ and $\left(\frac{d\text{R}}{dz}\right)$.

So that the equations become

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{(m+\text{M})\,x}{r^3} + \left(\frac{d\text{R}}{dx}\right) = 0 \\ \frac{d^2y}{dt^2} + \frac{(m+\text{M})\,y}{r^3} + \left(\frac{d\text{R}}{dy}\right) = 0 \\ \frac{d^2z}{dt^2} + \frac{(m+\text{M})\,z}{r^3} + \left(\frac{d\text{R}}{dz}\right) = 0 \end{aligned}\right\}; \quad (162)$$

R is called *the disturbing function*, because it alone involves $m'$, which is the mass of the body which disturbs the otherwise conic path of $m$ relatively to M.

The problem is manifestly that of the moon moving about the earth, the sun being the disturbing body: or of a planet moving about the sun, another planet being the disturbing body.

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## SECTION 5.—*The curvilinear motion of a particle in a resisting medium.*

296.] When a particle describes a curvilinear path in a resisting medium, the momentum of the molecules of the medium, which they have on account of their displacement and on account of the particle passing amongst them, is withdrawn from the moving particle, as was explained in Art. 219, and from it in the line in which it moves: the medium therefore has no effect in diminishing the velocity of the particle in the line which is normal to its path; and the loss takes place along the tangent to the curvilinear path: we proceed to consider the circumstances of a particle moving in such a resisting medium.

Let us consider the moving particle to be spherical in form, so that an equal surface is presented to the medium, whatever is the line in which the particle moves: and to take the general case, whatever is the law of the resistance of the medium, let R represent the velocity-increment (or decrement) which the resisting medium withdraws from the velocity of the particle $m$ in the line of its motion, that is, along the tangent of its curvilinear path in an unit of time; then if X, Y, Z are the three impressed velocity-increments, the equations of motion, referred to three rectangular axes, of a particle moving in space, are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} - \mathrm{R}\frac{dx}{ds} \\ \frac{d^2y}{dt^2} &= \mathrm{Y} - \mathrm{R}\frac{dy}{ds} \\ \frac{d^2z}{dt^2} &= \mathrm{Z} - \mathrm{R}\frac{dz}{ds} \end{aligned}\right\}, \tag{163}$$

since $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are the direction-cosines of the tangent of the curvilinear path, that is, of the line of action of R: now multiplying these severally by $2\,dx$, $2\,dy$, and $2\,dz$, and adding, we have,

$$\frac{2\,dx\,d^2x + 2\,dy\,d^2y + 2\,dz\,d^2z}{dt^2} = 2(\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) - 2\mathrm{R}\,ds; \tag{164}$$

let V = the velocity, when $t = 0$; and using the symbols indicative of definite integration according to the following form, so that the limits may be those corresponding to $t = t$ and to $t = 0$, we have

$$v^2 - \mathrm{V}^2 = 2\int_0^t (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) - 2\int_0^t \mathrm{R}\,ds; \tag{165}$$

Also because

$$\frac{d^2s}{dt^2} = \frac{dx}{ds}\frac{d^2x}{dt^2} + \frac{dy}{ds}\frac{d^2y}{dt^2} + \frac{dz}{ds}\frac{d^2z}{dt^2},$$

$$\therefore\quad \frac{d^2s}{dt^2} = \mathrm{X}\frac{dx}{ds} + \mathrm{Y}\frac{dy}{ds} + \mathrm{Z}\frac{dz}{ds} - \mathrm{R}; \tag{166}$$

that is, the expressed velocity-increment along the curve is that due to the impressed forces less that due to the resistance.

Also if P is the resultant of the impressed velocity-increments, then

$$\begin{aligned}
\mathrm{P}^2 &= \mathrm{X}^2+\mathrm{Y}^2+\mathrm{Z}^2 \\
&= \Big(\frac{d^2x}{dt^2}+\mathrm{R}\frac{dx}{ds}\Big)^2+\Big(\frac{d^2y}{dt^2}+\mathrm{R}\frac{dy}{ds}\Big)^2+\Big(\frac{d^2z}{dt^2}+\mathrm{R}\frac{dz}{ds}\Big)^2 \\
&= \frac{(d^2x)^2+(d^2y)^2+(d^2z)^2}{dt^4}+2\mathrm{R}\frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{ds\,dt^2}+\mathrm{R}^2 \\
&= \Big(\frac{v^2}{\rho}\Big)^2+\Big(\frac{d^2s}{dt^2}\Big)^2+2\mathrm{R}\frac{d^2s}{dt^2}+\mathrm{R}^2 \\
&= \Big(\frac{v^2}{\rho}\Big)^2+\Big(\frac{d^2s}{dt^2}+\mathrm{R}\Big)^2, \qquad (167)
\end{aligned}$$

if $\rho$ is the radius of absolute curvature of the path; that is, the resultant of the impressed velocity-increments consists of two components, the lines of action of which are at right angles to each other; and of which one acts along the principal normal to the curvilinear path, (see Art. 255), and is equal to $\frac{v^2}{\rho}$, and the other acts along the tangent and is the sum of the tangential expressed velocity-increment and the resistance. This resolution is therefore the tangential and the normal one, see Art. 254; and therefore if $\mathrm{T}$ = the tangential impressed velocity-increment, and $\mathrm{N}$ = the normal impressed velocity-increment,

$$\left.\begin{aligned}\mathrm{T} &= \frac{d^2s}{dt^2}+\mathrm{R} \\ \mathrm{N} &= \frac{v^2}{\rho}\end{aligned}\right\}. \qquad (168)$$

297.] If the motion is wholly in one plane we may take that plane to be the plane of $xy$; and if we resolve along the rectangular axes of $x$ and $y$, the equations of motion are,

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \mathrm{X}-\mathrm{R}\frac{dx}{ds} \\ \frac{d^2y}{dt^2} &= \mathrm{Y}-\mathrm{R}\frac{dy}{ds}\end{aligned}\right\}. \qquad (169)$$

And if we take the tangential and normal components, we have

$$\left.\begin{aligned}\frac{d^2s}{dt^2} &= \mathrm{T}-\mathrm{R} \\ \frac{v^2}{\rho} &= \mathrm{N}\end{aligned}\right\}; \qquad (170)$$

or,

$$\left.\begin{aligned}\frac{d^2s}{dt^2} &= \mathrm{X}\frac{dx}{ds}+\mathrm{Y}\frac{dy}{ds}-\mathrm{R} \\ \frac{v^2}{\rho} &= \mathrm{X}\frac{dy}{ds}-\mathrm{Y}\frac{dx}{ds}\end{aligned}\right\}; \qquad (171)$$

and
$$v^2-\mathrm{V}^2 = 2\int_0^t(\mathrm{X}\,dx+\mathrm{Y}\,dy)-2\int_0^t\mathrm{R}\,ds \tag{172}$$
$$= 2\int_0^t(\mathrm{T}-\mathrm{R})\,ds. \tag{173}$$

And if the motion is referred to a system of polar coordinates, and if $\mathrm{P}$ and $\mathrm{Q}$ are the radial and transversal components of the impressed velocity-increments, then from Art. 256, and because $\frac{dr}{ds}$ and $\frac{r\,d\theta}{ds}$ are respectively the sine and cosine of the angle between $r$ and the normal to the curve,

$$\left.\begin{aligned}\frac{d^2r}{dt^2}-r\frac{d\theta^2}{dt^2} &= \mathrm{P}-\mathrm{R}\frac{dr}{ds}\\ 2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2} &= \mathrm{Q}-\mathrm{R}\frac{r\,d\theta}{ds}\end{aligned}\right\}; \tag{174}$$

and therefore also

$$\left.\begin{aligned}\frac{d^2s}{dt^2} &= \mathrm{P}\frac{dr}{ds}+\mathrm{Q}\frac{r\,d\theta}{ds}-\mathrm{R}\\ \frac{v^2}{\rho} &= \mathrm{Q}\frac{dr}{ds}-\mathrm{P}\frac{r\,d\theta}{ds}\end{aligned}\right\}. \tag{175}$$

The first of these last two being multiplied by $2\,ds$ and integrated gives

$$v^2-\mathrm{V}^2 = 2\int_0^t(\mathrm{P}\,dr+\mathrm{Q}\,r\,d\theta-\mathrm{R}\,ds); \tag{176}$$

and these general formulæ are sufficient for the solution of all problems relating to motion in a resisting medium.

298.] A particle moves in a resisting medium under the action of forces parallel to the axes of $x$ and $y$; it is required to determine the law of resistance so that a given plane curve may be described.

Let the impressed forces parallel to the axes of $x$ and $y$ be $\mathrm{X}$ and $\mathrm{Y}$; and let $\mathrm{R}$ be the resistance; then by (171),

$$\mathrm{R} = \mathrm{X}\frac{dx}{ds}+\mathrm{Y}\frac{dy}{ds}-\frac{d^2s}{dt^2},$$

$$\frac{ds^2}{dt^2} = \rho\,\frac{\mathrm{X}\,dy-\mathrm{Y}\,dx}{ds};$$

$$\therefore\quad \frac{d^2s}{dt^2} = \frac{1}{2}\frac{d}{ds}\cdot\frac{\rho(\mathrm{X}\,dy-\mathrm{Y}\,dx)}{ds};$$

$$\therefore\quad \mathrm{R} = \frac{\mathrm{X}\,dx+\mathrm{Y}\,dy}{ds}-\frac{1}{2}\frac{d}{ds}\cdot\frac{\rho(\mathrm{X}\,dy-\mathrm{Y}\,dx)}{ds}; \tag{177}$$

into which expression $t$, which has been equicrescent, does not enter; we are therefore free to make any other variable equicrescent: and the expression will hereby become somewhat less complicated.

If, according to the law investigated in Art. 219, the resistance varies as the density and as the square of the velocity, and if the density also varies, and the law of its variation is to be discovered so that a given curve may be described, then if $\text{s}$ represents the varying density,

$$\text{R} = \text{s} \times v^2,$$

$$\therefore \quad \text{s} = \frac{\text{R}}{v^2}$$

$$= \frac{1}{\rho}\frac{\text{X}\,dx + \text{Y}\,dy}{\text{X}\,dy - \text{Y}\,dx} - \frac{1}{2}\frac{d}{ds}.\log\frac{\rho(\text{X}\,dy - \text{Y}\,dx)}{ds}. \qquad (178)$$

Ex. 1. A particle describes a parabola under the action of a constant force parallel to its principal axis: it is required to determine the law of resistance.

Let the equation to the parabola be, $y^2 = 4ax$; so that

$$\frac{dy}{2a} = \frac{dx}{y} = \frac{ds}{(y^2+4a^2)^{\frac{1}{2}}} = \frac{ds}{(4ax+4a^2)^{\frac{1}{2}}},$$

$$\rho = \frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

In this case $\text{Y} = 0$, $\text{X} = \text{a constant} = k$ (say); therefore by (177),

$$\text{R} = k\frac{dx}{ds} - k\frac{d}{ds}(x+a),$$

$$= 0;$$

that is, in vacuo only does a particle moving under the action of a constant force parallel to the axis of $x$ describe a parabola. See Art. 276, Ex. 3.

Ex. 2. A particle moves in a circle under the action of a constant force in parallel lines, and the resistance of the medium varies as the density and the square of the velocity: it is required to determine the law of variation of the density of the medium.

Let the equation to the circle be, $x^2+y^2 = a^2$; therefore

$$\frac{dx}{y} = -\frac{dy}{x} = \frac{ds}{a};$$

and let the line of action of the constant force be parallel to the axis of $y$: so that

$$\text{X} = 0, \qquad \text{Y} = \text{a constant} = -k \text{ (say)};$$

therefore from (178), $\qquad \text{s} = \dfrac{3x}{2ay}.$

299.] A particle moves in a resisting medium under the action of a central force P; it is required to find the law of resistance, so that a given curve may be described.

From (175), if Q = 0, we have

$$\frac{d^2s}{dt^2} = \mathrm{P}\frac{dr}{ds} - \mathrm{R}, \qquad \frac{ds^2}{dt^2} = -\mathrm{P}\rho\frac{r\,d\theta}{ds};$$

$$\therefore \quad \frac{d^2s}{dt^2} = -\frac{1}{2}\frac{d}{ds}.\mathrm{P}\rho\frac{r\,d\theta}{ds};$$

$$\therefore \quad \mathrm{R} = \mathrm{P}\frac{dr}{ds} + \frac{1}{2}\frac{d}{ds}.\frac{\mathrm{P}\rho r\,d\theta}{ds}; \tag{179}$$

and since $\rho = \frac{r\,dr}{dp}$, and $\frac{r\,d\theta}{ds} = \frac{p}{r}$,

$$\therefore \quad \mathrm{R} = \mathrm{P}\frac{dr}{ds} + \frac{1}{2}\frac{d}{ds}\left(\mathrm{P}p\frac{dr}{dp}\right) \tag{180}$$

$$= \frac{1}{2p^2}\frac{d}{ds}\left(\mathrm{P}p^3\frac{dr}{dp}\right). \tag{181}$$

$$\therefore \quad \mathrm{P} = \frac{dp}{p^3\,dr}\int 2p^2\mathrm{R}\,ds, \tag{182}$$

whereby, if the law of resistance and the equation of the curve are given, the central force P may be found.

Again, if the resistance varies as the density and as the square of the velocity, and if the density also varies, and the law of its variation is to be investigated, so that a given curve may be described, then if S represents the density,

$$\mathrm{R} = \mathrm{S} \times v^2,$$

$$\therefore \quad \mathrm{S} = \frac{\mathrm{R}}{v^2}$$

$$= -\frac{1}{2}\frac{d}{ds}\log\left(\mathrm{P}p^3\frac{dr}{dp}\right); \tag{183}$$

and if the density of the medium is given, and the central force is to be discovered, then from this last equation we have,

$$\mathrm{P} = \frac{h^2}{p^3}\frac{dp}{dr}e^{-2\int \mathrm{S}\,ds}, \tag{184}$$

where $h^2$ is a constant introduced in integration.

And (184) may be put under the following elegant form:

$$\text{Let} \quad u = \frac{1}{r}, \qquad \therefore \quad \frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2};$$

$$\therefore \quad -\frac{2\,dp}{p^3} = 2\left(u + \frac{d^2u}{d\theta^2}\right)du$$

$$= -2\left(u + \frac{d^2u}{d\theta^2}\right)\frac{dr}{r^2};$$

$$\therefore \quad \frac{dp}{p^3\,dr} = \left(u + \frac{d^2u}{d\theta^2}\right) u^2;$$

$$\therefore \quad \text{P} = h^2u^3\left(u + \frac{d^2u}{d\theta^2}\right) e^{-2\int s\,ds}. \tag{185}$$

Ex. 1. A particle moves in the circumference of a circle under the attraction of a central force whose origin is a point in the circumference, and the law of which varies as the $n$th power of the distance; it is required to determine the law of the density and the resistance of the medium.

Let the radius of the circle be $a$, and let the pole be at the centre of force, then

$$r^2 = 2\,ap, \quad \text{and} \quad \text{P} = -\mu\, r^n; \quad \text{also} \quad r\,dr = (r^2 - p^2)^{\frac{1}{2}}\,ds;$$

therefore from (181),

$$\text{R} = \frac{-\mu\,(n+5)\,r^{n+4}}{16\,a^2p^2}\,\frac{dr}{ds}$$

$$= \frac{-\mu\,(n+5)\,r^n\,(4\,a^2 - r^2)^{\frac{1}{2}}}{8\,a}; \tag{186}$$

and from (183),
$$\text{s} = -\frac{(n+5)\,(4\,a^2 - r^2)^{\frac{1}{2}}}{4\,ar}. \tag{187}$$

It appears therefore that if $n = -5$, that is, if the central force varies inversely as the fifth power of the distance, $\text{R} = 0$ and $\text{s} = 0$; that is, the particle must move in vacuo; also if $r = 2\,a$, whatever $n$ is, the resistance and the density vanish.

300.] To determine the motion of a projectile under the action of gravity in a medium of which the resistance varies directly as the velocity and of which the density is uniform.

Let the velocity of projection be $u$, and let the point of projection be taken as the origin: let the horizontal plane through it be that of $xy$, and let the axis of $z$ be measured in a direction contrary to that of the action of gravity: let the resistance be $k\dfrac{ds}{dt}$, of which the components along the coordinate axes are

$$k\frac{dx}{dt}, \quad k\frac{dy}{dt}, \quad k\frac{dz}{dt};$$

so that the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k\frac{dx}{dt} \\ \frac{d^2y}{dt^2} &= -k\frac{dy}{dt} \\ \frac{d^2z}{dt^2} &= -g - k\frac{dz}{dt} \end{aligned}\right\}; \tag{188}$$

from the first and second of which we have

$$dy\, d^2x - dx\, d^2y = 0,$$

$$\therefore \quad \frac{d^2x}{dx} = \frac{d^2y}{dy}, \qquad \therefore \quad \log\frac{dx}{a} = \log\frac{dy}{b};$$

$$\therefore \quad \frac{x}{a} = \frac{y}{b}, \tag{189}$$

which is the equation to a plane passing through the axis of $z$, in which therefore the motion of the particle takes place.

Since then the motion takes place in one plane, let us assume that to be the plane of $xy$; and let the point of projection be the origin, and let the horizontal line through the origin be the axis of $x$; and let the angle between the line of projection and the axis of $x = a$: and let the axis of $y$ be taken in a direction the opposite of that of gravity: so that the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k\frac{dx}{dt} \\ \frac{d^2y}{dt^2} &= -g-k\frac{dy}{dt} \end{aligned}\right\}; \tag{190}$$

from the former of which we have by definite integration

$$\frac{dx}{dt} - u\cos a = -kx; \tag{191}$$

$$\therefore \quad \frac{dx}{u\cos a - kx} = dt, \qquad \log\frac{u\cos a - kx}{u\cos a} = -kt;$$

$$\therefore \quad x = \frac{u\cos a}{k}\{1-e^{-kt}\}, \tag{192}$$

the limits of integration being the values corresponding to $t = t$ and to $t = 0$: and from the latter of (190) we have,

$$\frac{dy}{dt} - u\sin a = -gt-ky,$$

$$\frac{dy}{dt} + ky = u\sin a - gt; \tag{193}$$

$$\begin{aligned} \therefore \quad y &= \left(\frac{d}{dt}+k\right)^{-1}(u\sin a - gt) \\ &= e^{-kt}\int_0^t e^{kt}(u\sin a - gt)\, dt \\ &= e^{-kt}\left[\frac{(u\sin a - gt)}{k}e^{kt} + \frac{g}{k^2}e^{kt}\right]_0^t; \end{aligned}$$

$$\therefore \quad ky+gt = \left(u\sin a + \frac{g}{k}\right)(1-e^{-kt}); \tag{194}$$

and eliminating $t$ by means of (192) and (194) we have,

$$y = x\tan a + \frac{gx}{ku\cos a} + \frac{g}{k^2}\log\left(1-\frac{kx}{u\cos a}\right); \qquad (195)$$

which is the equation to the path of the projectile.

301.] If we expand the logarithmic term in the preceding expression, the equation becomes

$$y = x\tan a + \frac{gx}{ku\cos a} - \frac{g}{k^2}\left\{\frac{kx}{u\cos a} + \frac{k^2x^2}{2u^2(\cos a)^2} + \frac{k^3x^3}{3u^3(\cos a)^3} + \dots\right\}$$

$$= x\tan a - \frac{gx^2}{2u^2(\cos a)^2} - \frac{gkx^3}{3u^3(\cos a)^3} - \dots \qquad (196)$$

of which series all terms, except the first two, contain $k$ (the coefficient of resistance); and if $k=0$, the equation is that to the parabolic path which is the trajectory of a projectile in vacuo, see Article 281; and thus the terms on the right-hand side of (196), after the first two, express the excess of the ordinate of the parabola described in vacuo over the ordinate of the curve whose equation is (196): the trajectory (196) therefore is of a form somewhat parabolic, but the curve recedes from the directrix more than a parabola.

Also by reason of (191), $\frac{dx}{dt} = 0$ if $x = \frac{u\cos a}{k}$; for this value of $x$ therefore the horizontal velocity vanishes, and the projectile moves in a vertical path; and therefore a vertical line, at a distance $= \frac{u\cos a}{k}$ from the point of projection, is an asymptote to the curve.

Also from (195) $\frac{dy}{dx} = 0$, that is, the projectile comes to its highest point when

$$0 = \tan a + \frac{g}{ku\cos a} - \frac{g}{k(u\cos a - kx)}, \qquad (197)$$

that is, when $x = \dfrac{u^2\sin a\cos a}{g+ku\sin a}$,

and $\quad y = \dfrac{u\sin a}{k} + \dfrac{g}{k^2}\log\dfrac{g}{g+ku\sin a}$.

Also from (197), if $x = -\infty$, $\frac{dy}{dx} = \tan a + \frac{g}{ku\cos a}$; that is, the curve being produced backwards through the origin continually approaches to a certain definite angle with the axis of $x$.

302.] But the most important application of this theory is that of gunnery, in which the motion takes place in air, the resistance of which (at least approximately) varies as the square of the velocity; and thus the following problem offers itself:

A particle, or a spherical homogeneous ball, acted on by gravity, is projected with a given velocity $u$ in a given line, in a medium of which the resistance varies as the square of the velocity and the density is uniform: it is required to determine the circumstances of motion.

It may be shewn, by a process similar to that at the commencement of Art. 300, that the motion takes place in one plane; and therefore we may assume that plane to be the plane of $xy$.

Let $u =$ the velocity of projection; and let a horizontal and a vertical line through the point of projection be the axes respectively of $x$ and $y$: let the axis of $y$ be taken upwards: let $a$ be the angle between the axis of $x$ and the line of projection: and let the resistance of the medium be $k\frac{ds^2}{dt^2}$; then, as the line of action of this is the tangent of the curve, its components are

$$k\frac{ds}{dt}\frac{dx}{dt}, \text{ and } k\frac{ds}{dt}\frac{dy}{dt};$$

and as $k$, see Art. 219, varies directly as the density of the medium, as the surface which the moving ball presents to the medium, and inversely as the mass of the ball, then, as the ball is spherical and the density is uniform, $k$ is constant. Thus the equations of motion are

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -k\frac{ds}{dt}\frac{dx}{dt}\\ \frac{d^2y}{dt^2} &= -g-k\frac{ds}{dt}\frac{dy}{dt}\end{aligned}\right\}. \tag{198}$$

Integrating the first of these, and taking the limits corresponding to $t = t$ and to $t = 0$, we have

$$\frac{\frac{d^2x}{dt^2}}{\frac{dx}{dt}} = -k\frac{ds}{dt}, \qquad \therefore \log\frac{\frac{dx}{dt}}{u\cos a} = -ks;$$

$$\therefore \frac{dx}{dt} = u\cos a\, e^{-ks}. \tag{199}$$

Again from (198), and transforming the equations so that $t$ is not equicrescent, we have

$$\frac{d^2y\,dx - d^2x\,dy}{dt^2} = -g\,dx;$$

therefore from (199),

$$\frac{d^2y\,dx-d^2x\,dy}{dx^2} = -\frac{g}{u^2(\cos a)^2}\,e^{2ks}\,dx; \qquad (200)$$

$$\therefore \quad d.\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = -\frac{g}{u^2(\cos a)^2}\,e^{2ks}\,ds; \qquad (201)$$

and integrating, and taking the limits corresponding to $t=t$ and to $t=0$, we have

$$\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}$$

$$-\tan a\sec a-\log(\tan a+\sec a) = -\frac{g}{ku^2(\cos a)^2}(e^{2ks}-1); \qquad (202)$$

and for convenience let us substitute,

$$\tan a\sec a+\log(\tan a+\sec a)+\frac{g}{ku^2(\cos a)^2} = \text{c}; \qquad (203)$$

so that (201) becomes,

$$\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\} = \text{c}-\frac{g}{ku^2(\cos a)^2}\,e^{2ks} \qquad (204)$$

$$= \text{c}+\frac{1}{k}\frac{d}{dx}\left(\frac{dy}{dx}\right);$$

$$\therefore \quad \frac{d.\dfrac{dy}{dx}}{\dfrac{dy}{dx}\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left\{\dfrac{dy}{dx}+\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}-\text{c}} = k\,dx, \qquad (205)$$

$$\frac{\dfrac{dy}{dx}\,d.\dfrac{dy}{dx}}{\dfrac{dy}{dx}\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}+\log\left\{\dfrac{dy}{dx}+\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}-\text{c}} = k\,dy; \qquad (206)$$

from which equations, were it possible to integrate them, $x$ and $y$ might be found in terms of $\frac{dy}{dx}$; and if $\frac{dy}{dx}$ were eliminated from the two integrals, the resulting equation in terms of $x$ and $y$ would be that of the required trajectory.

But as the equations are not integrable, we must deduce from them in their present forms such results as are possible.

Equating the values of $e^{ks}$ which are given in equations (199) and (204), we have

$$dt = -\frac{1}{(kg)^{\frac{1}{2}}}\,\frac{d.\dfrac{dy}{dx}}{\left\{\text{c}-\dfrac{dy}{dx}\left(1+\dfrac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left(\dfrac{dy}{dx}+\left\{1+\dfrac{dy^2}{dx^2}\right\}^{\frac{1}{2}}\right)\right\}^{\frac{1}{2}}}; \qquad (207)$$

whence might the time be found in terms of $\frac{dy}{dx}$: also squaring (205) and (206), and dividing by the square of (207), we have

$$\frac{ds^2}{dt^2} = \frac{g}{k}\,\frac{1+\frac{dy^2}{dx^2}}{c-\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}}; \tag{208}$$

which equation gives the velocity in terms of $\frac{dy}{dx}$.

303.] When $\frac{dy}{dx} = -\infty$,

$c-\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}-\log\left\{\frac{dy}{dx}+\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}\right\}$ becomes $-\frac{dy^2}{dx^2}$;
so that (205) becomes,

$$k\,dx = -\frac{d.\frac{dy}{dx}}{\frac{dy^2}{dx^2}}; \tag{209}$$

$\therefore\quad k(x-a) = \frac{dx}{dy}$, where $a$ is an arbitrary constant:
therefore if $\frac{dy}{dx} = -\infty$, $x = a =$ a constant; that is, the line, whose equation is $x = a$, is an asymptote to the curve. Also under the same supposition from (208), we have

$$(\text{vel.})^2 = \frac{g}{k}; \tag{210}$$

that is, the velocity of the projectile, as it falls down the descending branch of the trajectory, approaches to the constant limit $\left(\frac{g}{k}\right)^{\frac{1}{2}}$.

And at the highest point of the path, when $\frac{dy}{dx} = 0$, $(\text{vel.})^2 = \frac{g}{k\text{c}}$.

Thus the path of the projectile is a curve of the form delineated in fig. 124, where OA $= a$.

304.] The case however frequently occurs in practice wherein the angle of projection is very small; and where the ball rises very little above the horizontal line; and therefore $\frac{dy}{dx}$ being very small, we may throughout the path on the upper side of the axis of $x$ neglect powers of $\frac{dy}{dx}$ higher than the first. In this case then

$$ds = dx; \qquad \therefore \quad s = x:$$

so that (200) becomes,

$$d.\frac{dy}{dx} = -\frac{g}{u^2(\cos a)^2} e^{2kx} dx;$$

$$\therefore \quad \frac{dy}{dx} - \tan a = -\frac{g}{2ku^2(\cos a)^2}(e^{2kx}-1);$$

$$\therefore \quad y = x\tan a + \frac{gx}{2ku^2(\cos a)^2} - \frac{g}{4k^2u^2(\cos a)^2}(e^{2kx}-1); \tag{211}$$

and expanding in a series the last term, we have

$$y = x\tan a - \frac{gx^2}{2u^2(\cos a)^2} - \frac{gkx^3}{3u^2(\cos a)^2} - \dots \tag{212}$$

which equation, if the terms involving $k$ are omitted, is that of a parabola, which is the path of the projectile in vacuo. It appears therefore that the ordinate of the actual curve is that of the parabola diminished by a quantity which is the sum of all the terms of the right-hand member of the last equation except the first two.

Also from (199), $\frac{dx}{dt} = u\cos a\, e^{-kx}$;

$$\therefore \quad ku\cos a\, t = e^{kx}-1; \tag{213}$$

which gives the time in terms of the abscissa.

# CHAPTER XI.

## THE FREE MOTION OF PARTICLES, UNDER THE ACTION OF CENTRAL FORCES.

### SECTION 1.—*General investigations; determination of laws of force and other circumstances of motion in given orbits.*

305.] A *central force* is that of which the source of influence is at a certain point, towards which it attracts or from which it repels any particle of matter within reach of its action; and according as the action of it is attraction or repulsion, so is it called an attractive or a repulsive force. The forces, whose effects will be considered, are supposed to be functions of the distance between their centres and the particle on which they act, and not to be functions explicitly of either the time or the velocity; the case in which the line of motion of the particle is coincident with that of the action of the force has been considered in Chapter VIII; and it remains for us now to discuss the case in which the line of motion of the particle is inclined at any angle to the line of action of the central force. The principles and equations of Chapter IX are sufficient for the inquiry, and have indeed been applied to the subject in Arts. 287, 288, 293, 295, in the form of rectangular coordinate resolution: but as the method of resolution into radial and transversal components is more convenient, and as formulæ different to any heretofore employed are deducible from them; and moreover as nature presents to us more instances of this than of any other kind of dynamical action, it is desirable to devote a separate Chapter to the inquiry: and in the course of it we shall take occasion to exhibit the first elements of celestial mechanics in the form of the simple elliptic orbit which a planet undisturbed would describe about its primary.

306.] Let $m$ be the mass of the moving particle, and let $(x, y, z)$ be its position at the time $t$; let the centre of force, which we suppose to be fixed, be the origin of coordinates: and let P represent the central force, that is, the impressed velocity-increment in an unit of time: let $r$ be the distance of $m$ from the centre of force at the time $t$: then we suppose P to be a function of $r$; let P be attractive, so that the equations of motion in terms of velocity-increments are,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -\frac{\text{P}x}{r} \\ \frac{d^2y}{dt^2} &= -\frac{\text{P}y}{r} \\ \frac{d^2z}{dt^2} &= -\frac{\text{P}z}{r} \end{aligned}\right\}. \tag{1}$$

Now in the first place I shall shew that the moving particle is always in one and the same plane, and which passes through the centre of force.

From the second and third of (1) we have,

$$y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2} = 0;$$

$$\left.\begin{aligned} \therefore \quad \text{integrating} \quad & y\frac{dz}{dt} - z\frac{dy}{dt} = h_1 \\ \text{similarly} \quad & z\frac{dx}{dt} - x\frac{dz}{dt} = h_2 \\ & x\frac{dy}{dt} - y\frac{dx}{dt} = h_3 \end{aligned}\right\}; \tag{2}$$

therefore multiplying these equations severally by $x, y, z$, and adding,

$$h_1 x + h_2 y + h_3 z = 0; \tag{3}$$

which is the equation to a plane passing through the origin, which is the centre of force, and in which therefore $m$ always is. The orbit therefore of $m$, as the trajectory is called, is a plane curve.

This fact too is evident by the principle of sufficient reason: because every reason which can be urged why $m$ should move out of the plane, which contains the centre and two consecutive points of the path, on one side may be shewn to be equally valid, why it should leave the plane towards the other side.

We may therefore, without loss of generality, suppose the plane in which $m$ moves to be that of $xy$.

307.] Let P represent the central force and be attractive; let the centre of force be the origin, $(x, y)$ the position of $m$ at the time $t$; $r =$ the distance of $m$ from the origin: then the equations of motion in terms of velocity-increments are,

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{\mathrm{P}x}{r} \\ \frac{d^2y}{dt^2} &= -\frac{\mathrm{P}y}{r}\end{aligned}\right\}; \tag{4}$$

therefore multiplying the former by $y$, and the latter by $x$, and subtracting,

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0; \tag{5}$$

therefore adding and subtracting $\frac{dx}{dt}\frac{dy}{dt}$, and integrating, we have,

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h; \tag{6}$$

where $h$ is an undetermined constant. We proceed to explain (6).

If $p$ is the perpendicular from the origin on the tangent, by Vol. I, Art. 186, (44),

$$x\,dy - y\,dx = p\,ds;$$

therefore from (6),

$$\frac{ds}{dt} = \frac{h}{p}; \tag{7}$$

that is, the velocity at any point of the orbit varies inversely as the perpendicular on the tangent at that point from the centre of force.

Also transforming (6) into its equivalent in terms of polar coordinates, we have

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}; \quad \therefore \quad \left.\begin{aligned} dx &= dr\cos\theta - r\sin\theta\,d\theta \\ dy &= dr\sin\theta + r\cos\theta\,d\theta \end{aligned}\right\}; \tag{8}$$

$$\therefore \quad x\,dy - y\,dx = r^2\,d\theta; \tag{9}$$

therefore from (6),

$$r^2\,d\theta = h\,dt. \tag{10}$$

Now $r^2\,d\theta$ is twice the sectorial area which the radius vector of $m$ describes in the time $dt$, and as it is proportional to $dt$ by (10), we infer that the sectorial areas described by the radius vector of $m$ are proportional to the times of describing them: or in other and equivalent words, Equal sectorial areas are described in equal times. Hence also it appears that

$h$ = twice the sectorial area described in one unit of time. (11)

Let us give a geometrical proof and interpretation of the theorems (7) and (10). Suppose S to be the centre of force, fig. 125, P to be the position of $m$ at the time $t$, and the element PQ to be its path-element in the time $dt$; and let $t$ be equicrescent; and let SY $= p$ be the perpendicular from S on PQ produced. Now let PQ be produced to R′ so that QR′ $=$ PQ; then if no force acted, $m$ would at the end of the second $dt$ be at R′; but suppose, when $m$ is at Q, the central force to act impulsively and to draw $m$ over a distance Q$r$ in the time $dt$; then at the end of $dt$, if R′R is equal and parallel to Q$r$, and the parallelogram $r$R′ is completed, $m$ is at R; by a similar process and construction it may be shewn that $m$ is at T at the end of the third $dt$, and so on: now since PQ $=$ QR′, therefore the triangles SPQ, SQR′ are equal, and because SRQ, SR′Q are on the same base SQ and between the same parallels, SRQ $=$ SR′Q; therefore SPQ $=$ SRQ: similarly it may be shewn that SRQ $=$ STR $=$ SUT, $= \ldots$; and thus the sectorial triangles which correspond to equal $dt$'s are equal. And the same result is true in the limit, when the polygon which is drawn in the figure becomes a continuous curve, and the central force acts continuously; and thus under the action of a central force equal sectorial areas are described in equal times.

Also let $h =$ twice the sectorial area described in an unit of time: let PQ $= ds$, SP $= r$, PSQ $= d\theta$, SY $= p$: then the triangle described in $dt$ units of time is PSQ; and the area of PSQ $= \dfrac{r^2\,d\theta}{2}$, see Vol. II, Art. 123; and also is equal to $\dfrac{1}{2}$ PQ $\times$ SY $= \dfrac{p\,ds}{2}$;

$$\therefore\quad \frac{h}{2}\,dt = \frac{r^2\,d\theta}{2} = \frac{p\,ds}{2}; \qquad \therefore\quad h = r^2\,\frac{d\theta}{dt} = p\,\frac{ds}{dt};$$

and therefore as equal sectorial areas are described in equal times, so does the velocity vary inversely as the perpendicular from the centre of force on the tangent.

Also since $\dfrac{d\theta}{dt} = \dfrac{h}{r^2}$, it appears that the ratio of the increase of $\theta$ to that of $t$, varies inversely as the square of the radius vector at the point.

Also if $t$ is the time during which the particle passes from a point in its orbit corresponding to $\theta_1$ to another point corresponding to $\theta_0$, then since

$$dt = \frac{r^2}{h}\,d\theta,$$

$$\therefore\quad t = \frac{1}{h}\int_{\theta_0}^{\theta_1} r^2\,d\theta; \tag{12}$$

whereby in a given orbit the time may be found in terms of the angle through which the radius vector of the particle has moved. The means of determining $h$ in a given orbit and under a given absolute force will be shewn hereafter.

Thus also if $\text{A}$ = the whole area enclosed by a curve through which $m$ has passed, and if $\text{T}$ is the whole time, or the periodic time as it is called,

$$\text{T} = \frac{2\text{A}}{h}. \tag{13}$$

308.] Again multiplying the first of (4) by $2dx$, and the second by $2dy$, and adding, we have

$$\frac{2\,dx\,d^2x+2\,dy\,d^2y}{dt^2} = -\frac{2\text{P}(x\,dx+y\,dy)}{r}; \tag{14}$$

but since $x^2+y^2 = r^2$, $\therefore\ x\,dx+y\,dy = r\,dr$;

$$\therefore\ \frac{2\,dx\,d^2x+2\,dy\,d^2y}{dt^2} = -2\text{P}\,dr. \tag{15}$$

Let $\text{V}$ be the velocity at a given point, at which $r = \text{R}$ (say); therefore integrating (15), we have

$$\frac{ds^2}{dt^2} - \text{V}^2 = -2\int_{\text{R}}^{r} \text{P}\,dr; \tag{16}$$

$$\therefore\ (\text{velocity})^2 = \text{V}^2 - 2\int_{\text{R}}^{r} \text{P}\,dr. \tag{17}$$

Let $v$ be the velocity at the point to which $r$ corresponds: and let $m$ be the mass of the moving particle: then

$$mv^2 - m\text{V}^2 = -2m\int_{\text{R}}^{r} \text{P}\,dr; \tag{18}$$

and if $\text{P}$ is a function of the distance of $m$ from the centre of force, it appears from this equation that the increase of vis viva of the particle in passing from one point to another depends on the coordinates of the two points, and not on the path which the particle has described in the passage: the change in vis viva therefore depends only on the distance through which the force has acted in its own line of action. This is another instance of a general principle of vis viva which we shall have occasion to investigate hereafter.

From (17) it appears that the velocity is the same at all points which are equally distant from the centre; for if $r = \text{R}$, the velocity $= \text{V}$: and thus if the orbit is a reentering curve, the particle always in its successive revolutions passes through the same point with the same velocity.

Now equating the values of the velocity $\left(=\frac{ds}{dt}\right)$ in (7) and (16), we have

$$\frac{h^2}{p^2} = \text{v}^2 - 2\int_{\text{R}}^{r} \text{P}\, dr; \tag{19}$$

therefore differentiating we have,

$$\frac{-2h^2}{p^3}\, dp = -2\text{P}\, dr;$$

$$\therefore \quad \text{P} = \frac{h^2}{p^3}\frac{dp}{dr}. \tag{20}$$

And if $\quad r = \frac{1}{u}, \qquad dr = -\frac{du}{u^2};$

and Vol. I, Art. 221, (19), $\frac{1}{p^2} = u^2 + \frac{du^2}{d\theta^2};$

$$\therefore \quad \frac{-2\,dp}{p^3} = 2\,du\left(u + \frac{d^2u}{d\theta^2}\right);$$

therefore (19) becomes

$$\text{P} = h^2u^2\left\{\frac{d^2u}{d\theta^2} + u\right\}; \tag{21}$$

and thus from either (20) or (21) may the law of central force be determined, under the action of which a particle moves in a given curve. And from (12) or (13) may the time, which is occupied by its passage through a given arc, or through the whole curve, if the curve is closed, be found. And from (7) may the velocity at any point in the orbit be determined.

Also because that part of the radius vector at any point of a curve referred to polar coordinates which is intercepted by the circle of curvature, or the chord of the circle of curvature, as it is called, see Vol. I, Art. 252, is equal to $2p\frac{dr}{dp}$, equation (20) gives,

$$\begin{aligned} \frac{h^2}{p^2} = (\text{vel.})^2 &= p\text{P}\frac{dr}{dp} \\ &= 2\text{P} \times \frac{\text{chord of circle of curvature}}{4}; \end{aligned} \tag{22}$$

and comparing this with (33), Art. 220, it appears that a particle at rest on the curve, and moving from it towards the centre of force under the action of the force continuing constant, acquires the velocity which the particle has in its curvilinear course, when it has moved through one-fourth of the chord of the circle of curvature.

I may finally observe that the results at which we have arrived are deducible from the equations (37), (38), Art. 256, where the impressed velocity-increments are resolved into their radial and transversal components: for here we have a central force only: therefore Q = 0, and therefore from (39), Art. 256,

$$\frac{r^2\,d\theta}{dt} = \text{a constant} = h \text{ (say)};$$

$$\therefore \quad \frac{d^2r}{dt^2} - \frac{h^2}{r^3} = -\text{P}. \qquad (23)$$

A point in an orbit at which the curve is at right angles to the radius vector is called an apse; the radius vector at an apse is called an apsidal distance; and the angle between two consecutive apsidal distances is called an apsidal angle of the orbit. The analytical character of an apse is manifestly

$$\frac{du}{d\theta} = 0, \quad \text{or} = \infty.$$

309.] Examples illustrative of the preceding equations.

Ex. 1. It is required to find the law of force, the velocity, and the periodic time, in an elliptic orbit, when the centre of force is in the focus.

Let the equation to the ellipse, the focus being the pole, be

$$r = \frac{a(1-e^2)}{1+e\cos\theta}; \qquad (24)$$

$$\therefore \quad u = \frac{1+e\cos\theta}{a(1-e^2)},$$

$$\frac{du}{d\theta} = -\frac{e\sin\theta}{a(1-e^2)}, \qquad \frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{a(1-e^2)};$$

$$\therefore \quad \frac{d^2u}{d\theta^2} + u = \frac{1}{a(1-e^2)},$$

$$\begin{aligned} \text{P} &= h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) \\ &= \frac{h^2u^2}{a(1-e^2)} \\ &= \frac{h^2}{a(1-e^2)}\frac{1}{r^2}; \end{aligned} \qquad (25)$$

therefore the force varies inversely as the square of the distance, and is attractive, as appears by its sign, and by the convention as to signs which was assumed in Art. 306. Let $\mu$ be the absolute force of the central force, then

$$\text{P} = \frac{\mu}{r^2};$$

and $$\mu = \frac{h^2}{a(1-e^2)}, \qquad \therefore \quad h = \{\mu a(1-e^2)\}^{\frac{1}{2}}; \qquad (26)$$

so that $h$ is given in terms of the absolute force, (which is the mass of the attracting body, or the sum of the masses of the attracting and attracted bodies, if the motion is relative), and of the quantities which determine the magnitude of the orbit.

Also since
$$\begin{aligned}\frac{1}{p^2} &= u^2 + \frac{du^2}{d\theta^2} \\ &= \frac{2au-1}{a^2(1-e^2)};\end{aligned}$$

$$\begin{aligned}\therefore \quad (\text{vel.})^2 &= \frac{h^2}{p^2} \\ &= \frac{\mu(2au-1)}{a}. \qquad (27)\end{aligned}$$

If $\text{T}$ is the periodic time in the elliptic orbit, then, as the area of the ellipse $= \pi ab = \pi a^2(1-e^2)^{\frac{1}{2}}$, therefore by (13),

$$\begin{aligned}\text{T} &= \frac{2\pi a^2(1-e^2)^{\frac{1}{2}}}{\{\mu a(1-e^2)\}^{\frac{1}{2}}} \\ &= \frac{2\pi}{\mu^{\frac{1}{2}}} a^{\frac{3}{2}}; \qquad (28)\end{aligned}$$

therefore the periodic time varies as the square root of the cube (as the sesquiplicate power) of the major axis. As these results will come under consideration hereafter, it is unnecessary now to comment on them.

Ex. 2. To find the law of force and the velocity in a parabola, the focus of which is the centre of force.

Let the equation to the parabola be

$$r = \frac{2a}{1+\cos\theta}; \qquad (29)$$

so that $\theta$ is measured from the line joining the focus and the vertex.

$$\begin{aligned}\therefore \quad \text{P} &= h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) \\ &= \frac{h^2u^2}{2a} \\ &= \frac{h^2}{2a}\frac{1}{r^2}; \qquad (30)\end{aligned}$$

therefore the central force varies inversely as the square of the distance, and is attractive. Let $\mu =$ the absolute force;

$$\therefore \quad \mu = \frac{h^2}{2a}, \qquad h^2 = 2a\mu; \qquad (31)$$

$$\therefore \quad (\text{velocity})^2 = \frac{h^2}{p^2}$$
$$= h^2\left(u^2 + \frac{du^2}{d\theta^2}\right)$$
$$= \frac{2\mu}{r}. \tag{32}$$

Also let $t$ be the time during which the particle moves from a point corresponding to $\theta_0$ to another point corresponding to $\theta_n$; then by (12),

$$t = \frac{(2a)^{\frac{3}{2}}}{(\mu)^{\frac{1}{2}}}\int_{\theta_0}^{\theta_n} \frac{d\theta}{(1+\cos\theta)^2}$$
$$= \frac{a^{\frac{3}{2}}}{(2\mu)^{\frac{1}{2}}}\int_{\theta_0}^{\theta_n}\left(\sec\frac{\theta}{2}\right)^4 d\theta$$
$$= \left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}\left\{\tan\frac{\theta_n}{2} - \tan\frac{\theta_0}{2} + \frac{1}{3}\left(\tan\frac{\theta_n}{2}\right)^3 - \frac{1}{3}\left(\tan\frac{\theta_0}{2}\right)^3\right\}. \tag{33}$$

And this value for $t$ may be expressed in terms of $r_n$ and $r_0$, the focal radii vectores corresponding to $\theta_n$ and to $\theta_0$, and of the chord $c$ (say) which joins their extremities.

For the sake of more convenient symbols, let $\tan\frac{\theta_n}{2} = t_n$, $\tan\frac{\theta_0}{2} = t_0$; so that (33) becomes,

$$t = \left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}\left\{t_n - t_0 + \frac{t_n^3 - t_0^3}{3}\right\}$$
$$= \left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}(t_n - t_0)\left(1 + \frac{t_n^2 + t_n t_0 + t_0^2}{3}\right).$$

By a substitution due to C. F. Gauss, let

$$1 + \frac{(t_n + t_0)^2}{4} = \eta^2;$$

$$\therefore \quad t = \left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}(t_n - t_0)\left(\eta^2 + \frac{(t_n - t_0)^2}{12}\right)$$
$$= \frac{1}{3}\left(\frac{2a^3}{\mu}\right)^{\frac{1}{2}}\left\{\left(\eta + \frac{t_n - t_0}{2}\right)^3 - \left(\eta - \frac{t_n - t_0}{2}\right)^3\right\}. \tag{34}$$

But if $c$ is the chord joining the extremities of $r_n$ and $r_0$,

$$c^2 = r_n^2 - 2r_n r_0 \cos(\theta_n - \theta_0) + r_0^2$$
$$= (r_n\cos\theta_n - r_0\cos\theta_0)^2 + (r_n\sin\theta_n - r_0\sin\theta_0)^2.$$

Also $\quad r_n = a(1 + t_n^2), \qquad r_0 = a(1 + t_0^2),$

and $\quad \cos\theta_n = \dfrac{1 - t_n^2}{1 + t_n^2}, \qquad \sin\theta_n = \dfrac{2t_n}{1 + t_n^2};$

and similar values are true for $\cos\theta_0$ and $\sin\theta_0$; therefore

$$\begin{aligned} c^2 &= 4a^2(t_n-t_0)^2\left\{\frac{(t_n+t_0)^2}{4}+1\right\} \\ &= 4a^2(t_n-t_0)^2\eta^2; \\ \therefore\quad c &= 2a(t_n-t_0)\eta; \end{aligned}$$

$$\begin{aligned} \therefore\quad r_n+r_0+c &= 2a\left\{1+\frac{t_n^2+t_0^2}{2}+(t_n-t_0)\eta\right\} \\ &= 2a\left\{\eta+\frac{t_n-t_0}{2}\right\}^2; \end{aligned}$$

similarly
$$r_n+r_0-c = 2a\left\{\eta-\frac{t_n-t_0}{2}\right\}^2;$$

therefore substituting in (34),

$$t = \frac{1}{6\mu^{\frac{1}{2}}}\{(r_n+r_0+c)^{\frac{3}{2}}-(r_n+r_0-c)^{\frac{3}{2}}\}. \tag{35}$$

This Theorem is generally known by the name of Lambert's Theorem.

Ex. 3. If the equation of a hyperbola, of which the focus is the pole, is

$$r = \frac{a(e^2-1)}{1+e\cos\theta};$$

then
$$\text{P} = \frac{h^2}{a(e^2-1)}\frac{1}{r^2}, \qquad (\text{vel.})^2 = \frac{\mu(2au+1)}{a}.$$

310.] Ex. 4. A particle moves in an ellipse about a centre of force in the centre; it is required to find the law of force, the velocity at any point of the orbit, and the periodic time.

The equation to the ellipse is,

$$\frac{(\cos\theta)^2}{a^2}+\frac{(\sin\theta)^2}{b^2} = u^2; \tag{36}$$

$$\therefore\quad u\frac{du}{d\theta} = \left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin\theta\cos\theta, \tag{37}$$

$$u\frac{d^2u}{d\theta^2}+\frac{du^2}{d\theta^2} = \left(\frac{1}{b^2}-\frac{1}{a^2}\right)\{(\cos\theta)^2-(\sin\theta)^2\}. \tag{38}$$

But from (36),

$$(\cos\theta)^2 = \frac{u^2-\dfrac{1}{b^2}}{\dfrac{1}{a^2}-\dfrac{1}{b^2}}, \qquad (\sin\theta)^2 = \frac{u^2-\dfrac{1}{a^2}}{\dfrac{1}{b^2}-\dfrac{1}{a^2}};$$

therefore substituting these values in (38), and also from (37), we shall find

$$\text{P} = \frac{h^2}{a^2 b^2 u}$$

$$= \frac{h^2}{a^2 b^2} r; \tag{39}$$

therefore the force varies directly as the distance, and is attractive: and if $\mu =$ the absolute force, $h^2 = \mu a^2 b^2$. Also

$$(\text{velocity})^2 = \mu(a^2 + b^2 - r^2). \tag{40}$$

And if $\text{T} =$ the periodic time, by (14),

$$\text{T} = \frac{2\pi ab}{h}$$

$$= \frac{2\pi}{\mu^{\frac{1}{2}}}; \tag{41}$$

that is, the periodic time is independent of the magnitude of the ellipse, and depends only on the absolute central force.

And the time in which the particle passes through an arc which subtends a given angle at the centre may thus be found. Let the arc begin at the extremity of the major axis; then if $t =$ the time required, from (12) we have

$$t = \frac{ab}{\mu^{\frac{1}{2}}} \int_0^{\theta} \frac{d\theta}{a^2 (\sin\theta)^2 + b^2 (\cos\theta)^2}$$

$$= \frac{1}{\mu^{\frac{1}{2}}} \tan^{-1}\left(\frac{a \tan\theta}{b}\right); \tag{42}$$

and thus if $\theta = \frac{\pi}{2}$, $t = \frac{\pi}{2\mu^{\frac{1}{2}}}$; where $t$ is one-fourth of the periodic time: and thus the whole time is the same as that given in (41).

Ex. 5. In the hyperbola described by a particle under the action of a central force in its centre, and of which the equation is,

$$\frac{(\cos\theta)^2}{a^2} - \frac{(\sin\theta)^2}{b^2} = u^2,$$

$$\text{P} = -\frac{h^2}{a^2 b^2} r = -\mu r; \tag{43}$$

that is, the central force varies directly as the distance, and is repulsive: also

$$(\text{velocity})^2 = \mu(r^2 - a^2 + b^2); \tag{44}$$

and the time from the extremity of the transverse axis

$$= \frac{1}{2\mu^{\frac{1}{2}}} \log \frac{b + a\tan\theta}{b - a\tan\theta}. \tag{45}$$

311.] Ex. 6. Let the particle move in a circle; and

(1) Let the centre of force be in the centre: let $a =$ the radius: then the equation to the circle is

$$r = a;$$

$$\therefore \quad u = \frac{1}{a}, \qquad \frac{du}{d\theta} = \frac{d^2u}{d\theta^2} = 0, \qquad \therefore \quad \mathrm{P} = \frac{h^2}{a^3},$$

$$(\text{velocity})^2 = \frac{h^2}{a^2} = \mathrm{V}^2 \text{ (say)};$$

therefore the central force is constant, but varies inversely as the cube of the radius as we pass from one circle to another: also the velocity is constant, and varies inversely as the radius: and if $\mathrm{T} =$ the periodic time,

$$\mathrm{T} = \frac{2\pi a}{\mathrm{V}} = \frac{2\pi a^2}{h}.$$

Also $\mathrm{P} = \frac{\mathrm{V}^2}{a}$; now as $\mathrm{V}^2$ is the velocity and $a$ is the radius of the circle, $\frac{\mathrm{V}^2}{a}$, see Art. 255, is the centrifugal force: the central force is therefore equal to the centrifugal force in the circle; the central force, that is, draws the particle towards the centre over a space equal to that by which the centrifugal force (so to speak) removes it from the centre: and as the velocity is constant, no part of the central force acts either to increase or to diminish the velocity in the circular path.

(2) Let the centre of force be on the circumference of the circle: and let the equation be

$$r = 2a\cos\theta, \qquad \therefore \quad 2au = \sec\theta,$$

$$2a\frac{du}{d\theta} = \sec\theta\tan\theta, \qquad 2a\frac{d^2u}{d\theta^2} = \sec\theta(\tan\theta)^2 + (\sec\theta)^3;$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = 8a^2u^3 - u;$$

$$\therefore \quad \mathrm{P} = 8a^2h^2u^5$$

$$= \frac{8a^2h^2}{r^5}; \tag{46}$$

that is, the force varies inversely as the fifth power of the distance, and is attractive; and if $\mu =$ the absolute force, $\mu = 8a^2h^2$;

$$\therefore \quad h^2 = \frac{\mu}{8a^2};$$

also $$(\text{velocity})^2 = \frac{\mu r^4}{2}.$$

And if T = the periodic time,

$$\text{T} = \frac{2^{\frac{5}{2}}\pi a^3}{\mu^{\frac{1}{2}}}. \tag{47}$$

And if $t$ is the time of the motion of the particle from the extremity of the diameter to the point corresponding to $\theta$, then

$$t = 4a^3\left(\frac{2}{\mu}\right)^{\frac{1}{2}}\left(\theta+\frac{\sin 2\theta}{2}\right).$$

(3) Let the centre of force be at any point within or without the circle: and suppose it to be at S, see fig. 126; and let C be the centre of the circle, SC $= c$, CA $= a$, SP $= r$, SY $= p$: then because SC$^2$ = SP$^2$ − 2 SP.CP cos SPC + CP$^2$,

$$\therefore\quad c^2 = r^2-2ap+a^2; \tag{48}$$

$$\therefore\quad \frac{dr}{dp} = \frac{a}{r};$$

therefore from (20), $$\text{P} = \frac{8a^2h^2r}{(r^2+a^2-c^2)^3}, \tag{49}$$

312.] Ex. 7. It is required to find the law of force, the velocity, and the periodic time in, (1) the Lemniscata of Bernoulli: (2) the Cardioid.

(1) $r^2 = a^2\cos 2\theta$.

$$\therefore\quad a^2u^2 = \sec 2\theta;$$

$$a^2u\frac{du}{d\theta} = \sec 2\theta\tan 2\theta$$

$$= a^2u^2\tan 2\theta;$$

$$\therefore\quad \frac{du}{d\theta} = u\tan 2\theta;$$

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan 2\theta+2u(\sec 2\theta)^2;$$

$$\therefore\quad \text{P} = \frac{3h^2a^4}{r^7}; \tag{50}$$

and therefore the force varies inversely as the seventh power of the distance, and is attractive: and if $\mu$ = the absolute force, $\mu = 3h^2a^4$,

$$\therefore\quad h^2 = \frac{\mu}{3a^4}.$$

Thus $$(\text{velocity})^2 = \frac{\mu}{3r^6};$$

and if $t$ = the time from an apse, then from (12),

$$t = \left(\frac{3}{\mu}\right)^{\frac{1}{2}} a^4 \int_0^{\theta} \cos 2\theta \, d\theta$$

$$= \left(\frac{3}{\mu}\right)^{\frac{1}{2}} \frac{a^4}{2} \sin 2\theta; \tag{51}$$

and therefore the time to the node $= \left(\frac{3}{\mu}\right)^{\frac{1}{2}} \frac{a^4}{2}$; and the time of describing one loop $= \left(\frac{3}{\mu}\right)^{\frac{1}{2}} a^4$.

(2) the equation to the Cardioid is

$$r = a(1+\cos\theta)$$

$$= 2a\left(\cos\frac{\theta}{2}\right)^2;$$

$$\therefore \quad 2au = \left(\sec\frac{\theta}{2}\right)^2;$$

$$\therefore \quad 2a\frac{du}{d\theta} = \left(\sec\frac{\theta}{2}\right)^2 \tan\frac{\theta}{2}$$

$$= 2au\tan\frac{\theta}{2};$$

$$\therefore \quad \frac{du}{d\theta} = u\tan\frac{\theta}{2};$$

$$\frac{d^2u}{d\theta^2} = u\left(\tan\frac{\theta}{2}\right)^2 + \frac{u}{2}\left(\sec\frac{\theta}{2}\right)^2$$

$$= 3au^2 - u;$$

$$\therefore \quad P = \frac{3ah^2}{r^4}; \tag{52}$$

and therefore the force is attractive and varies inversely as the fourth power of the distance; and if $\mu$ is the absolute force, $\mu = 3ah^2$,

$$\therefore \quad h^2 = \frac{\mu}{3a}.$$

Therefore $\qquad (\text{velocity})^2 = \dfrac{2\mu}{3ar^3}.$

And if $t =$ the time from the apse to the point corresponding to $\theta = \theta$, then

$$t = \frac{\mu^{\frac{1}{2}} a^{\frac{5}{2}}}{3^{\frac{1}{2}}} \int_0^{\theta} (1+\cos\theta)^2 \, d\theta$$

$$= \frac{\mu^{\frac{1}{2}} a^{\frac{5}{2}}}{3^{\frac{1}{2}}} \left\{\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right\}; \tag{53}$$

and therefore if $\theta = \pi$, the time from the apse to the pole $= \dfrac{(3\mu a^5)^{\frac{1}{2}}}{2}\pi$; and therefore if $T =$ the periodic time,

$$T = (3\mu a^5)^{\frac{1}{2}}\pi.$$

313.] It is required to find the law of force in, (1) the curve whose equation is, $r = 2a\cos n\theta$; (2) the curve whose equation is, $r = \dfrac{2a}{1-e\cos n\theta}$.

(1) The equation of the first curve in terms of $u$ is,

$$2au = \sec n\theta;$$

$$\therefore\quad 2a\frac{du}{d\theta} = n\sec n\theta\tan n\theta;$$

$$\begin{aligned} 2a\frac{d^2u}{d\theta^2} &= n^2\sec n\theta(\tan n\theta)^2 + n^2(\sec n\theta)^3 \\ &= 2au\,n^2(8a^2u^2-1); \end{aligned}$$

$$\therefore\quad \text{P} = \frac{8a^2h^2n^2}{r^5} + \frac{(1-n^2)h^2}{r^3}; \tag{54}$$

and therefore the force varies partly as the inverse fifth power, and partly as the inverse cube, of the distance.

And if $t =$ the time in which the particle moves from the point corresponding to $\theta = 0$, to that corresponding to $\theta = \theta$,

$$\begin{aligned} t &= \frac{4a^2}{h}\int_0^\theta (\cos n\theta)^2\,d\theta \\ &= \frac{2a^2}{h}\left(\theta + \frac{\sin 2n\theta}{2n}\right). \end{aligned}$$

As to (54) it is to be observed that the second term of the equivalent of P disappears, if $n = 1$; and that in this case the central force varies as the inverse fifth power of the distance. Now if $n = 1$, the equation to the orbit is $r = 2a\cos\theta$; that is, is the equation of a circle, of which the pole is on the circumference and the prime radius passes through the centre: and when $\theta = 0$, $r = 2a$ in both the circle and the given curve. A process of tracing such a curve, and of representing the motion of a particle on it, is hereby suggested to us. In fig. 127 take a line SX for a prime radius; on it take $\text{SCA} = 2a$; and on SA as a diameter describe a semicircle SQA: then since $r = 2a$, when $\theta = 0$, in the equation to the orbit, the point A is common to the circle, and to the orbit: let us suppose $n$ to be less than unity: and let AP be the curve of the orbit: on it take any point P; join SP: then as $\text{PSX} = \theta$, $\text{SP} = 2a\cos n.\text{PSX}$; let the angle $\text{PSA}' = n.\text{PSX}$, and make $\text{SA}' = \text{SA} = 2a$; on SA′ as a diameter describe a semicircle; then the semicircle will pass through P, because by the property of the semicircle $\text{SP} = \text{SA}'\cos\text{PSA}' = 2a\cos n\theta$, and thus SP is the same for both the semicircle and the curve of the orbit. In the same manner may every point P be shewn to be on a semicircle, the diameter of which has a varying position, and as $PSA = \theta$, $PSA' = n\theta$, therefore $ASA' = (1-n)\theta$: therefore while P has moved over an arc of the orbit subtending an angle $\theta$ at S, SA′ has moved through an angle $(1-n)\theta$: and therefore the ratio of the angular velocity of P, to that of the revolving diameter, is as $1 : 1-n$; and, according to our system, SA′ revolves in the same direction as P moves. If however $n$ is greater than 1, SA′ revolves in a direction the opposite of that in which SP revolves. In either case the moving particle $m$ may be represented as moving in a semicircle, the diameter of which revolves about the pole S with an angular velocity bearing a constant ratio to the angular velocity of $m$; and the orbit of $m$ is for this reason called a *revolving circle*.

(2) Again in the second curve which is given,

$$2au = 1 - e\cos n\theta;$$

$$2a\frac{d^2u}{d\theta^2} = en^2\cos n\theta;$$

$$\therefore \quad P = \frac{h^2n^2}{2a}\frac{1}{r^2} + \frac{(1-n^2)h^2}{r^3}; \qquad (55)$$

therefore the central force is compounded of two different forces, of which one varies inversely as the square of the distance, and the other varies inversely as the cube of the distance.

Now the second term of the right-hand member of (55) disappears if $n = 1$, and in that case the central force varies inversely as the square of the distance; but if $n = 1$, the equation to the orbit becomes

$$r = \frac{2a}{1 - e\cos\theta}; \qquad (56)$$

the equation to a conic, of which the focus is the pole, and the prime radius is the principal axis: and to fix our thoughts I will suppose $e$ less than unity, so that the conic is an ellipse: now if $\theta = 0$, $r = \frac{2a}{1-e}$, in both the conic and the orbit curve; so that if SX, fig. 128, is a prime radius, and the line SA is taken on it such that $SA = \frac{2a}{1-e}$, then A is a point common to the conic and the orbit. And to fix our thoughts, let us also suppose $n$ to be less than unity; and let us suppose the curve APQ to be that of the orbit, of which P is any point, and $PSX = \theta$: take an angle $PSA' = n\theta$, and let $SA' = SA$, and produce A′S to B′, so that SB$'=\dfrac{2a}{1+e}$; and on A$'$B$'$ as a major axis with S as the focus, describe an ellipse; then the radius vector of the ellipse corresponding to the angle PSA$'$

$$= \frac{2a}{1-e\cos \text{PSA}'}$$

$$= \frac{2a}{1-e\cos n\theta};$$

therefore the radius vector of the ellipse is equal to that of the orbit, and therefore the point P which is on the orbit is also on the ellipse. In a similar way it may be shewn that every point on the orbit is on an ellipse, the major axis of which has a varying position: and as PSA $=\theta$, PSA$'=n\theta$, therefore A$'$SA $=(1-n)\theta$; that is, as $m$ has passed over an arc subtending an angle $\theta$ at S, so has the major axis of the ellipse passed through an angle $(1-n)\theta$; the angular velocity therefore of the former is to that of the latter as $1 : 1-n$: and SA$'$ revolves in a direction the same as that of $m$ if $n$ is less than 1, but in an opposite direction if $n$ is greater than 1. In either case the moving particle may be represented as moving in an ellipse, the major axis of which revolves about the centre of force with an angular velocity bearing a constant ratio to that of the moving particle: the orbit of $m$ is for this reason called a *revolving ellipse*.

Also since in the orbit

$$u = \frac{1-e\cos n\theta}{2a}, \qquad \therefore\ \frac{du}{d\theta} = \frac{ne\sin n\theta}{2a};$$

now if $\dfrac{du}{d\theta}=0$, the corresponding point in the orbit is an apse: and the line drawn from the pole to an apse is the apsidal distance: therefore the orbit has an apse, whenever $\sin n\theta = 0$: that is, when

$$\theta = 0, \qquad \theta = \frac{\pi}{n}, \qquad \theta = \frac{2\pi}{n}, \ldots;$$

therefore the angle between two successive apsidal distances $=\dfrac{\pi}{n}$.

314.] By processes similar to those employed above let it be shewn that in the orbits whose equations are the following, viz.,

(1) $r = \dfrac{2a}{e^{n\theta} \pm e^{-n\theta}}$, (2) $r = a^{\theta}$,

(3) $r = \dfrac{a}{\theta}$, (4) $r = a\sec n\theta$,

the central force varies inversely as the cube of the distance: that in the lituus, whose equation is $a^2u^2 = \theta$, the force varies partly directly as the distance, and partly inversely as the cube of the distance: and that in the involute of the circle, of which the equation is, $r^2 = a^2+p^2$,

$$\text{P} = \frac{h^2 r}{(r^2-a^2)^2}.$$

315.] As an accurate comprehension of the expressions for P (the central force) is of great importance in the understanding of the phænomena of the action of a central force, I think it desirable to insert the following proof which is founded on first principles.

Let $m$ be the mass of a particle moving in a certain orbit under the action of a central force, the impressed velocity-increment of which along the radius vector is represented by P: let P, fig. 129, be the place of $m$ at the time $t$, and let $\text{PQ} = ds$ be the length-element described by $m$ in the time $dt$; and let QS be the length-element described in the succeeding $dt$. Now if no central force acted, $m$ would in $dt$ pass through QR in the line PQ produced, instead of passing through QS: but suppose a force, whose source is in O, to act on $m$ at Q, and to cause it to pass through QV in the line QO in the time $dt$; then at the end of the second $dt$, $m$ is at S, QS being the diagonal of the parallelogram of which QV and QR are two containing and adjacent sides; it is our purpose to estimate the effect of the central force as expressed in the deflexion of $m$ from its rectilinear path.

We may consider the central force P to be constant during the time $dt$ of its action on $m$, whereby it draws $m$ over the space QV; and therefore by (31), Art. 220,

$$2.\text{QV} = \text{P}\,dt^2. \tag{57}$$

Let QΠ be the radius of curvature of the curve at Q, and let QN be the projection of QV on it: therefore, as it has been proved in Art. 257, if $\rho = \text{Q}\Pi$,

$$ds^2 = 2\rho \times \text{QN}.$$

Now
$$\begin{aligned}\text{QV} &= \text{QN} \sec \text{OQ}\Pi \\ &= \frac{ds^2}{2\rho}\frac{r}{p} \\ &= \frac{ds^2\,dp}{2p\,dr}; \text{ (see Art. 251, Vol. I.)}\end{aligned}$$

therefore from (57), $$\text{P} = \frac{ds^2}{dt^2}\frac{dp}{p\,dr}$$

$$= \frac{h^2}{p^3}\frac{dp}{dr}, \qquad (58)$$

because by reason of (7), $\frac{ds^2}{dt^2} = \frac{h^2}{p^2}$: and this expression is the same as that before deduced analytically in Art. 308. The preceding process is nearly identical with that employed by Newton in Prop. VI, Section 2, Book I, of the Principia.

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## SECTION 2.—*The determination of orbits, and of their dimensions and position, when the laws of central force and other circumstances of motion are given.*

316.] In the previous Section the law of force and other circumstances of motion have been determined, when the equation to the orbit and the position of the centre of force have been given: it is our purpose now to inquire into the converse problem; and let it be observed, that for a complete determination of the orbit, when the law and centre of force are given, four constants, or what are, by means of the limits of the integral or otherwise, equivalent to four constants are required: this is evident from the form of the differential equations (4), which are two simultaneous differential equations of the second order, and the complete integral of each of which requires two constants: or again the equation (21) contains an undetermined constant $h$; and being of the second order, two more undetermined constants will be introduced during the process of integration: and one other constant will be required in the integral of (12), by means of which the time at which $m$ is at any point of the orbit may be found. The conditions which will for the most part be given in the following examples are, (1) the distance from the centre of force of the point where $m$ is at a given time; (2) the line in which $m$ is moving at the time, and the inclination of that line to the corresponding radius vector; and (3) the velocity with which $m$ is moving at the given time: the time at which all these circumstances are given is called the epoch; and in terms of them, the constants, or the limits of the integrals, can always be expressed.

317.] A particle $m$ is projected with a given velocity, in a given line, from a given point, and moves under the action of a central force, which varies directly as the distance and is attractive: it is required to determine the equation to the orbit, and the circumstances of motion.

The plane in which $m$ moves is manifestly that passing through the centre of force, and the point of projection, and which contains the line of projection.

Let the centre of force be the origin; let $\text{R} =$ the distance of the point of projection from the centre of force; $\text{V} =$ the velocity of projection; and let us suppose that $m$ is projected from an apse, so that the line of V is perpendicular to R: then since generally

$$\text{the velocity} = \frac{h}{p}; \qquad \therefore \quad \text{V} = \frac{h}{\text{R}};$$

$$\therefore \quad h = \text{VR}. \tag{59}$$

Let $t = 0$, when $m$ is projected with the velocity V: let $\mu =$ the absolute force of the central force: then since the force varies directly as the distance and is attractive,

$$\text{P} = \mu r = \frac{\mu}{u}. \tag{60}$$

Now from (21) we have,

$$\text{P} = h^2 u^2 \left\{ \frac{d^2 u}{d\theta^2} + u \right\};$$

therefore in this case

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2 u^3}; \tag{61}$$

multiplying by $2du$, and integrating, and taking the limits corresponding to $t = t$ and to $t = 0$, and observing that

$$(\text{vel.})^2 = h^2 \left(u^2 + \frac{du^2}{d\theta^2}\right),$$

we have,

$$\frac{du^2}{d\theta^2} + u^2 - \frac{\text{V}^2}{h^2} = -\frac{\mu}{h^2 u^2} + \frac{\mu \text{R}^2}{h^2}; \tag{62}$$

$$\therefore \quad (\text{the vel.})^2 = \text{V}^2 + \mu \text{R}^2 - \frac{\mu}{u^2}$$

$$= \text{V}^2 + \mu(\text{R}^2 - r^2); \tag{63}$$

and therefore the velocity is the greatest and least, according as $r$ is the least or the greatest.

And replacing $h$ in (62) by its value from (59), we have,

$$\frac{du^2}{d\theta^2} + u^2 - \frac{1}{\text{R}^2} = -\frac{\mu}{\text{V}^2 \text{R}^2 u^2} + \frac{\mu}{\text{V}^2};$$

$$\therefore \quad u^2\frac{du^2}{d\theta^2}+\left(u^2-\frac{\mathrm{V}^2+\mu\mathrm{R}^2}{2\mathrm{V}^2\mathrm{R}^2}\right)^2=\left(\frac{\mathrm{V}^2-\mu\mathrm{R}^2}{2\mathrm{V}^2\mathrm{R}^2}\right)^2;$$

$$\frac{2u\,du}{\left\{\left(\frac{\mathrm{V}^2-\mu\mathrm{R}^2}{2\mathrm{V}^2\mathrm{R}^2}\right)^2-\left(u^2-\frac{\mathrm{V}^2+\mu\mathrm{R}^2}{2\mathrm{V}^2\mathrm{R}^2}\right)^2\right\}^{\frac{1}{2}}}=2\,d\theta;$$

therefore integrating, and taking the limits corresponding to $t=t$ and to $t=0$, and assuming that the prime radius coincides with R, we have

$$\sin^{-1}\frac{2\mathrm{V}^2\mathrm{R}^2u^2-(\mathrm{V}^2+\mu\mathrm{R}^2)}{\mathrm{V}^2-\mu\mathrm{R}^2}-\sin^{-1}1=2\theta;$$

$$\therefore \quad 2\mathrm{V}^2\mathrm{R}^2u^2-(\mathrm{V}^2+\mu\mathrm{R}^2)=(\mathrm{V}^2-\mu\mathrm{R}^2)\cos 2\theta; \tag{64}$$

and changing to rectangular coordinates, we have

$$2\mathrm{V}^2\mathrm{R}^2-(\mathrm{V}^2+\mu\mathrm{R}^2)(x^2+y^2)=(\mathrm{V}^2-\mu\mathrm{R}^2)(x^2-y^2);$$

$$\therefore \quad \frac{x^2}{\mathrm{R}^2}+\frac{\mu y^2}{\mathrm{V}^2}=1; \tag{65}$$

the equation to an ellipse, of which the centre is the origin; $\mathrm{R}=$ the semi-axis parallel to the axis of $x$, $\frac{\mathrm{V}}{\mu^{\frac{1}{2}}}=$ the semi-axis parallel to the axis of $y$; and which is the orbit in which $m$ moves.

From (63) it appears that the velocity $=\mathrm{V}$, when $r=\mathrm{R}$; and $=\mathrm{R}\mu^{\frac{1}{2}}$, when $r=\frac{\mathrm{V}}{\mu^{\frac{1}{2}}}$: these therefore are the velocities at the extremities of the principal axes of the ellipse.

The point of projection is the extremity of the major or minor axes of the ellipse according as $\mathrm{R}^2$ is greater or less than $\frac{\mathrm{V}^2}{\mu}$; or in other words, as

$$\mathrm{V}^2 \text{ is less than, or greater than, } \mu\mathrm{R}^2. \tag{66}$$

Now suppose $m$ to be placed at a point at a distance R from the centre of force and to move in a rectilinear course towards it, then, as appears from Art. 232, the velocity of $m$ when it arrives at the centre is $\mathrm{R}\mu^{\frac{1}{2}}$; and therefore as the velocity of projection is greater than or less than this, so is the point of projection the extremity of the minor or the major axis.

And if the velocity of projection is equal to that which would be acquired by $m$ moving in the rectilinear path into the centre, then

$$\mathrm{V}^2=\mu\mathrm{R}^2, \tag{67}$$

and the orbit is a circle.

The periodic time is, by Art. 310, equation (41), equal to $\frac{2\pi}{\mu^{\frac{1}{2}}}$, and is independent of the magnitude of the ellipse. And if $t =$ the time from the point of projection to the point on the orbit corresponding to $\theta$, then as in (42),

$$t = \frac{1}{\mu^{\frac{1}{2}}}\tan^{-1}\left(\frac{\mathrm{R}\mu^{\frac{1}{2}}}{\mathrm{V}}\tan\theta\right); \tag{68}$$

and thus the circumstances of the orbit are completely determined.

I may observe that if the central force is repulsive, the sign of $\mu$ will be changed throughout the preceding investigations; and that the orbit will be a hyperbola referred to the centre as pole, of which the equation will be

$$\frac{x^2}{\mathrm{R}^2} - \frac{\mu y^2}{\mathrm{V}^2} = 1; \tag{69}$$

and
$$(\text{velocity})^2 = \mathrm{V}^2 + \mu(r^2 - \mathrm{R}^2); \tag{70}$$

$$t = \frac{1}{2\mu^{\frac{1}{2}}}\log\frac{\mathrm{V}+\mathrm{R}\mu^{\frac{1}{2}}\tan\theta}{\mathrm{V}-\mathrm{R}\mu^{\frac{1}{2}}\tan\theta}. \tag{71}$$

318.] In Article 233, (2), it is said that according to the principles of the undulatory theory of light, the force acting on a displaced molecule of ether, and bringing it back to its original position of rest, varies directly as the distance through which the molecule has been displaced; and it is also said that *generally* the line of motion of the particle, when it is brought within the action of this force, is not in the line joining the particle and its original position of rest. Here then are the circumstances required in the preceding Article; a particle moves in a given line, with a given velocity, and is acted on by an attractive force varying directly as the distance: its orbit therefore is an ellipse; and the periodic time in the ellipse is independent of the magnitude of it, and depends only on the absolute force in the centre; therefore the ethereal molecules move in ellipses which are in planes perpendicular to the line of propagation of a ray, and if the absolute force of the central force is the same for all the ellipses the periodic time is the same, but if it varies, the periodic time varies inversely as its square root. Now in this case, as in that of rectilinear motion, the intensity of light is supposed to depend on the magnitude of the ellipse, and the colour of light on the periodic time of the orbit: hence it appears that the variations of the intensity and of the colour are independent of each other; and this fact is in accordance with observation.

Hence also we have a physical explanation of other kinds of polarised light: if all the major axes (say) of the ellipses of ethereal motion are parallel to each other, the light is said to be elliptically-polarised: if the ellipses become circles, we have circularly-polarised light: if the azimuth of the major axes of the ellipses rotates uniformly, we have another modification of the ethereal vibrations. The further investigations however of such changes must be sought for in treatises where the phænomena of light are specially inquired into.

319.] A particle $m$ is projected with a given velocity from a given point and in a given line; and moves under the action of a central attractive force, which varies inversely as the square of the distance: it is required to determine the equation to the orbit, and the other circumstances of motion.

The plane in which $m$ moves is manifestly that which contains the point of projection, the centre of force, and the line in which $m$ is projected. Let the centre of force be the origin; $\mathrm{V} =$ the velocity of projection; $\mathrm{R} =$ the distance of the point of projection from the origin; $a =$ the angle between $\mathrm{R}$ and the line of projection; then since generally the velocity $= \frac{h}{p}$, and at the point of projection $p = \mathrm{R} \sin a$,

$$\therefore \quad \mathrm{V} = \frac{h}{\mathrm{R} \sin a};$$

$$\therefore \quad h = \mathrm{VR} \sin a; \tag{72}$$

and let $t = 0$, when $m$ is projected from the given point.

Let $\mu =$ the absolute force of the central force: then since the central force $\mathrm{P}$ varies inversely as the square of the distance, and is attractive,

$$\mathrm{P} = \frac{\mu}{r^2} = \mu u^2, \tag{73}$$

so that (21) becomes

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu}{h^2}. \tag{74}$$

Multiplying through by $2\,du$, integrating, taking the limits corresponding to $t = t$ and to $t = 0$, and observing that

$$(\text{velocity})^2 = h^2 \left(u^2 + \frac{du^2}{d\theta^2}\right),$$

we have,
$$\frac{du^2}{d\theta^2}+u^2-\frac{\mathrm{V}^2}{h^2}=\frac{2\mu u}{h^2}-\frac{2\mu}{h^2\mathrm{R}}; \qquad (75)$$

therefore,
$$(\text{velocity})^2 = \mathrm{V}^2+2\mu u-\frac{2\mu}{\mathrm{R}}$$
$$= \mathrm{V}^2+2\mu\left(\frac{1}{r}-\frac{1}{\mathrm{R}}\right); \qquad (76)$$

and thus the velocity is the greatest or least according as $r$ is the least or the greatest.

In (75) replacing $h$ by its value in (72) we have,
$$\frac{du^2}{d\theta^2}+u^2-\frac{1}{(\mathrm{R}\sin a)^2}=\frac{2\mu u}{(\mathrm{R}\mathrm{V}\sin a)^2}-\frac{2\mu}{\mathrm{R}^3(\mathrm{V}\sin a)^2}. \qquad (77)$$

To express this in a simpler form, let
$$\frac{\mu}{(\mathrm{R}\mathrm{V}\sin a)^2}=b, \qquad \frac{\mathrm{V}^2\mathrm{R}-2\mu}{\mathrm{R}^3\mathrm{V}^2(\sin a)^2}+\frac{\mu^2}{(\mathrm{R}\mathrm{V}\sin a)^4}=c^2; \qquad (78)$$

so that (77) becomes
$$\frac{du^2}{d\theta^2}=c^2-(u-b)^2;$$
$$\therefore \quad \frac{-du}{\{c^2-(u-b)^2\}^{\frac{1}{2}}}=d\theta, \qquad (79)$$

the negative sign being taken in the extraction of the square root, because we will assume that $r$ and $\theta$ simultaneously increase and decrease. Therefore integrating, and leaving the position of the prime radius undetermined, so that it may be determined by means of subsequent considerations; and thus introducing an arbitrary constant $\gamma$, we have
$$\cos^{-1}\frac{u-b}{c}=\theta-\gamma, \qquad (80)$$
$$\therefore \quad u=b+c\cos(\theta-\gamma); \qquad (81)$$

and restoring the equivalents of $b$ and $c$,
$$u=\frac{\mu}{(\mathrm{R}\mathrm{V}\sin a)^2}+\left\{\frac{\mathrm{V}^2\mathrm{R}-2\mu}{\mathrm{R}^3\mathrm{V}^2(\sin a)^2}+\frac{\mu^2}{(\mathrm{R}\mathrm{V}\sin a)^4}\right\}^{\frac{1}{2}}\cos(\theta-\gamma); \qquad (82)$$

and this is the equation to a conic, of which the focus is the pole. For if $e$ is the eccentricity of a conic, if $r$ is the focal radius vector, and $\phi =$ the angle between $r$ and the principal axis, and measured, say, from that point of the conic which is nearest to the focus,
$$\frac{1}{r}=u=\frac{1+e\cos\phi}{2k}; \qquad (83)$$

and comparing this with (82), it appears that

$$(1) \qquad e^2 = 1 + \frac{(\mathrm{v}^2 \mathrm{R} - 2\mu)\, \mathrm{R}\, \mathrm{v}^2 (\sin a)^2}{\mu^2}, \tag{84}$$

$$(2) \qquad \theta - \gamma = \phi. \tag{85}$$

Now from (84) $e$ is less than, equal to, or greater than, unity according as $\mathrm{v}^2 \mathrm{R} - 2\mu$ is negative, zero, or positive; the orbit therefore is an ellipse, a parabola, or a hyperbola, with the centre of force at the focus, according as $\mathrm{v}^2$ is less than, equal to, or greater than, $\frac{2\mu}{\mathrm{R}}$. The interpretation of this discriminating condition is as follows. Suppose a particle to move in a rectilinear path from an infinite distance towards the centre of force; then the equation of such a motion is

$$\frac{d^2 x}{dt^2} = -\frac{\mu}{x^2}, \tag{86}$$

where $x =$ the distance of the particle from the centre of force at the time $t$. Multiplying both sides of the equation by $2\,dx$, and integrating, we have

$$\frac{dx^2}{dt^2} = \frac{2\mu}{x}; \tag{87}$$

and therefore the square of the velocity at a distance $\mathrm{R}$ from the centre $= \frac{2\mu}{\mathrm{R}}$. Thus it appears that according as the velocity with which the particle is projected at a distance $\mathrm{R}$ from the centre of force is less than, equal to, or greater than, that which would be acquired by the particle moving from an infinite distance to that point under the action of the central force, so will the orbit be an ellipse, a parabola, or a hyperbola, with the centre of force in the focus. The species of conic therefore does not depend on the position of the line in which the particle is projected, but on the velocity of projection in reference to the distance of the point of projection from the centre of force. I may also observe that, by reason of (17), according as the velocity of projection is less than, equal to, or greater than, that from an infinite distance at the point of projection, so will it be at all points of the orbit. Thus if a particle moves in a parabola with the centre of force in the focus, the velocity at every point of the orbit is equal to that which would be acquired by the particle moving in a rectilinear path from an infinite distance under the action of the central force to the point on the orbit. This result has also been proved before in equation (32), Article 309.

And with respect to $\gamma$, the undetermined constant which is introduced at the integration of (79), from (85) it appears that $\theta-\gamma$ is the angle between the focal radius vector $r$ and that part of the principal axis which is between the focus and the point of the orbit which is nearest to the focus; therefore $\gamma$ is the angle between the prime radius and that part of the principal axis; and therefore if $\gamma = 0$, the principal axis is the prime radius. For the present I shall suppose this to be the case; and I shall consider each of the three conics separately.

320.] Let us in the first place consider the ellipse in which

$$\mathrm{v}^2 \text{ is less than } \frac{2\mu}{\mathrm{R}};$$

so that by reason of (84), if $e =$ the eccentricity,

$$e^2 = 1 - \frac{(2\mu - \mathrm{v}^2\mathrm{R})\,\mathrm{R}\mathrm{v}^2(\sin a)^2}{\mu^2}. \tag{88}$$

Now the equation of an ellipse, where $r =$ the focal radius vector, $\theta$ is measured from the shortest segment of the major axis, $2a =$ the major axis, $e =$ the eccentricity, is

$$r = \frac{a(1-e^2)}{1+e\cos\theta};$$

$$\therefore \quad u = \frac{1}{a(1-e^2)} + \frac{e}{a(1-e^2)}\cos\theta; \tag{89}$$

comparing which with (82), we have $e$ as in (88), and

$$a = \frac{\mu\mathrm{R}}{2\mu - \mathrm{v}^2\mathrm{R}}; \tag{90}$$

and thus the major axis of the ellipse is also independent of the angle between the line in which the particle is projected and the line joining the point of projection and the centre of force.

Let $\beta =$ the angle between the major axis of the ellipse and $\mathrm{R}$; then from (82), $u = \frac{1}{\mathrm{R}}$, when $\theta - \gamma = \beta$, and we have,

$$\sin\beta = \frac{\mathrm{R}\mathrm{v}^2 \sin a \cos a}{\{\mu^2 - (2\mu - \mathrm{v}^2\mathrm{R})\,\mathrm{R}\mathrm{v}^2(\sin a)^2\}^{\frac{1}{2}}}; \tag{91}$$

which equation determines the position of the major axis of the ellipse with reference to the given line $\mathrm{R}$.

Thus the position and the dimensions of the elliptic orbit are completely determined, on the supposition that the initial circumstances are given. Fig. 130 explains the several quantities which we have introduced. Let B be the point of projection;

SB = R; BY the line along which the particle is projected with the velocity V; SBY $= \alpha$, the angle of projection; BSA $= \beta$; SP $= r$; PSA $= \theta$; SY $=$ SB sin SBY $=$ R sin $\alpha$; if $\alpha = 90°$, the particle is projected from an apse, that is, from one or other of the extremities of the major axis of the ellipse.

It remains for us to investigate the time during which the moving particle passes from one to another point in the orbit; and let us suppose $t = 0$, when $m$ is at that extremity of the major axis which is nearest to the focus. Now since the quantities which determine the magnitude of the ellipse have been expressed in terms of the initial circumstances of motion, we may assume that $a$ and $e$ are known in the equation,

$$r = \frac{a(1-e^2)}{1+e\cos\theta}; \tag{92}$$

and by Ex. 1, Art. 309, (26),

$$h = \{\mu a(1-e^2)\}^{\frac{1}{2}}; \tag{93}$$

therefore by (10),

$$dt = \frac{r^2}{h}\,d\theta$$

$$= \frac{r^2\,d\theta}{\{\mu a(1-e^2)\}^{\frac{1}{2}}}; \tag{94}$$

but from (92),

$$d\theta = \frac{a(1-e^2)^{\frac{1}{2}}\,dr}{r\{a^2e^2-(r-a)^2\}^{\frac{1}{2}}}; \tag{95}$$

$$\therefore \quad dt = \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \frac{r\,dr}{\{a^2e^2-(r-a)^2\}^{\frac{1}{2}}}; \tag{96}$$

$$\therefore \quad t = \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \int_{a(1-e)}^{r} \frac{r\,dr}{\{a^2e^2-(r-a)^2\}^{\frac{1}{2}}}$$

$$= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \left[a\sin^{-1}\frac{r-a}{ae} - \{a^2e^2-(r-a)^2\}^{\frac{1}{2}}\right]_{a(1-e)}^{r}$$

$$= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \left\{a\cos^{-1}\frac{a-r}{ae} - \{a^2e^2-(a-r)^2\}^{\frac{1}{2}}\right\}; \tag{97}$$

which gives the value of $t$ corresponding to any value of $r$. At the farther extremity of the major axis, $r = a(1+e)$, the time corresponding to which is the semi periodic time: therefore

$$\text{the semi periodic time} = \frac{\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}; \tag{98}$$

and at the extremity of the minor axis, $r = a$; in which case

$$t = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\left\{\frac{\pi}{2} - e\right\};$$

and therefore the time from the extremity of the minor axis to the farther extremity of the major axis is

$$\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\left\{\frac{\pi}{2}+e\right\}.$$

And thus

$$\text{the periodic time} = \frac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}; \tag{99}$$

the same result as (28).

321.] The expression for the time given in (97) admits of the following simplification: let

$$\cos^{-1}\frac{a-r}{ae} = u,$$

$$\therefore \quad r = a(1-e\cos u); \tag{100}$$

$u$ being an auxiliary angle, the geometrical meaning of which we shall presently investigate: then substituting in (97), we have

$$t = \left(\frac{a}{\mu}\right)^{\frac{1}{2}}\{au-ae\sin u\}$$

$$= \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\{u-e\sin u\}; \tag{101}$$

and for convenience of notation, let

$$\frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} = \frac{1}{n}; \tag{102}$$

$$\therefore \quad nt = u-e\sin u. \tag{103}$$

Hence when $u$ is increased by $2\pi$, that is, when the particle has passed through the whole orbit, $t$ is increased by $\frac{2\pi}{n}$; $\frac{2\pi}{n}$ therefore is the periodic time of the orbit.

As $r$ and $t$ are given in terms of $u$, let us investigate also the relation between $\theta$ and $u$; equating the values of $r$ given in (92) and in (100), we have

$$\frac{1-e^2}{1+e\cos\theta} = 1-e\cos u;$$

$$\therefore \quad 1+\cos\theta = \frac{(1-e)(1+\cos u)}{1-e\cos u};$$

$$1-\cos\theta = \frac{(1+e)(1-\cos u)}{1-e\cos u};$$

$$\therefore \quad \frac{1-\cos\theta}{1+\cos\theta} = \frac{1+e}{1-e}\,\frac{1-\cos u}{1+\cos u};$$

$$\tan\frac{\theta}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}}\tan\frac{u}{2}. \tag{104}$$

The value of $t$ in terms of $\theta$ is so complicated that it is unnecessary to insert it. The geometrical meaning of these several quantities is interpreted by the aid of fig. 131. On the major axis ACA′ of the elliptic orbit, describe a semicircle: let P be the point on the ellipse corresponding to which are $\text{SP} = r$, $\text{PSA} = \theta$; let the ordinate PM be drawn, and let it be produced so as to cut the circle in Q: draw QC to C the centre of the circle. Let $\text{CA} = \text{CQ} = \text{CA}' = a$: then by a property of the ellipse,

$$\text{SP} = a - e.\text{CM};$$

$$\therefore \quad r = a - ae \cos \text{QCM}; \tag{105}$$

therefore comparing this with (100), $\text{QCM} = u$.

When a particle moves in an elliptic orbit under the action of a force in the focus, it is evident that as equal areas are described in equal times, the angles abutting at S are not described uniformly; and for this reason angles measured from SA are called *anomalies;* $\theta$ is called the *true anomaly*, $u$ the *eccentric anomaly*, and $nt$ the *mean anomaly*.

From (99) it appears that the periodic time of $m$ in the elliptic orbit is independent of the eccentricity of the ellipse, and is therefore the same as that in a circle whose radius is $a$; but in this latter case $e = 0$, $r = a$, $\theta = u = nt$; thus equal angles are described in equal times, and the particle moves uniformly in the circle: hence $nt$ represents the arc of the circle AQA′, which would be described *uniformly* by a particle in the same time as that in which the elliptic arc is described, both the particles being together at A; and therefore also at A′, because then $u = \pi$ and $\sin u = 0$, and thus from (103) the time from A to $\text{A}' = \frac{\pi}{n}$; $n$ is called the *mean motion* of $m$: now as these two particles start from A simultaneously and in the same direction, one along the circular and the other along the elliptic orbit, the latter is before its mean place from A to A′, because $\sin u$ is positive in the first two quadrants, and therefore $u$ is greater than $nt$; and is behind its mean place from A′ to A, because $\sin u$ is negative in the third and fourth quadrants. It is for this reason that $nt$ is called the mean anomaly, and also that $u$ which depends on the eccentricity is called the eccentric anomaly. Also as the velocity varies inversely as the perpendicular from the centre of force on the tangent, the velocity at A in the elliptic orbit is the greatest, and at A′ is the least: the particle therefore leaves A with a velocity greater than its mean velocity, and therefore gets before its mean place, but is at A′ at its mean time: and as it leaves A′ with a velocity less than its mean velocity it is behind its mean place, until its velocity increasing it arrives at A′ at its mean time and with a velocity greater than its mean velocity.

One case of elliptic motion under the action of a force varying inversely as the square of the distance requires notice. Suppose $e = 0$, that is, suppose the orbit to be a circle; then from (88) we have

$$\{\mu - \mathrm{R}\mathrm{V}^2 (\sin \alpha)^2\}^2 + \mathrm{R}^2\mathrm{V}^4 (\sin \alpha \cos \alpha)^2 = 0;$$

which can be satisfied only when $\alpha = 90°$, and $\mathrm{V}^2 = \frac{\mu}{\mathrm{R}}$: in which case $m$ is projected at an apse; and if $\mathrm{P}' =$ the central force at the point of projection, $\mathrm{P}' = \frac{\mu}{\mathrm{R}^2}$: and therefore

$$\mathrm{P}' = \frac{\mathrm{V}^2}{\mathrm{R}};$$

but by (31), Art. 255, $\frac{\mathrm{V}^2}{\mathrm{R}}$ is the centrifugal force at the point of projection; therefore at that point the central force is equal to it: if therefore $m$ is projected from an apse with a velocity such that the centrifugal force is equal to the central force, the particle continues throughout its motion at the same distance from the centre of force, and the orbit is a circle. We shall hereafter point out the general cause of this circumstance.

322.] Let us in the next place consider the parabola in which, see Art. 319,

$$\mathrm{V}^2 = \frac{2\mu}{\mathrm{R}}. \qquad (106)$$

Now the equation to the parabola, of which $4a$ is the latus rectum, and the focus is the pole, and the line from the focus to the vertex is the prime radius, is

$$r = \frac{2a}{1 + \cos \theta}; \qquad (107)$$

and comparing this with the form which (82) takes, when $\mathrm{V}^2\mathrm{R} = 2\mu$, we have,

$$u = \frac{1}{2\mathrm{R} (\sin \alpha)^2} \{1 + \cos \theta\}; \qquad (108)$$

$$\therefore \quad a = \mathrm{R} (\sin \alpha)^2; \qquad (109)$$

and if $\beta =$ the angle between the vertex and R, then $u = \frac{1}{\mathrm{R}}$

when $\theta = \beta$; in which case, from (108), $\beta = 180° - 2\alpha$. Thus the position and latus rectum of the parabola are completely determined. If $\alpha = 90°$, the particle is projected from an apse, and therefore from the vertex of the parabola.

The time has been found in terms of $\theta$ in Art. 309.

323.] And lastly let us consider the hyperbola: in which

$$\text{V}^2 \text{ is greater than } \frac{2\mu}{\text{R}};$$

and if $e$ is the eccentricity of the curve,

$$e^2 = 1 + \frac{(\text{V}^2\text{R} - 2\mu)\,\text{R}\,\text{V}^2 (\sin\alpha)^2}{\mu^2}. \tag{110}$$

Now the equation of the hyperbola, if $\theta$ is measured from the extremity of the transverse axis which is nearest to the focus where the pole is, is

$$r = \frac{a(e^2-1)}{1+e\cos\theta}; \tag{111}$$

$$u = \frac{1}{a(e^2-1)} + \frac{e}{a(e^2-1)}\cos\theta; \tag{112}$$

comparing which with (82), we have

$$a = \frac{\mu\,\text{R}}{\text{V}^2\text{R} - 2\mu}; \tag{113}$$

and if $\beta$ is the angle between the transverse axis and R, $\beta$ may be determined by a process similar to that by which equation (91) is found.

Thus the position and size of the orbit is completely determined.

And the time may thus be found. The equation to the hyperbola being (111), we have, as in Art. 309,

$$h = \{\mu a(e^2-1)\}^{\frac{1}{2}}.$$

Therefore by (12), Art. 307, if $t$ is the time from the vertex of the hyperbola,

$$\begin{aligned} t &= \int_0^\theta \frac{r^2\,d\theta}{\{\mu a(e^2-1)\}^{\frac{1}{2}}} \\ &= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \int_{a(e-1)}^r \frac{r\,dr}{\{(r+a)^2 - a^2e^2\}^{\frac{1}{2}}} \\ &= \left(\frac{a}{\mu}\right)^{\frac{1}{2}} \left\{ \{(r+a)^2 - a^2e^2\}^{\frac{1}{2}} - a\log\frac{r+a+\{(r+a)^2-a^2e^2\}^{\frac{1}{2}}}{ae} \right\}. \end{aligned} \tag{114}$$

To simplify this expression, let $\epsilon$ be the Napierian base, and let

$$\frac{r+a+\{(r+a)^2-a^2e^2\}^{\frac{1}{2}}}{ae} = \epsilon^u;$$

$$\therefore \quad t = \frac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}\left\{\frac{e}{2}(\epsilon^u-\epsilon^{-u})-u\right\}. \tag{115}$$

Hence, in conclusion, it appears that if a particle is projected with a given velocity, and moves under the action of a central force which varies inversely as the square of the distance, the path which it describes will be either an ellipse, a parabola, or a hyperbola: and that the species of curve depends on the velocity of projection. The physical application of the preceding results is deferred to Section 3 of the present Chapter.

324.] A particle is moving under the action of a central force, which varies inversely as the cube of the distance: it is required to determine the nature of the several orbits which it can describe.

Let us suppose the force to be attractive, and $\mu$ to be the absolute central force: so that for the central force we have,

$$\text{P} = \frac{\mu}{r^3} = \mu u^3. \tag{116}$$

Let $\text{V} =$ the velocity of $m$ at a point whose distance from the pole is $\text{R}$; and let $a =$ the angle between $\text{R}$ and the line in which $m$ is moving: so that

$$h = \text{V}\,\text{R} \sin a; \tag{117}$$

and thus $h$ is given in terms of known quantities. Substituting (116) in (21), we have

$$\frac{d^2u}{d\theta^2}+u = \frac{\mu}{h^2}u.$$

And multiplying by $2du$ and integrating, and taking the limits corresponding to $r = r$ and to $r = \text{R}$, we have

$$h^2\left(\frac{du^2}{d\theta^2}+u^2\right)-\text{V}^2 = \mu u^2-\frac{\mu}{\text{R}^2};$$

$$\therefore \quad \frac{du^2}{d\theta^2} = \frac{\mu-h^2}{h^2}u^2+\frac{\text{V}^2\text{R}^2-\mu}{h^2\text{R}^2}. \tag{118}$$

Now if a particle moves in a rectilinear path from an infinite distance, under the action of a central force which varies inversely as the cube of the distance, towards the centre of force, then if $\text{V}$ is the velocity at a distance $\text{R}$ from the centre,

$$\text{V}^2 = \frac{\mu}{\text{R}^2};$$

so that $\text{V}^2\text{R}^2-\mu$ is positive, zero, or negative, according as the velocity corresponding to R is greater than, equal to, or less than, that acquired in moving from an infinite distance.

Now (118) admits of different cases, according as the coefficients in its right-hand member are positive, zero, or negative; and replacing $h^2$ by its value in (117), we have

$$\mu - h^2 = \mu - \mathrm{V}^2 \mathrm{R}^2 (\sin a)^2. \tag{119}$$

(1) Let $\mu - h^2$ be a positive quantity; and let

$$\frac{\mu - h^2}{h^2} = n^2;$$

($a$) and suppose $\mathrm{V}^2\mathrm{R}^2$ to be greater than $\mu$; and let

$$\frac{\mathrm{V}^2 \mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2} = n^2 c^2;$$

so that (118) becomes,

$$\frac{du^2}{d\theta^2} = n^2 (u^2 + c^2);$$

$$\therefore \quad u = \frac{1}{r} = \frac{c}{2} \{e^{n(\theta-\gamma)} - e^{-n(\theta-\gamma)}\}. \tag{120}$$

($\beta$) suppose $\mathrm{V}^2\mathrm{R}^2 = \mu$; then $\mu - h^2 = \mathrm{V}^2\mathrm{R}^2 (\cos a)^2$; and (118) becomes,

$$\frac{du^2}{d\theta^2} = (\cot a)^2 u^2;$$

$$\therefore \quad r = a e^{\pm \theta \cot a}. \tag{121}$$

($\gamma$) let $\mathrm{V}^2\mathrm{R}^2$ be less than $\mu$; and let

$$\frac{\mathrm{V}^2 \mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2} = -n^2 c^2.$$

so that (118) becomes,

$$\frac{du^2}{d\theta^2} = n^2 (u^2 - c^2);$$

$$\therefore \quad u = \frac{1}{r} = \frac{c}{2} \{e^{n(\theta-\gamma)} + e^{-n(\theta-\gamma)}\}. \tag{122}$$

(2) Let $\mu - h^2 = 0$; so that from (119), $\mu = \mathrm{V}^2 \mathrm{R}^2 (\sin a)^2$; and therefore

$$\frac{\mathrm{V}^2 \mathrm{R}^2 - \mu}{h^2 \mathrm{R}^2} = \frac{(\cot a)^2}{\mathrm{R}^2};$$

so that (118) becomes,

$$\frac{du}{d\theta} = \frac{\cot a}{\mathrm{R}};$$

$$\therefore \quad r = \frac{\mathrm{R} \tan a}{\theta - \gamma}; \tag{123}$$

which is the equation to the reciprocal spiral.

(3) Let $\mu-h^2$, that is, $\mu-\mathrm{R}^2\mathrm{V}^2(\sin a)^2$, be a negative quantity; and let

$$\frac{\mu-h^2}{h^2} = -n^2;$$

therefore $\mathrm{V}^2\mathrm{R}^2-\mu$ is a positive quantity: and let us therefore suppose

$$\frac{\mathrm{V}^2\mathrm{R}^2-\mu}{h^2\mathrm{R}^2} = n^2c^2;$$

so that (118) becomes,

$$\frac{du^2}{d\theta^2} = n^2(c^2-u^2);$$

$$\therefore\quad u = \frac{1}{r} = c\cos n(\theta-\gamma). \tag{124}$$

If the central force is repulsive, the sign of $\mu$ must be changed; in which case, if $\frac{-\mu-h^2}{h^2} = -n^2$, and $\frac{\mathrm{V}^2\mathrm{R}^2+\mu}{h^2\mathrm{R}^2} = n^2c^2$, equation (118) becomes,

$$\frac{du^2}{d\theta^2} = n^2(c^2-u^2);$$

$$\therefore\quad u = \frac{1}{r} = c\cos n(\theta-\gamma). \tag{125}$$

There are therefore generally five different orbits in which a particle may move under the action of a central force which varies inversely as the cube of the distance, and to which the equations are, if the prime radius is judiciously chosen,

$$(1)\quad r = \frac{2a}{e^{n\theta}-e^{-n\theta}},\qquad (2)\quad r = a^{\theta},\qquad (3)\quad r = \frac{2a}{e^{n\theta}+e^{-n\theta}},$$

$$(4)\quad r = \frac{a}{\theta},\qquad\qquad (5)\quad r = \frac{2a}{\cos n\theta}.$$

One case of the preceding requires notice: in (2), wherein $\mu = h^2$; if $\mathrm{V}^2\mathrm{R}^2 = \mu$, then $\cot a = 0$, and the equation of the orbit becomes,

$$\frac{du}{d\theta} = 0;\qquad \therefore\quad u = \frac{1}{r} = \frac{1}{\mathrm{R}};$$

$$\therefore\quad r = \mathrm{R}; \tag{126}$$

which is the equation of a circle, the centre of which is the pole. Since in this case

$$\mathrm{V}^2\mathrm{R}^2 = \mu;\qquad \therefore\quad \frac{\mathrm{V}^2}{\mathrm{R}} = \frac{\mu}{\mathrm{R}^3}.$$

From these results it appears that, since $\cot a = 0$, $m$ is projected in a line perpendicular to R; and at the point of projection the centrifugal force, of which $\frac{\mathrm{V}^2}{\mathrm{R}}$ is the representative, is equal to the central force, that is, to $\frac{\mu}{\text{R}^3}$; and as the orbit is circular, this equality of the central and centrifugal forces will hold good at every point of it.

325.] A particle is moving under the action of a central force which varies inversely as the fifth power of the distance, and is attractive: it is required to determine the nature of the orbits.

Let R be the distance from the centre of the point of projection, $a$ the angle between R and the line of projection, V = the velocity of projection: then

$$h = \text{V} \, \text{R} \sin a;$$

and thus $h$ is given in terms of known quantities. And for the central force we have,

$$\text{P} = \frac{\mu}{r^5} = \mu u^5;$$

so that (21) becomes,

$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu u^3}{h^2};$$

$$\therefore \quad h^2 \left(\frac{du^2}{d\theta^2} + u^2\right) - \text{V}^2 = \frac{\mu u^4}{2} - \frac{\mu}{2\text{R}^4};$$

$$\therefore \quad h^2 \frac{du^2}{d\theta^2} = \frac{\mu u^4}{2} - h^2 u^2 + \text{V}^2 - \frac{\mu}{2\text{R}^4}. \tag{127}$$

Now suppose $\text{V}^2 - \frac{\mu}{2\text{R}^4} = 0$, in which case the velocity of projection is equal to that acquired at a distance R in moving towards the centre of force from an infinite distance, then (127) becomes,

$$\frac{du^2}{d\theta^2} = \frac{\mu u^4}{2h^2} - u^2.$$

Let $\frac{\mu}{2h^2} = a^2$, and substituting $u = \frac{1}{r}$, we have,

$$\therefore \quad \frac{-dr}{(a^2 - r^2)^{\frac{1}{2}}} = d\theta; \qquad \therefore \quad \cos^{-1}\frac{r}{a} = \theta - \gamma;$$

$$\therefore \quad r = a \cos(\theta - \gamma), \tag{128}$$

the equation of a circle, on the circumference of which the pole is, whose diameter is inclined at an angle $\gamma$ to the prime radius, and of which the diameter is $a$.

Again (127) may be integrated when the right-hand member is a complete square; in which case, we have

$$\text{V}^2 - \frac{\mu}{2\text{R}^4} = \frac{h^4}{2\mu};$$

and therefore (127) becomes,

$$h\frac{du}{d\theta} = u^2\left(\frac{\mu}{2}\right)^{\frac{1}{2}} - \frac{h^2}{(2\mu)^{\frac{1}{2}}}$$

$$= \left(\frac{\mu}{2}\right)^{\frac{1}{2}}\left\{u^2 - \frac{h^2}{\mu}\right\}$$

$$= \left(\frac{\mu}{2}\right)^{\frac{1}{2}}\{u^2 - a^2\},$$

if $a^2 = \frac{h^2}{\mu}$; $\quad \therefore \quad \log\frac{u-a}{u+a} = 2^{\frac{1}{2}}(\theta-\gamma)$;

$$\therefore \quad r = \frac{\mu^{\frac{1}{2}}}{h}\,\frac{1-e^{2^{\frac{1}{2}}(\theta-\gamma)}}{1+e^{2^{\frac{1}{2}}(\theta-\gamma)}}; \tag{129}$$

which is the equation to the orbit.

326.] Let us now consider certain general properties of the equation (21) of central orbits. We have

$$\frac{d^2u}{d\theta^2} + u = \frac{\mathrm{P}}{h^2u^2}. \tag{130}$$

If at all points of the orbit, $\mathrm{P} = h^2u^3$, then $\frac{d^2u}{d\theta^2} = 0$, and we have the orbit (123) of Article 324, that is, the reciprocal spiral. But if there is an apse at any one point of the orbit, $\frac{du}{d\theta} = 0$ at all points, and the orbit is a circle: in which case

$$\mathrm{P} = \frac{h^2}{r^3}$$

$$= \frac{(\text{velocity})^2}{r}$$

$$= \text{the centrifugal force}, \tag{131}$$

since $r$ is at every point of the orbit perpendicular to the tangent at the point. Hence, whatever is the law of force, if the particle is moving at an apse, and its velocity and distance from the centre are such that the centrifugal force is equal to the central force, the particle moves in a circle. In this case the result is the singular solution of the differential equation (130), because it arises, not from any particular values given to the arbitrary constants of integration, but from the variable P in the original equation being replaced by a constant quantity. We have had instances of these singular solutions in the preceding Articles.

327.] The equation of the orbit can always be found in terms of $r$ and $\theta$, if the central force P varies inversely as the $n$th power of the distance, and if the velocity of projection is equal to that acquired at the point in moving from an infinite distance towards the centre of force.

In this case, $\mathrm{P} = \frac{\mu}{r^n} = \mu u^n$;

so that (130) becomes,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\, u^{n-2}. \tag{132}$$

Now to calculate the velocity acquired in moving from an infinite distance to a point at a distance $\mathrm{R}$ from the centre of force;

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^n}; \qquad \therefore \quad \frac{dx^2}{dt^2} = \frac{2\mu}{(n-1)\,x^{n-1}};$$

therefore if $\mathrm{V}$ is the velocity at the distance $\mathrm{R}$,

$$\mathrm{V}^2 = \frac{2\mu}{(n-1)\,\mathrm{R}^{n-1}}; \tag{133}$$

and integrating (132), and taking the limits corresponding to $r = r$ and to $r = \mathrm{R}$, we have

$$h^2\left(\frac{du^2}{d\theta^2} + u^2\right) - \mathrm{V}^2 = \frac{2\mu}{n-1}\, u^{n-1} - \frac{2\mu}{(n-1)\,\mathrm{R}^{n-1}}; \tag{134}$$

therefore by reason of (133),

$$\frac{du^2}{d\theta^2} + u^2 = \frac{2\mu}{h^2(n-1)}\, u^{n-1}. \tag{135}$$

Let $\frac{2\mu}{h^2(n-1)} = a^{n-3}$;

$$\therefore \quad \frac{du}{(a^{n-3}\,u^{n-1} - u^2)^{\frac{1}{2}}} = d\theta;$$

$$\frac{du}{u^{\frac{n-1}{2}}\{a^{n-3} - u^{-(n-3)}\}^{\frac{1}{2}}} = d\theta;$$

$$\frac{-d.u^{-\frac{n-3}{2}}}{\{a^{n-3} - u^{-(n-3)}\}^{\frac{1}{2}}} = \frac{n-3}{2}\, d\theta;$$

$$\therefore \quad \cos^{-1}\frac{u^{-\frac{n-3}{2}}}{a^{\frac{n-3}{2}}} = \frac{n-3}{2}\,(\theta - \gamma);$$

$$\therefore \quad r^{\frac{n-3}{2}} = a^{\frac{n-3}{2}} \cos\frac{n-3}{2}\,(\theta - \gamma); \tag{136}$$

which is the equation to the orbit: also (135) shews that the velocity at every point of the orbit is equal to that which would be acquired in moving to that point from an infinite distance.

$n = 2$, $r = \dfrac{2a}{1+\cos\theta}$, the equation of a Parabola.

$n = 3$, then by means of (135), $r = a^{\theta}$, the Logarithmic Spiral.

$n = 4$, $r = \dfrac{a}{2}(1+\cos\theta)$, the equation of a Cardioid.

$n = 5$, $r = a\cos\theta$, the equation of a Circle.

$n = 6$, $r^3 = \dfrac{a^3}{2}(1+\cos 3\theta)$.

$n = 7$, $r^2 = a^2\cos 2\theta$, the equation of a Lemniscata.

and so for other values of $n$.

328.] If the form of P is

$$\text{P} = \frac{\mu}{r^3} + \frac{k^2}{r^2} f(\theta) \tag{137}$$

$$= \mu u^3 + k^2 u^2 f(\theta),$$

where $f(\theta)$ represents any function of $\theta$; then equation (130) becomes,

$$\frac{d^2u}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right)u = \frac{k^2}{h^2} f(\theta); \tag{138}$$

or

$$\frac{d^2u}{d\theta^2} + n^2 u = a^2 f(\theta); \tag{139}$$

which is a linear differential equation of the second order, and of which, by Art. 352, Vol. II, the integral is,

$$u = \frac{1}{r} = \frac{a^2}{n}\left\{\sin n\theta \int f(\theta)\cos n\theta\, d\theta - \cos n\theta \int f(\theta)\sin n\theta\, d\theta\right\} + \text{C}\sin(n\theta - \gamma); \tag{140}$$

where C and $\gamma$ are constants undetermined, and dependent on the initial circumstances of the motion.

Suppose $f(\theta) = 1$, so that the central force, see (137), varies partly inversely as the square, and partly inversely as the cube, of the distance, then (140) becomes

$$\frac{1}{r} = \frac{a^2}{n^2} + \text{C}\sin(n\theta - \gamma);$$

which represents a revolving conic: see Art. 313.

And if $f(\theta) = 1$, and $\mu = 0$, so that $n = 1$, then

$$\frac{1}{r} = a^2 + \text{C}\sin(\theta - \gamma);$$

which is the equation of a conic.

And the differential equation (130) can always be reduced to simple quadrature, if the central force is a homogeneous function of the second order in terms of $x$ and $y$, because in that case

$$\text{P} = \frac{f(\theta)}{r^2}.$$

Also again if

$$\text{P} = \frac{\mu}{r^3} + \mu' r$$

$$= \mu u^3 + \frac{\mu'}{u};$$

then
$$\frac{d^2 u}{d\theta^2} + u = \frac{\mu u}{h^2} + \frac{\mu'}{h^2 u^3};$$

$$\therefore \quad \frac{d^2 u}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right) u = \frac{\mu'}{h^2 u^3};$$

therefore multiplying by $2du$, integrating, and introducing an arbitrary constant,

$$\frac{du^2}{d\theta^2} + \left(1 - \frac{\mu}{h^2}\right) u^2 = -\frac{\mu'}{h^2 u^2} + \text{C};$$

the integral of which is of the form

$$u^2 = \frac{1}{r^2} = a + k \cos 2n(\theta - \gamma). \tag{141}$$

If $n = 1$, that is, if $\mu = 0$, the central force varies directly as the distance, and (141) is the equation of an ellipse, the centre of which is at the centre of force.

329.] Some few problems are added, so that the results of the preceding Articles may be exhibited in forms somewhat different to those which have been already discussed.

Ex. 1. A particle $m$ moves under the action of a central force; and its velocity varies inversely as the $n$th power of the distance from the centre of force: it is required to determine the law of force, and the equation of the orbit.

Let $v$ and $\text{v}$ be the velocities at the distances $r$ and $\text{R}$ respectively; and let

$$v = \frac{k}{r^n};$$

then by (16), Art. 308,

$$\frac{k^2}{r^{2n}} - \text{v}^2 = -2\int_{\text{R}}^{r} \text{P}\, dr;$$

therefore differentiating,

$$-\frac{2nk^2}{r^{2n+1}} = -2\text{P}; \qquad \therefore \quad \text{P} = \frac{nk^2}{r^{2n+1}};$$

which gives the law of attracting force.

Also since

$$(\text{velocity})^2 = h^2\left(u^2+\frac{du^2}{d\theta^2}\right),$$

$$\therefore \quad k^2u^{2n} = h^2\left(u^2+\frac{du^2}{d\theta^2}\right);$$

of which the integral is,

$$r^{n-1} = \frac{k}{h}\cos(n-1)(\theta-\gamma). \tag{142}$$

Ex. 2. A particle $m$ is moving in an ellipse, at the focus of which is the centre of force, and as $m$ passes successively through the apse which is nearer to the centre of force, the absolute force is increased in the ratio of $n$ to 1; it is required to determine the nature of the orbit after $p$ passages through the apse.

Since $m$ moves in an ellipse, the velocity at every point of the orbit, and therefore at the apse, is less than that due to an infinite distance; see Art. 319. Now the velocity at the apse is always the same, see Art. 308, in the successive passages through it, whatever is the value of the absolute force: and therefore if the absolute force is increased, the velocity at it becomes proportionately less and less than that acquired from an infinite distance; the orbit therefore is still an ellipse.

And as the velocity at the nearer apse is unchanged, by the increase of $\mu$, $h$ is the same in the orbit after the $p$th increase as it was in the original ellipse. Let $2a$ and $e$ be the major axis and the eccentricity of the original ellipse, $2a_p$ and $e_p$ those of the final ellipse: then equating the values of $h$, see (93), Art. 320,

$$\mu a(1-e^2) = n^p\mu a_p(1-e_p^2);$$

also as the distance of the apse from the focus is the same in both orbits,

$$a(1-e) = a_p(1-e_p),$$

$$\therefore \quad 1+e = n^p(1+e_p),$$

which gives the eccentricity of the final ellipse.

Thus as $n$ is greater than unity, the eccentricity becomes less and less, and $=0$, when

$$p = \frac{\log(1+e)}{\log n},$$

in which case the orbit is a circle: and as the revolutions continue, the orbit again becomes an ellipse, but the apse which was the nearer becomes that farther from the focus: and ultimately, when $p=\infty$, that is, after an infinite number of revolutions, $e_p = -1$; in which case from (90), $2a_p$ becomes equal to the distance between the focus and the apse, and thus the particle falls into the centre of force.

Ex. 3. A particle under the action of a central force which varies partly inversely as the cube, and partly inversely as the fifth power, of the distance, is projected from a given point with the velocity which would be acquired in motion from an infinite distance, at $\tan^{-1} 2^{\frac{1}{2}}$ with the distance, and the forces are equal at the point of projection. It is required to determine the orbit.

Let $R$ be the distance of the point of projection from the centre of force: and let

$$\begin{aligned} P &= \frac{\mu}{r^3} + \frac{\mu'}{r^5} \\ &= \mu u^3 + \mu' u^5; \end{aligned}$$

and since the forces are equal at the point of projection,

$$\frac{\mu}{R^3} = \frac{\mu'}{R^5}, \qquad \therefore \quad \mu' = \mu R^2.$$

Let $V =$ the velocity of projection: therefore

$$\begin{aligned} V^2 &= \frac{\mu}{R^2} + \frac{\mu'}{2R^4} \\ &= \frac{3\mu}{2R^2}; \\ \therefore \quad h^2 &= V^2 R^2 (\sin \tan^{-1} 2^{\frac{1}{2}})^2 \\ &= \mu. \end{aligned}$$

The equation of motion is

$$h^2 \left(\frac{d^2u}{d\theta^2} + u\right) = \mu u + \mu' u^3;$$

$$\therefore \quad h^2 \left(\frac{du^2}{d\theta^2} + u^2\right) - V^2 = \mu u^2 + \frac{\mu' u^4}{2} - \frac{\mu}{R^2} - \frac{\mu'}{2R^4};$$

$$\begin{aligned} \therefore \quad h^2 \left(\frac{du^2}{d\theta^2} + u^2\right) &= \mu u^2 + \frac{\mu'}{2} u^4 \\ &= \mu u^2 + \frac{\mu R^2 u^4}{2}; \end{aligned}$$

$$\frac{du^2}{d\theta^2} = \frac{R^2}{2} u^4, \qquad \therefore \quad r = \frac{R}{2^{\frac{1}{2}}} (\theta + 2^{\frac{1}{2}}),$$

if $R$ coincides with the prime radius.

Ex. 4. A particle is projected from an apse, and moves under the action of an attractive central force which varies inversely as the seventh power of the distance; it is required to find the orbit, the velocity of projection being equal to $3^{-\frac{1}{2}} V_c$, where

$\text{v}_c$ is the velocity with which a particle projected from an apse would describe a circle about the centre of force in its centre.

In this case, $$\text{P} = \frac{\mu}{r^7} = \mu u^7;$$

therefore if $\text{v} =$ the velocity of projection, $\text{R} =$ the distance of projection, by Art. 326,

$$\text{v}_c^2 = \frac{\mu}{\text{R}^6}, \qquad \therefore \quad \text{v}^2 = \frac{\mu}{3\,\text{R}^6};$$

$$\therefore \quad h^2 = \text{v}^2\,\text{R}^2$$

$$= \frac{\mu}{3\,\text{R}^4}.$$

The equation of motion is,

$$h^2\left(\frac{d^2u}{d\theta^2}+u\right) = \mu u^5;$$

$$\therefore \quad h^2\left(\frac{du^2}{d\theta^2}+u^2\right)-\frac{\mu}{3\,\text{R}^6} = \frac{\mu}{3}u^6-\frac{\mu}{3\,\text{R}^6};$$

$$\therefore \quad \frac{du^2}{d\theta^2} = \text{R}^4u^6-u^2;$$

$$\therefore \quad r^2 = \text{R}^2\cos 2\theta,$$

if the prime radius coincides with R: and this is the equation of the lemniscata.

Ex. 5. From an apse at a distance $a$ from a given centre of force, which varies as any power of the distance, a particle is projected with a velocity very nearly equal to that which would be required for a circular orbit; it is required to determine the value of the radius vector which corresponds to the apse next to the point of projection.

Let P the force be represented by $u^2\phi(u)$; so that the equation of motion becomes,

$$\frac{d^2u}{d\theta^2}+u = \frac{\phi(u)}{h^2}. \tag{143}$$

Now if $\text{v}_c$ is the velocity for a circular orbit at the distance $a$,

$$\text{v}_c^2 = \frac{1}{a}\phi\left(\frac{1}{a}\right).$$

Let the velocity of projection $= \text{v}$; and let $\text{v} = (1+e)\,\text{v}_c$, where $e$ is a small quantity;

$$\therefore \quad \text{v}^2 = \frac{(1+e)^2}{a}\phi\left(\frac{1}{a}\right), \qquad h^2 = (1+e)^2 a\,\phi\left(\frac{1}{a}\right).$$

Let $a'$ be the value of the radius vector corresponding to the next apse; then from the equation of motion (143), by integration we have,

$$h^2\left(\frac{du^2}{d\theta^2}+u^2\right)-\text{v}^2 = 2\int_a^r \phi(u)\,du;$$

but $\frac{du}{d\theta}=0$, when $u=\frac{1}{a'}$; therefore

$$\frac{(1+e)^2}{a'^2}a\phi\left(\frac{1}{a}\right)-\frac{(1+e)^2}{a}\phi\left(\frac{1}{a}\right) = 2\int_{\frac{1}{a}}^{\frac{1}{a'}}\phi(u)\,du;$$

and taking the $\frac{1}{a}$-differential, we have

$$(1+e)^2\left\{\frac{a}{a'^2}\phi'\left(\frac{1}{a}\right)-\frac{a^2}{a'^2}\phi\left(\frac{1}{a}\right)-\phi\left(\frac{1}{a}\right)-\frac{1}{a}\phi'\left(\frac{1}{a}\right)\right\} = -2\phi\left(\frac{1}{a}\right);$$

$$\therefore\quad \frac{a^2}{a'^2}\left\{\phi\left(\frac{1}{a}\right)-\frac{1}{a}\phi'\left(\frac{1}{a}\right)\right\}+\phi\left(\frac{1}{a}\right)+\frac{1}{a}\phi'\left(\frac{1}{a}\right) = 2\phi\left(\frac{1}{a}\right)(1+e)^{-2}$$

$$= 2\phi\left(\frac{1}{a}\right)(1-2e),$$

omitting powers of $e$ higher than the first.

$$\therefore\quad \frac{a^2}{a'^2} = 1-\frac{4e\,\phi\left(\frac{1}{a}\right)}{\phi\left(\frac{1}{a}\right)-\frac{1}{a}\phi'\left(\frac{1}{a}\right)};$$

$$\therefore\quad a' = a\left\{1+\frac{2e\,\phi\left(\frac{1}{a}\right)}{\phi\left(\frac{1}{a}\right)-\frac{1}{a}\phi'\left(\frac{1}{a}\right)}\right\}. \tag{144}$$

If the force varies inversely as the square of the distance, then $\phi(u)=\mu$ (say); and

$$a' = a(1+2e).$$

Ex. 6. A particle moves under the action of a central force which varies as any power of the distance, and describes an orbit which is very nearly a circle; it is required to find the angle between two successive apsidal distances.

Let the law of force be represented by $u^2 f(u)$, so that

$$\text{P} = u^2 f(u).$$

Let $a$ be the value of $u$ at the point whence $m$ is projected; and let the velocity of projection be to that corresponding to a circular orbit as $1 : 1+e$;

$$\therefore\quad (1+e)^2\,\text{v}^2 = uf(u);$$

and let the particle be projected from an apse, so that

$$h^2 = \frac{f(a)}{a}(1+e)^{-2}. \tag{145}$$

Since $u$ is always nearly equal to $a$, let

$$u = a+z; \qquad \therefore \quad \frac{d^2u}{d\theta^2} = \frac{d^2z}{d\theta^2};$$

$$\begin{aligned} \therefore \quad f(u) &= f(a+z) \\ &= f(a)+f'(a)\,z, \end{aligned}$$

neglecting terms involving higher powers of $z$: so that the equation of motion becomes,

$$\begin{aligned} h^2\left(\frac{d^2z}{d\theta^2}+a+z\right) &= f(u) \\ &= f(a)+f'(a)\,z, \end{aligned}$$

$$\frac{d^2z}{d\theta^2}+\left(1-\frac{f'(a)}{h^2}\right)z = \frac{f(a)}{h^2}-a;$$

and substituting for $h^2$ from (145), and neglecting terms involving $e^2$, $e^3$, ... and $ez$, ... because $e$ and $z$ are infinitesimal, we have

$$\frac{d^2z}{d\theta^2}+\left\{1-\frac{af'(a)}{f(a)}\right\}z = 2ea;$$

and if $1-\dfrac{af'(a)}{f(a)} = n^2$, we have

$$\frac{d^2z}{d\theta^2}+n^2z = 2ea; \tag{146}$$

the integral of which is

$$z = \text{A}\sin(n\theta-\gamma)+\frac{2ea}{n^2};$$

and therefore the angle between two successive apsidal distances

$$= \frac{\pi}{n} = \frac{\pi}{\left\{1-\dfrac{af'(a)}{f(a)}\right\}^{\frac{1}{2}}}. \tag{147}$$

If the force varies inversely as the $n$th power of the distance, then

$$\begin{aligned} f(u) &= \mu u^{n-2}, \\ f'(u) &= \mu(n-2)\,u^{n-3}; \end{aligned}$$

therefore the angle between two successive apsidal distances

$$= \frac{\pi}{(3-n)^{\frac{1}{2}}}; \tag{148}$$

and therefore if $n = 2$, that is, if the force varies inversely as the square of the distance, the angle between two successive apses $= \pi$.

If $n = -1$, the angle $= \dfrac{\pi}{2}$; which accords with the results of Art. 317.

The preceding example is Prop. XLV, Section IX, of the first Book of Newton's Principia.

## SECTION 3.—*The Elements of Physical Astronomy.*

330.] The science of physical astronomy consists in the application of mechanical principles and equations to the motion, figure, and other circumstances of the celestial bodies; and, as such, it has been called Celestial Mechanics, or, as Laplace has named it in his great work, *Mécanique Céleste.* It is the most important application of the two preceding Sections; and I propose therefore to investigate the process by which our equations, deduced as they are from the abstract principles of dynamics, may be so applied; and also to prove some few elementary astronomical theorems which are immediately consequent upon them. I must observe that the explanations will be of motions as they are, and not as they appear to us on the surface of the earth: we are upon one of the planets which circulate about the sun, as secondaries about their primaries; and we are therefore subject to displacement; and to displacements of two kinds: one of which is due to the absolute motion of the earth in space, by reason of which its centre is carried forward about 19 miles every second of time, and the distance between its places at an interval of six months is about 190 millions of miles: and the other is owing to the diurnal rotation of the earth about its own axis, so that as the diameter of the earth is nearly 8000 miles, a person at the equator suffers a displacement to this extent in the course of every 12 hours, and of an observer at a place the latitude of which is $\lambda$, the displacement is 8000 miles $\times \cos\lambda$: these effects are called *parallax*: the former is the *annual*, the latter the *diurnal*, parallax: so that the motions and displacements of the heavenly bodies do not appear to us to be what they actually are. Thus another planet appears to us to have a motion sometimes in one direction, sometimes in another, and at other times appears to be stationary. In the following account, to anticipate the sequel, the sun, which is the centre of our system, will be supposed to be at rest, and the planets will be supposed to circulate around it; and the system of reference is said to be heliocentric.

Now the equations and results which have been proved in the preceding pages are, as I have said, founded on principles of the abstract science of motion: thus from the relation of moving matter to time and space have the cinematical principles been deduced: inertia has been assumed to be a property of matter as the subject of motion, and on it has been founded the principle of equalizing impressed and expressed momenta. Instances also have been quoted wherein cosmical matter fulfils these conditions, and, as the laws of inductive philosophy authorize, a property which is proved to be true of some is extended to all cosmical matter. Other members of the solar system however do not admit of direct experiment, and therefore the proof is, in a great measure, *a posteriori*. In the preceding pages these properties have been applied to various kinds of laws of impressed momenta: some have had their source at an infinite distance, so that the lines of action of the impressed momenta on a particle in its different positions are parallel: of some the source has been in a point or centre at a finite distance, which can be conveniently taken as the origin of reference: and thus we have shewn that the pure science is comprehensive enough for all kinds and for all laws of force. But as our object now is, the investigation of the motion of a system of bodies which exist in cosmos, the questions which first meet us are, What is the nature of the force which acts on these bodies? is it a central force? or is the centre, if there is one, at a distance so great that the lines of action of it must be considered parallel? And again, if the acting force is a central force, with its centre at a finite distance, what is the law of force? is it attractive or repulsive? is it periodic; that is, does the law of it change from time to time? and at regular intervals? and is it a central force varying as some power of the distance only? and what is the law according to which it varies? does it vary inversely as the square of the distance? or according to what, if any, power of the distance? And what is its absolute force? is that constant for each one body in its orbit? is it the same for all bodies in their orbits? or is it periodical? Surely to all these questions the reply must be sought in observation. *Hypotheses non fingamus.* Although when the nature and the law of force, which the Almighty Creator has impressed on cosmical matter, is known, we do perceive wisdom and fitness in it; yet it is not our prerogative to assert that such or such *must* be the law: it may be that the law of the inverse square of the distance is simple, because then spherical and spheroidal shells attract an internal particle equally in all directions, see Arts. 181, 185; because then a sphere composed of a series of concentric homogeneous shells attracts an external particle with an effect the same as if it were condensed into a particle at its centre, Art. 178: because it is the law of attraction of matter, when none of the influence is lost in the process of propagation, see Art. 183: because it is that law for which the angle between two successive apsidal distances is 180°, see Art. 329: yet no one of these reasons, nor all collectively, are of cogency such as to necessitate the existence of that law. We are therefore compelled to have recourse in the first place to observation, by which we may detect such facts as will enable us to discover the mechanical laws of which they are the effects. The necessary observation was made by John Kepler with marvellous industry and ingenuity, and the three laws which he discovered, and which have since been known by the name of Kepler's Laws, constitute the basis on which mechanical principles are applied to the solution of the problem of planetary and lunar motion. John Kepler was born in 1571: Galileo, to whom we owe the law of inertia, and therein the first correct principles of mechanics, was born in 1564: and Newton in 1642. Thus when Galileo had first stated in a true form the fundamental axioms of mechanics, and when John Kepler had enuntiated those laws, which, when translated into mathematical language, as we shall just now do, are pregnant (approximately) with all the results of physical astronomy, was born that great philosopher, Sir Isaac Newton; for whom it was reserved to construct the general science of which Galileo had given the axioms, and to apply it to the explanation of the solar system, the observed laws of which had been discovered by Kepler. This is one, and perhaps the most remarkable, instance of a great genius being brought into the world exactly at the time when, so to speak, the materials of his work were ready. The Greek geometers had sufficiently examined the properties of conics in addition to the ordinary elements of geometry: Galileo had laid the foundation of dynamics; Kepler had enuntiated the planetary laws: it remained for Newton to assimilate these elements, and out of them to form that structure of physical astronomy which the Principia contain: D'Alembert, Lagrange, and Laplace subsequently elaborated, and well nigh completed, the work which he began.

331.] Kepler's observations were at first chiefly confined to the planet Mars; and after long and assiduous study of its motion he enuntiated the first two of the three following laws: and subsequently his observations were extended, and he enuntiated the third law. The first two laws were extended by him analogically to all the planets: the third law he discovered from a comparison of the numbers which a table of the quantities corresponding to the several planets exhibited. And however unsatisfactory his proofs of these laws may seem to us, yet modern astronomy has completely verified them.

The three laws are—

I. *The sectoreal areas described by the radius vector of a planet in its motion about the sun are proportional to the times of describing them.*

II. *The orbits of the planets are ellipses, in one of the foci of which is the sun.*

III. *The squares of the periodic times of the planets are proportional to the cubes of the semi axes major (or mean distances*.)*

Let us translate these into their mathematical equivalents, and deduce from them such mechanical laws as they contain.

(1) By law I. equal areas are described in equal times: therefore when the areas and times are infinitesimal, the law still is true; in which case we have

$$r^2 d\theta = h\,dt; \tag{149}$$

and changing to rectangular coordinates,

$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\frac{y}{x};$$

$$\therefore \quad d\theta = \frac{x\,dy - y\,dx}{x^2+y^2};$$

so that (149) becomes, $x\,dy - y\,dx = h\,dt$;

therefore differentiating,

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0;$$

$$\therefore \quad \frac{\frac{d^2x}{dt^2}}{x} = \frac{\frac{d^2y}{dt^2}}{y} = \frac{-\text{P}}{r}, \tag{150}$$

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* In astromony if a quantity contains periodic and non-periodic terms, the value of it, when the periodic terms are omitted, is called its *mean* value. Thus the focal radius vector of an ellipse is, (100), Art. 321, $r = a(1 - e\cos u)$, where $\cos u$ is a periodical quantity: and thus $a$ is the mean value of $r$: $a$ is for this reason called the *mean* distance.

if $\mathrm{P}$ = the resultant expressed velocity-increment, and tends to diminish $x$ and $y$;

$$\therefore \quad \frac{d^2x}{dt^2} = -\mathrm{P}\frac{x}{r}, \qquad \frac{d^2y}{dt^2} = -\mathrm{P}\frac{y}{r}; \tag{151}$$

that is, the resolved expressed velocity-increments along the axes are the resolved parts of a central force $\mathrm{P}$; the force therefore, under the action of which the planets move, is a central force, of which the place of the sun is the centre.

(2) Since the orbit is an ellipse, having the sun's centre, which is the centre of force, in one of the foci, let the equation to it be,

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

If therefore $\mathrm{P}$ = the central force, by Art. 308, equation (21),

$$\begin{aligned}\mathrm{P} &= h^2u^2\left\{\frac{d^2u}{d\theta^2}+u\right\}\\ &= \frac{h^2u^2}{a(1-e^2)}\\ &= \frac{h^2}{a(1-e^2)}\frac{1}{r^2};\end{aligned} \tag{152}$$

and thus the central force varies inversely as the square of the distance. And therefore from Kepler's second law we infer, that the planets move about the sun under the action of a force in the sun's centre which varies inversely as the square of the distance: this law is called the law of gravitation, and the planets are said to gravitate towards the sun.

(3) Let $\mathrm{T}$ = the periodic time in the planet's orbit: then the third law asserts that

$$\mathrm{T}^2 = ca^3;$$

where $c$ is a constant. Now if $\mu$ = the absolute central force, from (152),

$$\mu = \frac{h^2}{a(1-e^2)}; \qquad \therefore \quad h^2 = \mu a(1-e^2).$$

Also from (13), Art. 307,

$$\begin{aligned}\mathrm{T}^2 &= \frac{4\pi^2a^4(1-e^2)}{h^2}\\ &= \frac{4\pi^2a^3}{\mu};\end{aligned}$$

and equating this to the value of $\mathrm{T}^2$ given by Kepler's third law, we have

$$\mu = \frac{4\pi^2}{c}; \tag{153}$$

and since $c$ is constant for all the planets, so is $\mu$: that is, the same absolute force in the centre acts on all the planets; and modified by the distance of the planets retains them all in their orbits about the sun. And here I cannot but quote the eloquent words of Sir John Herschel: "Of all the laws to which induction from pure observation has ever conducted man, this third law of Kepler may justly be regarded as the most remarkable, and the most pregnant with important consequences. When we contemplate the constituents of the planetary system from the point of view which this relation affords us, it is no longer mere analogy which strikes us, no longer a general resemblance among them as individuals independent of each other, and circulating about the sun, each according to its own peculiar nature, and connected with it by its own peculiar tie. The resemblance is now perceived to be a true *family* likeness; they are bound up in one chain; interwoven in one web of mutual relation and harmonious agreement; subjected to one pervading influence, which extends from the centre to the farthest limit of that great system, of which all of them, the earth included, must henceforth be regarded as members*."

Thus from these three laws of planetary motion we infer, (1) the nature of the force acting on the secondaries of the solar system; viz., it is a central force, of which the source is in the sun's centre: (2) the law of the central force: viz., it varies inversely as the square of the distance: (3) the fact that the absolute central force is the same for all the planets: and the proper work of physical astronomy is to deduce from these dynamical facts the results which they contain: the nature of the problem therefore is the same as that of those which have been investigated in the present Chapter.

332.] The preceding laws however are only approximately true; it is assumed in them that the sun is fixed, and the motion of only one body about the sun is supposed: whereas the sun is attracted by, and moves towards, the other bodies of its system: and the motion is relative, not absolute: thus as the absolute central force in Art. 293 is the sum of the masses of the attracting and the attracted bodies, so if $a$ is the mean distance of a planet's orbit, and T is the periodic time, by (28), Art. 309,

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* Outlines of Astronomy, Art. 488, 4th Edition London 1851.

$$\text{T} = \frac{2\pi a^{\frac{3}{2}}}{(\text{M}+m)^{\frac{1}{2}}}; \qquad (154)$$

and thus (the periodic time)$^2$ varies directly as the cube of the mean distance, and inversely as the sum of the masses of the sun and the planet. In the solar system however the numerical correction of Kepler's third law thus introduced is too small to be of any importance, the mass of Jupiter, the largest of the planets, being much less than a thousandth part of that of the sun. Again in the first two laws we have assumed one body to be moving about the sun, and attraction to exist between these two bodies only; whereas all the planets attract each other; and thus the elliptic orbit of each becomes disturbed. The method of investigation however in these and similar problems is, firstly to seek a solution in the simple, although inaccurate, form; and subsequently to correct it by the introduction of other elements which enter into it. Thus the elliptic motion of a planet is a first approximation to the actual orbit: and subsequently we prove that the size and position of the ellipse undergo certain changes, the magnitudes of which can be expressed in terms of the time.

Hereafter it will be shewn that when a body of finite dimensions moves in space, and its particles are under the action of many forces, so that it has motions both of translation and of rotation; so far as the motion of translation is concerned, we may consider all the impressed forces to be applied, each in its own intensity and in its own line of action at the centre of gravity of the body, and the motion of translation of the centre will be the same as it is in the actual motion of the body. By virtue of this theorem therefore we may consider the planets to be condensed into their centres of gravity, and thus to move as material particles about the sun.

333.] The motion of the centre of gravity of a planet is, by virtue of Art. 306, in one plane, which also contains the centre of the sun. Thus if lines are drawn from the centre of the sun to the centre of the earth in all its positions, and if these lines are produced to the heavenly vault, it is found that all of them are in one and the same plane; and the plane in which the centre of the earth moves is called *the plane of the ecliptic:* to it, for the present, the motions of the other planets are referred. Also the fixed stars which are in the plane of the ecliptic are called ecliptic stars. Of the ellipse in which the centre of the earth moves, the mean distance is about 95 millions of miles, and the eccentricity is 0.0167836*; so that the distance of the centre of the ellipse from the sun is about 1604442 miles; the eccentricity however is of such a small amount, that if the orbit is drawn with a major axis of 10 feet, and a circle is also drawn on the major axis as a diameter, the difference between the two will not be sensible at any part of the curves. Suppose now in fig. 132 the plane of the paper to be the plane of the ecliptic; and let us suppose the path which the centre of the earth describes about the sun to be (approximately) a circle, of which the centre is S, the place of the sun. The position of the plane of the ecliptic having been determined by observation, it is found that the earth's axis is always inclined at the same angle to that plane: that is, as the earth moves in its path about the sun, the axis about which it rotates always points (approximately) to the same star in the heavens: and this star receives the name of the polar star: the stars being at a distance so great that, neglecting small variations, the earth's axis during the whole of the earth's orbit about the sun being produced passes through the same star: hence we infer that the earth's axis is parallel to itself through the whole of the orbit: but as the polar star is not in the normal to the ecliptic, it appears that the axis of the earth's rotation is not perpendicular to the ecliptic: the mean value of the angle between it and the normal to the ecliptic is found to be 23° 27′ 30″. Now this fact gives rise to the seasons, and to the varying lengths of the day and the night on the earth: thus, in fig. 132, let S be the sun, and let A, B, C, D be four positions of the earth in its orbit: PQ being in each position the polar axis of the earth. It is evident that as the polar axis always retains its line parallel to itself, in two positions of the earth's orbit it is at right angles to the line drawn from the centre of the sun to the centre of the earth: let these positions be A and C; and at another position, say B, the angle between the polar axis towards the north, say P, and the sun's radius vector is the least acute angle, being 66° 32′ 30″; and at another position D, the angle is the greatest obtuse angle, being equal to 123° 27′ 30″. It is evident also that the positions A and C are at the extremities of a diameter of the earth's orbit, and that B and D are at the extremities of another diameter, which is perpendicular to the former. As the sun illumines only that part of the earth which is turned towards it, so is one half of the earth always enlightened, and the other is darkened: now in the position A, the diametral plane, which divides the enlightened and the darkened parts, passes through the two poles: and therefore as the earth during 24 hours revolves uniformly on its axis, so does every place on its surface describe half of its path in the light of the sun, and half of it out of it: thus the length of day and night is equal at every place on the surface of the earth; and hence the term *equinox;* and the earth in such a position is said to be at the equinox. As C is the other place in the orbit where the sun's radius vector is perpendicular to the earth's polar axis, so is C also the other equinox: the preceding circumstances are, it is to be observed, the physical definition of equinox; those positions of the earth in its orbit, at which the earth's polar axis is perpendicular to the heliocentric radius vector of the earth's centre. Let A be the position of the earth at the vernal equinox; C its position at the autumnal equinox.

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* See the synoptical table in Herschel's Outlines of Astronomy.

Thus B is the position of the earth in the summer, when POS is 66° 32′ 30″, and which is called the summer solstice: in this case, as the earth revolves about its axis, the north pole P, and all places within the circle described by B as the earth revolves, are within the enlightened part of the sphere during the whole of a revolution; so that the north pole and all places within the circle (the geographical arctic circle) which is distant 23° 27′ 30″ from the pole, have continual light for 24 hours: and the south pole Q and all places within 23° 27′ 30″ of the south pole (within the geographical antarctic circle) are in the darkened part of the sphere, and so have night during the 24 hours: and for all other places, the day decreases as we pass from the arctic to the antarctic circles, at the equator the length of the day being evidently 12 hours. It appears therefore that the north pole has during the summer continual light for six months, and that the south pole has continual darkness. In the fourth position, viz. D, of the earth, all the above phænomena are exactly inverted, the north pole and all places within the arctic circle are immersed in total darkness during the whole diurnal revolution, and all places within the antarctic circle have continual light; and we have the phænomena of the winter solstice.

334.] If the earth were a perfect sphere, the equinoxes and the solstices would every year take place at the same points of the path of the earth: but as the earth is an oblate spheroid and revolves about its polar axis, and as the axis is not perpendicular to the plane of the ecliptic, the sun unequally attracts the protuberant masses at the earth's equator, by reason of the difference of their distances from the sun, and thus produces a change in the position of the axis of rotation. The resultant effect of this inequality of attraction is to cause the equinoxes and solstices to take place at points in the orbit about 50.4″ in advance of the point of the previous year: let us confine our attention to the vernal equinox, that viz. when the earth has the position A. A similar effect is also produced by the attraction of the moon; and another of smaller amount, and in an opposite direction, by the planets. Thus the position A takes place each year 50.2″ nearer to D than it did in the preceding year. As the earth's axis has the same mean inclination to the normal of the ecliptic, the effect of this shifting of A, or of the precession of the vernal equinox, as it is called, is a change of the polar star; and so that in the course of as many years as 360° contains 50.2″, that is, in the course of 25868 years, the pole of the earth will describe a circle in the heavens about the pole of the ecliptic, the angular radius of which is 23° 27′ 30″; and thus the polar star is continually changing. After a lapse of 12000 years, says Sir John Herschel, the bright star $\alpha$ Lyræ will become the polar star of the earth. In addition to this regular precession of the equinoxes, there are also in the mathematical expression certain periodic terms, by reason of which the axis is sometimes in advance of, and sometimes behind, its mean place: and sometimes nearer to, and at other times farther from, the pole of the ecliptic. These variations are called Nutation; but I am unable with the present data to give a more complete account of them.

Now a certain line passing through S is required from which angles may be measured: it might at first sight be thought that the major axis of the earth's orbit would serve the purpose: but by reason of the disturbing action of other bodies which circulate about the sun, the position of the major axis changes so much that it is ill adapted to such an use: and the line which is best suited is that which passes through one of the equinoxes or one of the solstices: and it is usual to take that which passes through the vernal equinox: that is, the line drawn from S to the centre of the earth at the position A. The fixed stars in the plane of the ecliptic have been divided into twelve equal parts, each consisting of 30°, and which are known by the names of "the signs of the Zodiac:" and the constellation *Aries* began at the position of the vernal equinox at the time when the names of the zodiacal constellations were given: but by reason of the precessional motion of the equinox, the vernal equinox has now passed out of the constellation Aries and is nearly at the beginning of Pisces; the line however drawn from the centre of the sun to the centre of the earth at the vernal equinox still bears the name of "the line drawn through the first point of Aries," and the earth at the vernal equinox is said to be in "the first point of Aries." Angles measured on the plane of the ecliptic from the line drawn through the first point of Aries are called Longitudes, and those perpendicular to the plane of the ecliptic are called Latitudes; these terms being used heliocentrically, and thus affording a complete system of spherical coordinates. The effect of the precession of the equinox is to increase the longitudes of all the fixed stars annually by 50.2″; and thus the values of these quantities given in the astronomical tables become yearly farther and farther removed from the true values.

335.] Although the orbit of each planet is in a plane containing the sun's centre, yet all the orbits are not in one and the same plane: neither do all the planes of the planets intersect the ecliptic in one and the same line: and therefore for the determination of the plane in which a planet moves in reference to the ecliptic it is necessary to know (1) the angle of inclination of the plane of the planet's orbit to the plane of the ecliptic: (2) the longitude of the line of intersection of the two planes; this line of intersection is called the line of nodes; the point at which the planet passes through the plane of the ecliptic in its passage from the south to the north of the ecliptic being called the ascending node, and the other being called the descending node. For the determination of the position of the orbit in its own plane, it is necessary to know the position of the major axis of the ellipse with reference to the line of nodes: the points in the orbit which are at the greatest and least distances from the sun, and which are of course the extremities of the major axis, are called the aphelion and the perihelion of the planet; and the position of the major axis is usually determined by means of the angle between the line of nodes and the perihelion distance; but as the inclination of the planes of all the larger planets to the ecliptic is very small, the projection of this angle on the plane of the ecliptic is very nearly equal to the angle itself in its own plane: and as we are obliged to have recourse to approximations in all problems of this kind, "the longitude of perihelion" is commonly given; which is the angle between the line of Aries and the perihelion distance, and which is measured on the plane of the ecliptic as far as the line of nodes, and on the plane of the planet's orbit from that line to the perihelion distance. Again, the magnitude of the orbit is determined by means of the semi major axis or mean distance and the eccentricity. These five quantities, which are called "elements" of the orbit, viz. the inclination, the longitude of the ascending node, the longitude of the perihelion distance, the mean distance, and the eccentricity, completely determine the plane, the position, and the magnitude, of the ellipse in which a planet moves. For the determination of its position in its orbit at a given time, two more elements are required: (1) its position at a certain given time: (2) the mean motion, which assigns mean velocity; or, and this is equivalent, the periodic time: this last element, see equation (154), Art. 332, requires a knowledge of the mass of the planet. The time at which the position of a planet is given is called the epoch, and is in the ordinary tables of planetary elements taken to be Jan. 1, 1800. The other element is the mass. Thus these two together with the preceding five are sufficient for the complete determination of a planet in its orbit: the first five and the seventh must be found by means of observation: the sixth is evidently arbitrary, and must therefore be conventionally assigned.

336.] Now the problem of the planetary theory in physical astronomy is, the determination of the position of a planet at any time, when its position at some previous time is known by observation or otherwise: and the simplest form of the problem is, when the planet is assumed to be moving in its own plane about the sun, and to be undisturbed by any other body: and this is the only form that I shall venture to enter upon in this treatise; so that the inclination, the longitude of the line of nodes, and the latitude of the planet will not enter into consideration; we have therefore to determine (1) the angular distance of the planet in its own plane from a certain fixed line in that plane, which I shall assume to be the perihelion distance, the epoch being the time at which the planet is in its perihelion; and (2) the planet's radius vector, in terms of the time which has elapsed since the epoch. And from Art. 321, the equations which connect these quantities are,

$$nt = u - e \sin u, \tag{155}$$

$$\tan\frac{\theta}{2} = \left(\frac{1+e}{1-e}\right)^{\frac{1}{2}} \tan\frac{u}{2}, \tag{156}$$

$$r = a(1 - e\cos u), \tag{157}$$

which are three transcendental equations; in the first $u$ and $t$ are connected; and in the second and third $\theta$ and $r$ are given in terms of $u$; our object therefore is so to express $u$ in terms of $t$ from (155), that we may be able to express $\theta$ and $r$ in terms of $t$ also. The problem is known by the name of Kepler's Problem.

We know by observation that the eccentricity of the orbits of all the larger planets is very small; so that the functions which require expansion may be expressed in ascending powers of $e$, and thereby terms involving powers of $e$ higher than, say, the third may be omitted: $n$, it will be observed, is given in equation (102), Art. 321; and as the motion is relative, if $\text{M}$ = the mass of the sun, $m$ = the mass of the planet, $a$ = the mean distance,

$$n = \frac{(\text{M}+m)^{\frac{1}{2}}}{a^{\frac{3}{2}}}. \tag{158}$$

We must in the first place expand $\theta$ in terms of $u$. In (156) let

$$\frac{1+e}{1-e} = m^2;$$

and since
$$(-1)^{\frac{1}{2}} \tan x = \frac{e^{2x\sqrt{-1}} - 1}{e^{2x\sqrt{-1}} + 1},$$

therefore from (156),

$$\frac{e^{\theta\sqrt{-1}} - 1}{e^{\theta\sqrt{-1}} + 1} = m\,\frac{e^{u\sqrt{-1}} - 1}{e^{u\sqrt{-1}} + 1};$$

$$\therefore \quad e^{\theta\sqrt{-1}} = \frac{(m+1)e^{u\sqrt{-1}} - (m-1)}{(m+1) - (m-1)e^{u\sqrt{-1}}}. \tag{159}$$

Let $\dfrac{m-1}{m+1} = \lambda$;

$$\therefore \quad e^{\theta\sqrt{-1}} = e^{u\sqrt{-1}}\,\frac{1 - \lambda e^{-u\sqrt{-1}}}{1 - \lambda e^{u\sqrt{-1}}};$$

therefore taking the logarithms of both sides of the equation,

$$\begin{aligned}
\theta\sqrt{-1} &= u\sqrt{-1}+\log(1-\lambda e^{-u\sqrt{-1}})-\log(1-\lambda e^{u\sqrt{-1}})\\
&= u\sqrt{-1}-\{\lambda e^{-u\sqrt{-1}}+\frac{\lambda^2}{2}e^{-2u\sqrt{-1}}+\frac{\lambda^3}{3}e^{-3u\sqrt{-1}}+\dots\}\\
&\qquad\qquad +\{\lambda e^{u\sqrt{-1}}+\frac{\lambda^2}{2}e^{2u\sqrt{-1}}+\frac{\lambda^3}{3}e^{3u\sqrt{-1}}+\dots\}\\
&= u\sqrt{-1}+\lambda(e^{u\sqrt{-1}}-e^{-u\sqrt{-1}})\\
&\qquad +\frac{\lambda^2}{2}(e^{2u\sqrt{-1}}-e^{-2u\sqrt{-1}})+\frac{\lambda^3}{3}(e^{3u\sqrt{-1}}-e^{-3u\sqrt{-1}})+\dots\\
&= u\sqrt{-1}+2\lambda\sqrt{-1}\sin u\\
&\qquad +2\frac{\lambda^2}{2}\sqrt{-1}\sin 2u+2\frac{\lambda^3}{3}\sqrt{-1}\sin 3u+\dots;
\end{aligned}$$

$$\therefore\quad \theta = u+2\{\lambda\sin u+\frac{\lambda^2}{2}\sin 2u+\frac{\lambda^3}{3}\sin 3u+\dots\}. \tag{160}$$

But

$$\begin{aligned}
\lambda &= \frac{m-1}{m+1}\\
&= \frac{(1+e)^{\frac{1}{2}}-(1-e)^{\frac{1}{2}}}{(1+e)^{\frac{1}{2}}+(1-e)^{\frac{1}{2}}}\\
&= \frac{1-(1-e^2)^{\frac{1}{2}}}{e}\\
&= \frac{e}{2}+\frac{e^3}{8}+\dots;
\end{aligned} \tag{161}$$

Thus to express $\theta$ and $r$ in terms of $t$ we must find $\sin u$, $\sin 2u$, $\sin 3u, \dots$ and $\cos u$ in terms of $t$ by means of (155); and for this purpose the theorem of Lagrange, see Vol. I, Art. 82, is adapted, being indeed discovered for the solution of this very problem.

337.] Given $u=nt+e\sin u$, it is required to find $u$, $\sin u$, $\sin 2u$, $\sin 3u, \dots \cos u$, in terms of $t$ and $e$.

If

$$y = z+x\phi(y), \tag{162}$$

then by Lagrange's theorem,

$$\begin{aligned}
f(y) &= f(z)+f'(z)\,\phi(z)\frac{x}{1}\\
&\qquad +\frac{d}{dz}\Big\{f'(z)\{\phi(z)\}^2\Big\}\frac{x^2}{1.2}+\frac{d^2}{dz^2}\Big\{f'(z)\{\phi(z)\}^3\Big\}\frac{x^3}{1.2.3}+\dots;
\end{aligned}$$

and comparing (162) with $u=nt+e\sin u$, we have, by means of this series,

$$y = z+\sin z\frac{x}{1}+\frac{d}{dz}(\sin z)^2\frac{x^2}{1.2}+\frac{d^2}{dz^2}(\sin z)^3\frac{x^3}{1.2.3}+\dots$$

$$\sin y = \sin z + \cos z \sin z \frac{x}{1} + \frac{d}{dz}\{\cos z(\sin z)^2\}\frac{x^2}{1.2} + \ldots$$

$$\sin 2y = \sin 2z + 2\cos 2z \sin z \frac{x}{1} + \frac{d}{dz}\{2\cos 2z(\sin z)^2\}\frac{x^2}{1.2} + \ldots$$

$$\sin 3y = \sin 3z + 3\cos 3z \sin z \frac{x}{1} + \frac{d}{dz}\{3\cos 3z(\sin z)^2\}\frac{x^2}{1.2} + \ldots$$

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$$\cos y = \cos z - (\sin z)^2 \frac{x}{1} - \frac{d}{dz}(\sin z)^3 \frac{x^2}{1.2} - \frac{d^2}{dz^2}(\sin z)^4 \frac{x^3}{1.2.3} + \ldots$$

Therefore substituting

$$u = nt + \sin nt \frac{e}{1} + \sin 2nt \frac{e^2}{1.2} + (3\sin 3nt - \sin nt)\frac{e^3}{8} + \ldots$$

$$\sin u = \sin nt + \sin 2nt \frac{e}{2} + (3\sin 3nt - \sin nt)\frac{e^2}{8} + \ldots$$

$$\sin 2u = \sin 2nt + (\sin 3nt - \sin nt)\, e + \ldots$$

$$\sin 3u = \sin 3nt + \ldots$$

. . . . . . . . . . . . . . . . . . . . . . . .

$$\cos u = \cos nt - (1 - \cos 2nt)\frac{e}{2} + (3\cos 3nt - 3\cos nt)\frac{e^2}{8} + \ldots;$$

therefore substituting these values in (157) and (160), and also replacing $\lambda$ by its value from (161), and omitting terms involving powers of $e$ higher than the third, we have,

$$\theta = nt + 2e\sin nt + \frac{5e^2}{4}\sin 2nt + \frac{e^3}{24}(13\sin 3nt - 3\sin nt) + \ldots; \quad (163)$$

$$r = a\left\{1 - e\cos nt + \frac{e^2}{2}(1 - \cos 2nt) - \frac{3e^3}{8}(\cos 3nt - \cos nt) + \ldots\right\}, \quad (164)$$

which give us the true anomaly, and the radius vector, in terms of the mean anomaly or the time, the approximations being carried correctly as far as terms involving the cube of the eccentricity; and omitting periodic terms, we have for the mean values of $\theta$ and $r$,

$$\theta = nt, \text{ (the mean anomaly)}, \quad (165)$$

$$r = a\left(1 + \frac{e^2}{2}\right). \quad (166)$$

If the line from which the angles are measured is that drawn through the first point of Aries, so that we have longitude instead of anomaly; then, if $\theta$ is the longitude of the planet at the time $t$ measured in the manner explained in Art. 335, and if $\varpi$ is the longitude of the perihelion measured in the same manner, $\theta$ in (163) must be replaced by $\theta - \varpi$; and if the time begins, that is, if $t=0$, when the planet is at a place whose mean longitude is $\epsilon$, then in (163) and (164) $nt$ must be replaced by $nt+\epsilon-\varpi$. It is unnecessary however to write down the preceding equations when these substitutions have been made, as the form of them is of course unaltered; $\epsilon$ is called the epoch.

338.] The excess of the true anomaly at any time over the mean anomaly, that is, $\theta-nt$, is called by astronomers *the equation of the centre;* and is positive or negative according as the planet is in advance of or behind its mean place: the equation of the centre evidently is zero at perihelion and at aphelion. It is necessary however to investigate the point in the orbit at which it is a maximum. Since at this point

$$d(\theta-nt) = 0, \tag{167}$$

and since by (10) Art. 307, (93) Art. 320, and (102) Art. 321,

$$\begin{aligned} r^2\,d\theta &= h\,dt \\ &= na^2(1-e^2)^{\frac{1}{2}}\,dt, \end{aligned} \tag{168}$$

therefore from (167),

$$d(\theta-nt) = \frac{a^2(1-e^2)^{\frac{1}{2}}-r^2}{a^2(1-e^2)^{\frac{1}{2}}}\,d\theta = 0,$$

$$\text{if} \qquad r = a(1-e^2)^{\frac{1}{4}}; \tag{169}$$

that is, when $r$ is a mean proportional between the two principal semi-axes. And to determine the corresponding maximum value of the equation of the centre, let $\mathrm{E}$ represent it; then

$$\begin{aligned} \mathrm{E} &= \theta-nt; \\ &= \theta-u+e\sin u. \end{aligned} \tag{170}$$

But from the equation to the ellipse, and from (169)

$$\cos\theta = -\frac{3e}{4}-\frac{3e^3}{32},$$

$$\therefore \quad \theta = \frac{\pi}{2}+\frac{3e}{4}+\frac{21e^3}{128}+\dots \tag{171}$$

Also from (157) and (169),

$$\cos u = \frac{e}{4}+\frac{3e^3}{32}+\dots;$$

$$\therefore \quad u = \frac{\pi}{2}-\frac{e}{4}-\frac{37e^3}{384}, \tag{172}$$

$$e\sin u = e-\frac{e^3}{32};$$

$$\therefore \quad \text{E} = 2e + \frac{11e^3}{48} + \dots, \tag{173}$$

which is the maximum value of the equation of the centre. And since from (172) it appears that $u$ is at the corresponding point less than 90°, it follows that the equation is a maximum at a point before the planet comes to the extremity of the minor axis. It may also by a similar process be shewn that in the passage of the planet from aphelion to perihelion, the equation of the centre is a maximum at a point between the extremity of the minor axis and the perihelion.

Also since mean time depends on the earth's mean position in its orbit, that is, on $nt$, the equation of the centre is the difference between true solar time and mean solar time, in so far as the difference is due to the varying velocity of the earth in its orbit: for this reason $\theta - nt$ is also called the elliptic inequality.

339.] It is necessary for us still to consider certain properties of parabolic orbits, because there is reason to suppose that some comets move either in such orbits, or in ellipses of which the eccentricity is so nearly equal to unity that the orbits appear to us, so far as our calculations go when the comets are within observation, to be parabolic.

The elements of a parabolic orbit are (see Art. 335) only six in number; the inclination and the longitude of the line of nodes are required as in the elliptic orbit, whereby the position of the plane of the comet's orbit may be determined. The longitude of the perihelion distance, the mean motion, and the epoch are also required; and as the eccentricity is unity, the perihelion distance is sufficient for the determination of the magnitude of the orbit.

Let us investigate the relation between the time and the angle measured from the perihelion distance which is due to the time in an elliptic orbit of which the eccentricity is nearly unity; that is, when the orbit is nearly parabolic.

Let $\theta =$ the angle measured from the perihelion distance, and $r =$ the corresponding radius vector, so that

$$r = \frac{a(1-e^2)}{1+e\cos\theta}.$$

Let $p =$ the perihelion distance, so that

$$p = a(1-e);$$

$$\therefore \quad r = p(1+e)(1+e\cos\theta)^{-1}: \tag{174}$$

and if $t =$ the time during which $\theta$ from perihelion is described, then from Art. 309, (26),

$$\begin{aligned} h^2 &= \mu a(1-e^2) \\ &= \mu p(1+e); \end{aligned} \tag{175}$$

$$\begin{aligned} \therefore \quad dt &= \frac{r^2}{h}\,d\theta \\ &= \frac{p^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \frac{d\theta}{\left(1+e\left\{\left(\cos\frac{\theta}{2}\right)^2 - \left(\sin\frac{\theta}{2}\right)^2\right\}\right)^2} \\ &= \frac{p^{\frac{3}{2}}(1+e)^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \frac{d\theta}{\left\{(1+e)\left(\cos\frac{\theta}{2}\right)^2 + (1-e)\left(\sin\frac{\theta}{2}\right)^2\right\}^2} \\ &= \frac{2p^{\frac{3}{2}}}{\mu^{\frac{1}{2}}(1+e)^{\frac{1}{2}}} \frac{\left(\sec\frac{\theta}{2}\right)^2 d.\tan\frac{\theta}{2}}{\left\{1+\mathrm{E}\left(\tan\frac{\theta}{2}\right)^2\right\}^2}, \end{aligned} \tag{176}$$

if $\mathrm{E} = \dfrac{1-e}{1+e}$, and which is therefore a small quantity, when $e$ is nearly equal to unity. Therefore expanding (176), and omitting terms involving powers of $\mathrm{E}$ higher than $\mathrm{E}^3$, and integrating between the limits $\theta = \theta$ and $\theta = 0$, we have,

$$\begin{aligned} t = \frac{2p^{\frac{3}{2}}}{\mu^{\frac{1}{2}}(1+e)^{\frac{1}{2}}} \Big\{ \tan\frac{\theta}{2} &+ \frac{1-2\mathrm{E}}{3}\left(\tan\frac{\theta}{2}\right)^3 - \frac{\mathrm{E}(2-3\mathrm{E})}{5}\left(\tan\frac{\theta}{2}\right)^5 \\ &+ \frac{\mathrm{E}^2(3-4\mathrm{E})}{7}\left(\tan\frac{\theta}{2}\right)^7 - \frac{4\mathrm{E}^3}{9}\left(\tan\frac{\theta}{2}\right)^9 \Big\}. \end{aligned} \tag{177}$$

If the orbit becomes a parabola, then $e = 1$, $\mathrm{E} = 0$; and we have,

$$t = \frac{2^{\frac{1}{2}}p^{\frac{3}{2}}}{\mu^{\frac{1}{2}}} \left\{ \tan\frac{\theta}{2} + \frac{1}{3}\left(\tan\frac{\theta}{2}\right)^3 \right\}; \tag{178}$$

which result is the same as that before investigated in Art. 309.

The preceding expressions are those commonly used for the calculation of the true anomaly, and also of the true longitude in the case of comets which move in ellipses of eccentricities which are nearly equal to unity, or in parabolas. In the actual use of (178), the determination of $\theta$ in terms of $t$ requires the solution of a cubic equation which has only one real root. And to avoid the difficulty of the solution, a table is formed containing the values of $t$ corresponding to the values of $\theta$ in the parabola whose least focal distance is unity; and this being multiplied by $p^{\frac{3}{2}}$ gives the time corresponding to the anomaly $\theta$ in the given parabola.

As it is very uncertain whether any celestial bodies move in hyperbolic orbits, it is unnecessary for us to develope the equations which express the relation between the time and the longitude in such orbits.

340.] It remains for us still to indicate means by which the masses of the planets may be determined, at least approximately, from the laws of elliptic motion.

If $\text{T} =$ the periodic time of a planet about the sun, $a =$ the mean distance of the planet's orbit, if $m$ and $\text{M}$ are the masses of the planet and of the sun respectively, then by (154), Art. 332,

$$\text{T} = \frac{2\pi a^{\frac{3}{2}}}{(m+\text{M})^{\frac{1}{2}}}. \tag{179}$$

Let $\text{T}'$ be the periodic time of another planet of which the mass is $m'$ and of whose orbit the mean distance is $a'$; then

$$\text{T}' = \frac{2\pi a'^{\frac{3}{2}}}{(m'+\text{M})^{\frac{1}{2}}};$$

$$\therefore \quad \frac{\text{T}^2}{\text{T}'^2} = \left(\frac{a}{a'}\right)^3 \frac{m'+\text{M}}{m+\text{M}}. \tag{180}$$

But by Kepler's third law, the squares of the periodic times vary as the cubes of the mean distances; and it is found by observation that this law is true with very slight variations; it follows therefore that the second factor in the right-hand member of (180) is nearly constant: and thus the masses of the planets must in comparison of that of the sun be so small, that the preceding process does not enable us to determine the ratio between them.

Suppose however the planet to have a satellite; and $m'$ to be the mass of the satellite, $a'$ being the mean distance of its orbit about its primary, and $\text{T}'$ being its periodic time: then if $m$ is the mass of the planet,

$$\text{T}' = \frac{2\pi a'^{\frac{3}{2}}}{(m+m')^{\frac{1}{2}}};$$

therefore from (179),

$$\frac{\text{T}^2}{\text{T}'^2} = \left(\frac{a}{a'}\right)^3 \frac{m+m'}{\text{M}+m}. \tag{181}$$

Now if $m$ is so small in comparison of $\text{M}$ (the mass of the sun) that it may be neglected without very great error; and also if the mass of the satellite be similarly small in comparison of the mass of the planet; then (181) becomes,

$$\frac{T^2}{T'^2} = \left(\frac{a}{a'}\right)^3 \frac{m}{M}; \tag{182}$$

and thus T, T', $a$, $a'$ having been determined by observation, the ratio of $m$ to M is known. When this method is applied to the determination of the mass of Jupiter by means of its fourth satellite, it is found that

$$m = \frac{M}{1066.09}; \tag{183}$$

and thus may the masses of all the planets which have satellites be compared with the mass of the sun.

The masses however are only *compared* by this process: no one is absolutely determined: and to accomplish this object it is necessary to find the mass of at least one: and the one which naturally offers itself for the purpose is the earth: hence arises the necessity of direct observation of the figure, magnitude, and density of the earth. Many processes have been devised; but the most reliable is that of direct geodesic measurement: and arcs of meridian have been measured in England, France, Russia, India, and the south of Africa: and from them the form and the magnitude of the earth have been determined. The density has been directly investigated by means of Dr. Maskelyne's observations with the pendulum near to the Schehallien mountain in Scotland *. And also by experiments with leaden balls, which were conducted by Cavendish, and have been subsequently repeated with great care by the late Mr. Baily, in the time intervening between Oct. 1838 and May 1842; and from these experiments it appears that the mean density of the earth is equal to 5.66 times that of distilled water. But this is not the place for entering into a detailed explanation of such experiments and observations. The article on the Figure of the Earth in the Encyclopedia Metropolitana, by Mr. Airy, the astronomer royal, contains all necessary information on the former subject: and for that on the latter I must refer the reader to the Philosophical Transactions, and to Vol. XIV of the Memoirs of the Royal Astronomical Society.

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* See Philosophical Transactions, 1811.

## SECTION 4.—*The polar equation of motion of a disturbed planet.*

341.] In Article 295 we have calculated the equations of motion of a particle $m$ relatively to M, when both of the particles are acted on by another particle $m'$, the law of attraction of all three being that of the mass directly and of the square of the distance inversely. As the problem is evidently that of the moon moving about the earth, both being acted on by the sun, which is a disturbing body, it deserves farther consideration; and I therefore propose to indicate the process by which equations (162), Art. 295, are transformed into their equivalents in terms of polar coordinates.

Let us suppose the earth to be the central body $m$, relatively to which the motion of M, the moon, is calculated: let $m'$ be the mass of the sun: let the plane of the ecliptic be that of $xy$, so that the sun is always in that plane, and therefore its $z$-ordinate is zero: let the positions of M and $m'$ at the time $t$ respectively be $(x, y, z)$ and $(x', y')$: let $r$ and $r'$ be the radii vectores of M and of $m'$: firstly let the acting forces be resolved in and perpendicular to the ecliptic, which is the plane of $xy$; and, to fix the thoughts, let the supposed system be represented in fig. 133; P being the place of the moon, and P′ that of the sun at the time $t$. Let the velocity-increments acting on M in the line perpendicular to the plane of $xy$ be called the *orthogonal* velocity-increments. Let ON $= \rho$, the moon's curtate radius vector; NOM $= \theta$, the moon's longitude; P′OM′ $= \theta'$, the sun's longitude; also let $s =$ the tangent of the angle PON, which is the moon's latitude; let P = the impressed velocity-increment on M in the plane of the ecliptic along ON, and acting to diminish ON; similarly, let T = that which is in the same plane, and is perpendicular to ON, and tends to increase $\theta$; S = that which is orthogonal, and tends to diminish the angle PON, and thus to bring the moon nearer to the plane of the ecliptic. Then using the notation of Art. 295, if $\mu = m + \text{M}$,

$$\left.\begin{aligned} \text{P} &= \frac{\mu\rho}{r^3} + \cos\theta\left(\frac{d\text{R}}{dx}\right) + \sin\theta\left(\frac{d\text{R}}{dy}\right) \\ \text{T} &= \sin\theta\left(\frac{d\text{R}}{dx}\right) - \cos\theta\left(\frac{d\text{R}}{dy}\right) \\ \text{S} &= \frac{\mu z}{r^3} + \left(\frac{d\text{R}}{dz}\right) \end{aligned}\right\}. \qquad (184)$$

And since

$$\left.\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{aligned}\right\} \qquad \left.\begin{aligned} x' &= r'\cos\theta' \\ y' &= r'\sin\theta' \end{aligned}\right\}; \tag{185}$$

and since $s = \tan \text{PON}$, so that

$$z = \rho s, \tag{186}$$

we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta + \text{P} &= 0 \\ \frac{d^2x}{dt^2}\sin\theta - \frac{d^2y}{dt^2}\cos\theta + \text{T} &= 0 \\ \frac{d^2z}{dt^2} + \text{S} &= 0 \end{aligned}\right\}; \tag{187}$$

Let us first consider the equations as they refer to the projection of the moon's path on the plane of the ecliptic; for which purpose, from the first two of (187), we have

$$\left.\begin{aligned} \frac{d^2\rho}{dt^2} - \rho\frac{d\theta^2}{dt^2} + \text{P} &= 0 \\ \rho\frac{d^2\theta}{dt^2} + 2\frac{d\rho}{dt}\frac{d\theta}{dt} - \text{T} &= 0 \end{aligned}\right\}. \tag{188}$$

Multiplying the last of these equations by $\rho$, we have

$$\begin{aligned} \rho\text{T} &= \rho^2\frac{d^2\theta}{dt^2} + 2\rho\frac{d\rho}{dt}\frac{d\theta}{dt} \\ &= \frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right), \end{aligned} \tag{189}$$

$$\therefore \quad 2\rho^3\text{T}\,d\theta = 2\rho^2\frac{d\theta}{dt}\,d\left(\rho^2\frac{d\theta}{dt}\right);$$

therefore integrating, and introducing an arbitrary constant $h^2$, the meaning of which will have to be determined hereafter, we have,

$$\left(\rho^2\frac{d\theta}{dt}\right)^2 = h^2 + 2\int\rho^3\text{T}\,d\theta, \tag{190}$$

$$\therefore \quad \frac{d\theta^2}{dt^2} = \frac{h^2}{\rho^4} + \frac{2}{\rho^4}\int\rho^3\text{T}\,d\theta. \tag{191}$$

Let $\rho = \dfrac{1}{u}$, therefore

$$\frac{d\theta^2}{dt^2} = h^2u^4 + 2u^4\int\frac{\text{T}}{u^3}\,d\theta. \tag{192}$$

By means of this equation we can eliminate $t$ from the first of (188); and let us assume $\theta$ to be equicrescent in the new equations. Since

$$\rho = \frac{1}{u},$$

$$\therefore \quad \frac{d\rho}{dt} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\rho^2 \frac{d\theta}{dt}\frac{du}{d\theta};$$

$$\therefore \quad \frac{d^2\rho}{dt^2} = -\rho^2 \frac{d\theta}{dt}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} - \frac{du}{d\theta}\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right)$$

$$= -\frac{1}{u^2}\frac{d^2u}{d\theta^2}\frac{d\theta^2}{dt^2} - \frac{du}{d\theta}\frac{\mathrm{T}}{u};$$

therefore from (188),

$$\left(\frac{d^2u}{d\theta^2}+u\right)\frac{d\theta^2}{dt^2} + \mathrm{T}u\frac{du}{d\theta} - \mathrm{P}u^2 = 0; \tag{193}$$

and substituting for $\frac{d\theta^2}{dt^2}$ from (191), we have

$$\frac{d^2u}{d\theta^2} + u - \frac{\frac{\mathrm{P}}{u^2} - \frac{\mathrm{T}}{u^3}\frac{du}{d\theta}}{h^2 + 2\int\frac{\mathrm{T}}{u^3}d\theta} = 0, \tag{194}$$

which is the differential equation of the moon's curtate radius vector in terms of the longitude.

Again, taking the third equation of (187), since

$$z = \rho s = \frac{s}{u},$$

$$\frac{dz}{dt} = \left(u\frac{ds}{d\theta} - s\frac{du}{d\theta}\right)\rho^2\frac{d\theta}{dt},$$

$$\therefore \quad \frac{d^2z}{dt^2} = \left(u\frac{d^2s}{d\theta^2} - s\frac{d^2u}{d\theta^2}\right)\rho^2\frac{d\theta^2}{dt^2} + \left(u\frac{ds}{d\theta} - s\frac{du}{d\theta}\right)\frac{d}{dt}\left(\rho^2\frac{d\theta}{dt}\right);$$

and substituting from (189), (192) and (194), we have

$$\frac{d^2s}{d\theta^2} + s + \frac{\frac{\mathrm{S} - \mathrm{P}s}{u^3} + \frac{\mathrm{T}}{u^3}\frac{ds}{d\theta}}{h^2 + 2\int\frac{\mathrm{T}\,d\theta}{u^3}} = 0, \tag{195}$$

which is the differential equation to the moon's motion in latitude.

342.] In respect of the three equations (191), (194), (195) it is to be noticed that the values of P, T, S, as given in (184) and taken from Article 295, involve both the mass and the coordinates of the place of $m'$, the disturbing body: with this difference however, that T and S are functions of these quantities only, and that P contains also the quantity $\frac{\mu\rho}{r^3}$, which is the radial impressed velocity-increment of M on $m$; and thus, if the disturbing body is omitted, the first two equations become identical with (10) and (21) of the present Chapter. Now when substitutions are made for P, T and S in terms of the polar coordinates of M and $m'$, the preceding equations are not integrable; and we are obliged to have recourse to a method of approximation; and for this end, the disturbing function R is first expressed in terms of the polar coordinates of M and $m'$, and its partial derived-functions are expanded in ascending powers of small quantities, the values of which must be given by the observed data of the problem; thus, for instance, the mean distance of the sun from the earth is nearly 400 times as great as the mean distance of the moon; if therefore the distance of the earth from the sun is assumed as the standard of comparison, the distance of the moon from the earth will be such that higher powers of it may be neglected. And after these expansions have been performed, the integration-process takes place. But it is beyond the object of the present work to enter on these investigations.

Equation (195) however, when the disturbing forces are omitted, requires consideration: for we have,

$$\frac{d^2 s}{d\theta^2} + s = 0;$$

whence integrating, $s = k \sin(\theta - \gamma)$, (196)

where $k$ and $\gamma$ are two constants introduced in integration. Now $s$ is the tangent of the latitude, and $\theta$ is the longitude of the moon measured from the first point of Aries; and thus $\gamma$ is the longitude of the ascending node. Suppose then, in fig. 134, NMP to be a right angled spherical triangle, P being the place of the moon, N the ascending node, and NM the plane of the ecliptic, then $\text{NM} = \theta - \gamma$, $\text{MP} = \tan^{-1} s$; and by Napier's rules,

$$\sin \text{NM} = \tan \text{MP} \times \cot \text{PNM},$$

$$\text{or,} \quad \sin(\theta - \gamma) = s \times \cot \text{PNM}; \tag{197}$$

on comparing which with (196), $\tan \text{PNM} = k$, that is, is constant; and therefore the moon moves in a plane making a constant angle with the ecliptic; or, in other words, the moon would move in a plane, if it were undisturbed.

343.] In concluding the subject of central forces, and of the present Chapter, I propose to investigate some problems which are connected with them: and in the first place one which refers to certain conditions under which it has been imagined the system of the sun, earth, and moon might have been framed; and which is such, so it has been supposed, that more light might have been received by the earth from the moon during the night than in the present configuration. The following problem however is that which gives rise to the speculation:

Three particles or bodies M, $m$, $m'$ attract each other directly as their masses and inversely as the squares of their distances, and move in one plane; it is required to determine the relations between them, so that the equations of motion of the latter two relatively to the former one may have the same form as those of two particles mutually attracting each other according to the same law, and may consequently admit of complete solution.

Let M be the mass of the particle relatively to which the motions of the other two are to be calculated; let the plane in which they move be that of $xy$; let the positions of $m$ and $m'$ at the time $t$ be $(x, y)$, $(x', y')$; and $r$, $r'$ their distances from the origin, at which M is; and, to fix our thoughts, let us assume $r$ to be greater than $r'$; let $\rho =$ the distance between $m$ and $m'$: so that the equations of relative motion of $m$ and $m'$, see Article 295, are severally

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{(\text{M}+m)\,x}{r^3} + m'\left\{\frac{x'}{r'^3} + \frac{x-x'}{\rho^3}\right\} &= 0 \\ \frac{d^2y}{dt^2} + \frac{(\text{M}+m)\,y}{r^3} + m'\left\{\frac{y'}{r'^3} + \frac{y-y'}{\rho^3}\right\} &= 0 \end{aligned}\right\}, \qquad (198)$$

$$\left.\begin{aligned} \frac{d^2x'}{dt^2} + \frac{(\text{M}+m')\,x'}{r'^3} + m\left\{\frac{x}{r^3} + \frac{x'-x}{\rho^3}\right\} &= 0 \\ \frac{d^2y'}{dt^2} + \frac{(\text{M}+m')\,y'}{r'^3} + m\left\{\frac{y}{r^3} + \frac{y'-y}{\rho^3}\right\} &= 0 \end{aligned}\right\}. \qquad (199)$$

But if the motion of $m$ were that which it would be under the action of a single particle placed at the origin, its equations of motion would be of the form,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + \frac{kx}{r^3} &= 0 \\ \frac{d^2y}{dt^2} + \frac{ky}{r^3} &= 0 \end{aligned}\right\}, \qquad (200)$$

where $k$ is a constant. Now on comparing these with (198), we have

$$\frac{x'}{r'^3} + \frac{x-x'}{\rho^3} = \frac{k'x}{r^3},$$

$$\frac{y'}{r'^3} + \frac{y-y'}{\rho^3} = \frac{k'y}{r^3},$$

where $k'$ is another undetermined constant; therefore eliminating $k'$, we have

$$(yx'-xy')\left(\frac{1}{r'^3} - \frac{1}{\rho^3}\right) = 0. \tag{201}$$

By a similar process, from (199) we may deduce the equation,

$$(yx'-xy')\left(\frac{1}{r^3} - \frac{1}{\rho^3}\right) = 0; \tag{202}$$

which may be satisfied in either one or the other of two ways:

$$(1) \qquad \frac{x}{x'} = \frac{y}{y'}, \tag{203}$$

$$(2) \qquad r = r' = \rho. \tag{204}$$

344.] Let us consider the first case; it is manifest from (203) that the three particles are always in the same straight line; and we have

$$\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'},$$

and also $\qquad \rho = r - r';$

so that the first of (198) becomes

$$\frac{d^2x}{dt^2} + \frac{x}{r^3}\left\{\mathrm{M} + m + \frac{m'r^2}{r'^2} + \frac{m'r^2}{(r-r')^2}\right\} = 0. \tag{205}$$

Now if this motion of $m$ is to be the same as it would be under the action of a single central force emanating from the origin, the coefficient of $\frac{x}{r^3}$ in (205) must be constant, and therefore

$$\frac{r'}{r} = \text{a constant}.$$

Let us then assume the following: let

$$r = r'(1+p), \tag{206}$$

where $p$ is an undetermined constant; therefore

$$\left.\begin{array}{cc} x = x'(1+p), & y = y'(1+p), \\ \multicolumn{2}{c}{\rho = pr'} \end{array}\right\}, \tag{207}$$

and (205) becomes,

$$\frac{d^2x}{dt^2} + \frac{x}{r^3}\left\{\mathrm{M} + m + m'(1+p)^2\left(1 + \frac{1}{p^2}\right)\right\} = 0. \tag{208}$$

By a similar process the first of (199) becomes

$$\frac{d^2x'}{dt^2}+\frac{x}{r^3}\left\{(\text{M}+m')(1+p)^2+m\left(1-\frac{(1+p)^2}{p^2}\right)\right\}=0; \quad (209)$$

and from (207), $$\frac{d^2x}{dt^2}=(1+p)\frac{d^2x'}{dt^2}; \quad (210)$$

therefore, equating the coefficients of $\frac{x}{r^3}$, we have

$$\text{M}+m+m'(1+p)^2\left(1+\frac{1}{p^2}\right)$$
$$=(\text{M}+m')(1+p)^3+m(1+p)-m\frac{(1+p)^3}{p^2}; \quad (211)$$

which after expansion becomes,

$$(\text{M}+m')p^5+(3\text{M}+2m')p^4+(3\text{M}+m')p^3-(m'+3m)p^2$$
$$-(2m'+3m)p-(m+m')=0, \quad (212)$$

which is an equation of the fifth degree, and accordingly has one real root; and as the result is $+\infty$ when $p=\infty$, and is negative when $p=0$, the equation has a real positive root. Now the second equations of (198) and (199) will lead to the same equation. Thus the three particles will be always in a straight line, and their distances will be in the constant ratio given in (206): the paths therefore of $m$ and $m'$ will be conics similar to each other, and similarly placed with respect to the position of M, which is in the focus of each. For such a motion the initial positions and circumstances must be in accordance with the preceding conditions.

In ellipses described about a centre of force in the focus the squares of the periodic times are as the cubes of the mean distances; in this case however that law cannot hold, because the orbit described by the exterior particle has necessarily a larger mean distance, and yet the same periodic time.

Also for such a motion to be possible the circumstances at the initial or other epoch must be consistent: that is, the initial velocities of the two masses must have lines of action parallel to each other, and be directly as the distances of the particles from the central particle; because from (207),

$$\left.\begin{aligned}\frac{dx}{dt}&=(1+p)\frac{dx'}{dt}\\ \frac{dy}{dt}&=(1+p)\frac{dy'}{dt}\end{aligned}\right\}. \quad (213)$$

345.] This problem has acquired some consideration from the use which has been made of it in the argument concerning final causes. For if we suppose the three bodies $M$, $m'$, $m$ to be respectively the sun, the earth, and the moon, then, assuming the mass of $M$, the sun, to be unity, we have

$$m' = \frac{1}{329630}, \qquad m = \frac{1}{329630 \times 68.5}. \tag{214}$$

As these numbers are very small, so will a value of $p$ deduced from (212) also be very small: we may therefore, without any great error, substitute for (212),

$$(3M+m')p^3-(m+m') = 0,$$

$$\therefore \quad p = \left(\frac{m+m'}{3}\right)^{\frac{1}{3}}$$

$$= \frac{1}{100} \text{ (nearly)}; \tag{215}$$

that is, the distances from the sun of the earth and the moon would, in geocentric opposition, under the preceding assumptions, be as the numbers 100 to 101. And if these bodies were in such a position and at such relative distances, their magnitudes are such that the moon would never be eclipsed, and therefore she would always be *full* to the earth. It has been supposed that under such an arrangement, assuming that the moon were made by the Creator solely for the purpose of giving light to the earth, more light would have been given than under the present system; or, in other words, it is presumed that the benefit of the moon to the inhabitants of the earth would have been greater, if she had always been in geocentric opposition in this assumed system; and thus if turning as she does the same face to the earth, the other face had never received the light of the sun. But would such a system be stable? subject to perturbations from the planets &c. it would necessarily be: would these perturbations be confined within certain small limits? or would they go on increasing, and finally attain such magnitudes that the whole system would be permanently overthrown? This question I proceed to examine, and for the solution I am indebted to M. Liouville; it is contained in a paper appended to the Connaisance des Temps for the year 1845.

Let us suppose some slight derangement to take place; and to simplify the calculation I shall assume that the bodies $m'$ and $m$ do not leave the plane of $xy$; so that the equations of motion are (198) and (199).

Let us suppose that the relation of $x$ and $y$ to $x'$ and $y'$ is of a form similar to those of equations (207), that is, let

$$\left.\begin{aligned} x &= (1+p)(x'+\xi) \\ y &= (1+p)(y'+\eta) \end{aligned}\right\}, \tag{216}$$

where $p$ is a constant as heretofore; and $\xi$ and $\eta$ are the increments of $x'$ and $y'$ due to the action of the disturbing forces. When $t=0$, $\xi, \eta, \frac{d\xi}{dt}, \frac{d\eta}{dt}$ are not absolutely zero, but are very small. Our object is to determine the values of $\xi$ and $\eta$ as $t$ increases; because if they continue small, or involve periodical quantities only, the assumed system is stable: but if they indefinitely increase, the deviations of $x$ and $y$ from the relations assumed in (207) become greater and greater, and the system is unstable. We must therefore investigate the differential equations on which $\xi$ and $\eta$ depend, and assuming these quantities to be small, we shall neglect squares, cubes, and the higher powers of them and of their products.

Let us suppose the mass of the sun to be the unit-mass: so that $\text{M}=1$; and let us calculate the differential equations which refer to $m'$, the earth, viz. (199). Now

$$\begin{aligned} \frac{x}{r^3} &= \frac{(1+p)(x'+\xi)}{(1+p)^3\{(x'+\xi)^2+(y'+\eta)^2\}^{\frac{3}{2}}} \\ &= \frac{x'+\xi}{(1+p)^2 r'^3\left\{1+\frac{2(x'\xi+y'\eta)}{r'^2}\right\}^{\frac{3}{2}}} \\ &= \frac{x'+\xi}{(1+p)^2 r'^3}\left\{1-\frac{3(x'\xi+y'\eta)}{r'^2}\right\} \\ &= \frac{x'}{(1+p)^2 r'^3}+\frac{1}{(1+p)^2 r'^3}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}. \end{aligned}$$

Similarly,

$$\frac{x-x'}{\rho^3}=\frac{x'}{p^2 r'^3}+\frac{1+p}{p^3 r'^3}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\};$$

and substituting these in the first of (199), we have, since $\text{M}=1$,

$$\begin{aligned} &\frac{d^2x'}{dt^2}+\left(1+m'+\frac{m}{(1+p)^2}-\frac{m}{p^2}\right)\frac{x'}{r'^3} \\ &\quad+\left\{\frac{m}{(1+p)^2 r'^3}-\frac{m(1+p)}{p^3 r'^3}\right\}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}=0. \end{aligned} \tag{217}$$

Let $$1+m'+\frac{m}{(1+p)^2}-\frac{m}{p^2}=n'^2; \tag{218}$$
so that (217) becomes,

$$\frac{d^2x'}{dt^2}+n'^2\frac{x'}{r'^3}+\left\{\frac{m}{(1+p)^2r'^3}-\frac{m(1+p)}{p^3r'^3}\right\}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}=0. \tag{219}$$

From the first of (216),

$$\frac{d^2x}{dt^2}=(1+p)\left(\frac{d^2x'}{dt^2}+\frac{d^2\xi}{dt^2}\right), \tag{220}$$

and substituting in the first of (198), we have

$$\frac{d^2x'}{dt^2}+\frac{d^2\xi}{dt^2}+\frac{x'}{r'^3}\left\{\frac{1+m}{(1+p)^3}+\frac{m'(1+p^2)}{p^2(1+p)}\right\}$$
$$+\left\{\frac{1+m}{(1+p)^3r'^3}+\frac{m'}{p^3r'^3}\right\}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}=0. \tag{221}$$

By reason of (211), when M $=1$, and from (218),

$$\frac{1+m}{(1+p)^3}+\frac{m'(1+p^2)}{p^2(1+p)}=1+m'+\frac{m}{(1+p)^2}-\frac{m}{p^2}=n'^2; \tag{222}$$

so that (221) becomes

$$\frac{d^2x'}{dt^2}+\frac{d^2\xi}{dt^2}+\frac{n'^2x'}{r'^3}+\left\{\frac{1+m}{(1+p)^3r'^3}+\frac{m'}{p^3r'^2}\right\}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}=0; \tag{223}$$

from which subtracting (219), we have,

$$\frac{d^2\xi}{dt^2}+\left\{\frac{1-mp}{(1+p)^3r'^3}+\frac{m'+m(1+p)}{p^3r'^3}\right\}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}=0. \tag{224}$$

Let $$\frac{1-mp}{(1+p)^3}+\frac{m'+m(1+p)}{p^3}=n^2; \tag{225}$$

and therefore we have

$$\frac{d^2\xi}{dt^2}+\frac{n^2}{r'^3}\left\{\xi-\frac{3x'(x'\xi+y'\eta)}{r'^2}\right\}=0. \tag{226}$$

By a similar process from the second of (198) and of (199) we have,

$$\frac{d^2\eta}{dt^2}+\frac{n^2}{r'^3}\left\{\eta-\frac{3y'(x'\xi+y'\eta)}{r'^2}\right\}=0. \tag{227}$$

These equations in their present forms are not susceptible of integration, and we are obliged to have recourse to a process of approximation. For this purpose we must first express $x'$ and $y'$ in terms of $t$ under the supposition that the system is not deranged: in this case we have from (219), if we put $\xi=\eta=0$,

$$\left.\begin{array}{l}\dfrac{d^2x'}{dt^2}+\dfrac{n'^2x'}{r'^3}=0\\[2ex] \dfrac{d^2y'}{dt^2}+\dfrac{n'^2y'}{r'^3}=0\end{array}\right\}; \tag{228}$$

and similarly (for the second line)

whence by integration, as in Article 288, we shall arrive at the equation of a conic, which is the path of $m'$ when undisturbed. Let us however assume the conic to be a circle of which the radius is unity, so that $r'=1$; in which case the solution of (228) is,

$$\left.\begin{aligned} x' &= \cos(n't+\tau) \\ y' &= \sin(n't+\tau) \end{aligned}\right\}, \tag{229}$$

where $\tau$ is a constant dependent on the origin of the time. In (226), replacing $r'$ by 1, we have

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + n^2\{\xi - 3x'(x'\xi + y'\eta)\} &= 0 \\ \frac{d^2\eta}{dt^2} + n^2\{\eta - 3y'(x'\xi + y'\eta)\} &= 0 \end{aligned}\right\}. \tag{230}$$

Let
$$\left.\begin{aligned} x'\xi + y'\eta &= u \\ y'\xi - x'\eta &= v \end{aligned}\right\}, \tag{231}$$

so that, as $x'$, $y'$ are functions of $t$, the substitution is equivalent to the replacement of the original system of rectangular axes by a new system which moves simultaneously with $m$ and $m'$ in their relative motion about M. Now from (229)

$$\left.\begin{aligned} \frac{dx'}{dt} &= -n'y' \\ \frac{dy'}{dt} &= n'x' \end{aligned}\right\}, \tag{232}$$

$$\therefore\quad \left.\begin{aligned} \frac{du}{dt} &= x'\frac{d\xi}{dt} + y'\frac{d\eta}{dt} - n'v \\ \frac{dv}{dt} &= y'\frac{d\xi}{dt} - x'\frac{d\eta}{dt} + n'u \end{aligned}\right\}; \tag{233}$$

and differentiating again and substituting from (230) we have,

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + 2n'\frac{dv}{dt} - (2n^2+n'^2)u &= 0 \\ \frac{d^2v}{dt} - 2n'\frac{du}{dt} + (n^2-n'^2)v &= 0 \end{aligned}\right\}, \tag{234}$$

which are two simultaneous differential equations with constant coefficients.

Let us assume two particular integrals of these equations to be

$$u = e^{at}, \qquad v = \text{A}e^{at},$$

where A and $a$ are two undetermined constants. And eliminating A, we shall have,

$$a^4 + a^2(2n'^2 - n^2) - (n^2 - n'^2)(2n^2 + n'^2) = 0, \tag{235}$$

an equation whence $a^2$ may be determined; and the two values of $a^2$ will have the same or different signs according as $n^2-n'^2$ is negative or positive. In the case of the sun, the earth, and the moon, $p$, $m$, and $m'$ are small fractions, and approximately from (218) and (225) $n'=1$, $n=2$, so that $n^2-n'^2$ is positive; and therefore one value of $a^2$ is positive and the other is negative. Therefore the solutions of (234) will contain functions of $t$ consisting partly of periodical quantities involving circular functions, and partly of exponential quantities; the former may be confined within small limits; the latter will increase indefinitely as $t$ increases, and therefore the values of $u$ and $v$ will not be small; hence also the values of $\xi$ and $\eta$ will not continue small, and the system will depart more and more from the state which it had when $t=0$, and thus will be unstable.

And if we substitute the preceding values of $n$ and $n'$ in (235), $a^2$ is greater than 2; and since the time of the earth's revolution about the sun is, see (229), $\frac{2\pi}{n'}=2\pi$, it is plain that $e^{at}$ will increase with great rapidity; and thus the system will be very quickly overthrown. Under the supposed arrangement therefore, the moon would leave the earth, and the end which the suggesters of the hypothesis now considered had in view would be lost. I may also observe that the system would be unstable if the moon were placed in a position between the sun and the earth, in which case the moon would always be *new* to the earth; and the system would no more be continued in this state than in the other.

346.] Let us next consider the second case in which the equations of motion of one disturbed particle moving relatively to another are of the same form as those of a single particle under the action of a central force; that is, let us suppose equation (204) to be satisfied.

In this case $$r=r'=\rho,$$

and therefore the three particles are during the whole of the motion at equal distances apart: they are therefore at the three angles of an equilateral triangle; and the equations of motion of, say, $m$ become,

$$\left.\begin{aligned}\frac{d^2x}{dt^2}+(\mathrm{M}+m+m')\frac{x}{r^3}&=0\\ \frac{d^2y}{dt^2}+(\mathrm{M}+m+m')\frac{y}{r^3}&=0\end{aligned}\right\}, \qquad (236)$$

and the equations of motion of $m'$ are similar. And as these equations, when integrated, express a conic, it follows that the paths of $m$ and $m'$ are conics relatively to M, the position of which is in the focus of the paths of $m$ and $m'$; the initial circumstances of $m$ and $m'$ must of course be consistent with this permanent arrangement; and we must always have,

$$\frac{dr}{dt} = \frac{dr'}{dt} = \frac{d\rho}{dt}.$$

347.] Lagrange in the Mécanique Analytique, second part, Sect. VII, Chap. III, Art. 83, remarks that a conic, say, an ellipse, which would be described by a particle under the action of a force varying inversely as the square of the distance and tending to the focus of the ellipse, or under the action of a force varying directly as the distance and having its source in the centre of the ellipse, may also be described under the action of three similar forces which have their sources in the two foci and in the centre of the ellipse respectively; and he makes this remark, after he has proved that such forces yield a particular integral of the differential equation which expresses the motion of a particle under the action of two central forces, which vary inversely as the square of the distance, and whose centres are in two given points. This fact however is only a special application of the following more general law:

If many particles $m_1, m_2, \dots m_n$ which are respectively under the action of the force $\text{P}_1, \text{P}_2, \dots \text{P}_n$ are projected from a given point with the velocities respectively $\text{V}_1, \text{V}_2, \dots \text{V}_n$ along the same line and in the same direction; and if each of these particles describes the same path; then one particle M, projected with the velocity V from the same point, along the same line, and in the same direction as the $m$'s, will describe the same path, if the initial vis viva of M is equal to the sum of the initial vires vivæ of the $m$'s; that is, if

$$\text{M}\,\text{V}^2 = m_1 \text{V}_1^2 + m_2 \text{V}_2^2 + \dots + m_n \text{V}_n^2. \tag{237}$$

Let $(x, y, z)$ be the position of M at the time $t$; so that its expressed momentum-increments along the coordinate axes are

$$\text{M}\frac{d^2x}{dt^2}, \quad \text{M}\frac{d^2y}{dt^2}, \quad \text{M}\frac{d^2z}{dt^2};$$

and let $\text{X}_1, \text{Y}_1, \text{Z}_1, \text{X}_2, \text{Y}_2, \text{Z}_2, \dots \text{X}_n, \text{Y}_n, \text{Z}_n$ be the components of the impressed momentum-increments of the several forces $\text{P}_1, \text{P}_2, \dots \text{P}_n$; and let N be a certain normal force, the direction angles of the line of action of which are $\alpha, \beta, \gamma$, and which is such that M under the action of it and the P's describes the required path. Then the equations of motion of M are

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= X_1 + X_2 + \ldots + X_n + N\cos\alpha = \Sigma.X + N\cos\alpha \\ M\frac{d^2y}{dt^2} &= Y_1 + Y_2 + \ldots + Y_n + N\cos\beta = \Sigma.Y + N\cos\beta \\ M\frac{d^2z}{dt^2} &= Z_1 + Z_2 + \ldots + Z_n + N\cos\gamma = \Sigma.Z + N\cos\gamma \end{aligned}\right\}. \quad (238)$$

Multiplying these respectively by $2\,dx$, $2\,dy$, $2\,dz$, and supposing the velocities of $M, m_1, m_2, \ldots m_n$ at the time $t$ to be $v, v_1, v_2, \ldots v_n$, we have

$$d.Mv^2 = 2\,dx\,\Sigma.X + 2\,dy\,\Sigma.Y + 2\,dz\,\Sigma.Z, \qquad (239)$$

because $dx\cos\alpha + dy\cos\beta + dz\cos\gamma = 0$.

But the equations of motion of $m_1, m_2, \ldots m_n$ under the action of their respective forces yield the following equations:

$$\left.\begin{aligned} d.m_1v_1^2 &= 2(X_1\,dx + Y_1\,dy + Z_1\,dz) \\ d.m_2v_2^2 &= 2(X_2\,dx + Y_2\,dy + Z_2\,dz) \\ &\ldots\ldots\ldots\ldots \\ d.m_nv_n^2 &= 2(X_n\,dx + Y_n\,dy + Z_n\,dz) \end{aligned}\right\}, \qquad (240)$$

so that (239) becomes

$$d.Mv^2 = \Sigma.d.mv^2 = d.\Sigma.mv^2;$$

and therefore taking definite integrals, with limits corresponding to $t = t$ and to $t = 0$, we have

$$Mv^2 = \Sigma.mv^2; \qquad (241)$$

and therefore at all points of the path of M its vis viva is equal to the sum of the vires vivæ of the $m$'s in their separate motions.

Hence it follows that the normal force N assumed in equations (238) is zero, and consequently M, under the action of the several impressed momenta which act on $m_1, m_2, \ldots m_n$, will describe the same path as each of the $m$'s. This general proposition is due to M. Ossian Bonnet, and is given in the notes appended by M. Bertrand to the edition of the "Mécanique Analytique" of M. Lagrange, Vol. II, 1855.

# CHAPTER XII.

## THE CONSTRAINED MOTION OF PARTICLES, UNDER THE ACTION OF GIVEN FORCES.

### SECTION 1.—*The motion of a particle constrained to move on a given curved line.*

348.] THE subjects of motion thus far have been particles not constrained by any geometrical conditions; they have therefore been free to take in space such paths as are due to the action of the impressed forces: it remains now to investigate the motion of particles which are constrained; that is, which are subject to certain geometrical relations: such is the motion of a particle in a small tube, either smooth or rough, and the bore of which is supposed to be of the same size as the particle: of a small ring sliding on a curved wire, with or without friction: of a particle fastened to a string, or moving on a given surface. The principles of the science which were investigated in Chapter IX are of breadth sufficient for this inquiry. For in addition to the impressed momentum-increments on the moving particle, there will in general be a normal pressure or reaction of the curve or surface, arising from the deflexion of the particle from the path which it would have under the action of the impressed forces alone: and this pressure together with the expressed momentum-increments will be equal to the impressed momentum-increments.

349.] In this section I propose to consider the motion of a particle constrained to move on a given curve; such as that of a particle in a tube, or of a small ring on a wire, or of a particle fastened to a string. And, to take the general case, I shall suppose firstly the curve to be in space, and the motion to be free from friction.

Let the equations to the curve on which the particle, of mass $m$, is constrained to move, be

$$\text{F}_1(x, y, z) = 0, \qquad \text{F}_2(x, y, z) = 0; \tag{1}$$

and let $(x, y, z)$ be the place of $m$ at the time $t$: let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the coordinate components of the impressed velocity-increments; let $\mathrm{R}$ be the normal pressure, and let $\alpha, \beta, \gamma$ be the direction angles of its line of action; so that the equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= m\mathrm{X} - \mathrm{R}\cos\alpha \\ m\frac{d^2y}{dt^2} &= m\mathrm{Y} - \mathrm{R}\cos\beta \\ m\frac{d^2z}{dt^2} &= m\mathrm{Z} - \mathrm{R}\cos\gamma \end{aligned}\right\}; \tag{2}$$

and dividing by $m$, whereby the equations become expressed in terms of velocity-increments,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} - \frac{\mathrm{R}}{m}\cos\alpha \\ \frac{d^2y}{dt^2} &= \mathrm{Y} - \frac{\mathrm{R}}{m}\cos\beta \\ \frac{d^2z}{dt^2} &= \mathrm{Z} - \frac{\mathrm{R}}{m}\cos\gamma \end{aligned}\right\}. \tag{3}$$

The line of action of $\mathrm{R}$ cannot be definitely determined; we know no more of it, at present, than that it is perpendicular to the tangent of the curvilinear path at the place of the particle; so that we have

$$dx\cos\alpha + dy\cos\beta + dz\cos\gamma = 0: \tag{4}$$

neither do we know the *direction* in which $\mathrm{R}$ acts.

Now multiplying the equations of (3) severally by $2\,dx$, $2\,dy$, $2\,dz$, and adding, we have

$$\frac{2\,dx\,d^2x + 2\,dy\,d^2y + 2\,dz\,d^2z}{dt^2} = 2(\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz). \tag{5}$$

Let the circumstances of motion be considered at the times $t$ and $t_0$; and let us suppose that $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$ is such a function of $x, y, z$ as to admit of integration; and let $v$ and $v_0$ be the corresponding velocities; then integrating (5) we have,

$$v^2 - v_0^2 = 2\int_{t_0}^{t} (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz), \tag{6}$$

whereby the velocity is found by simple integration at any point of the path.

Also since

$$\frac{d^2s}{dt^2} = \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds} + \frac{d^2z}{dt^2}\frac{dz}{ds},$$

$$\therefore \quad \frac{d^2s}{dt^2} = \frac{\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz}{ds}, \tag{7}$$

and is therefore independent of the resistance of the curve; the velocity-increment therefore of the particle along the curvilinear path is the same as if the particle were moving freely along that path.

And since $v^2 = \frac{ds^2}{dt^2}$, we have from (6),

$$dt = \frac{\pm\, ds}{\left\{v_0^2 + 2\int_{t_0}^{t} (\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)\right\}^{\frac{1}{2}}}, \tag{8}$$

whence the time may be found in terms of the coordinates of the particle at $t$ and $t_0$.

And the normal pressure on the curve may thus be found: multiplying (3) severally by $\cos a$, $\cos \beta$, $\cos \gamma$, we have

$$\frac{\text{R}}{m} = \left(\text{X} - \frac{d^2x}{dt^2}\right)\cos a + \left(\text{Y} - \frac{d^2y}{dt^2}\right)\cos\beta + \left(\text{Z} - \frac{d^2z}{dt^2}\right)\cos\gamma. \tag{9}$$

Now suppose the line of action of the normal pressure to be the principal normal: then if $\rho =$ the radius of absolute curvature,

$$\cos a = \pm\rho\frac{d}{ds}\frac{dx}{ds}, \quad \cos\beta = \pm\rho\frac{d}{ds}\frac{dy}{ds}, \quad \cos\gamma = \pm\rho\frac{d}{ds}\frac{dz}{ds},$$

the double sign referring to the value of $\rho$, and which is to be positive or negative, according to the *direction* in which it is measured, so that it may finally bear a positive sign, because it is an absolute length. And replacing in (9) $\cos a$, $\cos\beta$, $\cos\gamma$ by these values, we have

$$\frac{\text{R}}{m} = \text{X}\cos a + \text{Y}\cos\beta + \text{Z}\cos\gamma \pm \rho\left\{\frac{d^2x}{dt^2}\frac{d}{ds}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{d}{ds}\frac{dy}{ds} + \frac{d^2z}{dt^2}\frac{d}{ds}\frac{dz}{ds}\right\},$$

$$\therefore \quad \text{R} = m\left\{\text{X}\cos a + \text{Y}\cos\beta + \text{Z}\cos\gamma \pm \frac{v^2}{\rho}\right\}. \tag{10}$$

$\frac{v^2}{\rho}$ is the centrifugal force; see Art. 255: thus if the normal pressure on the curve acts along the radius of absolute curvature, it is equal to the algebraical sum of the resolved parts of the impressed momentum-increments along that radius of absolute curvature and of the centrifugal force.

And the normal pressure is along the principal normal when the curve is described by the moving particle without the action of any continually impressed forces: for in this case,

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{\text{R}}{m}\cos a\\ \frac{d^2y}{dt^2} &= -\frac{\text{R}}{m}\cos\beta\\ \frac{d^2z}{dt^2} &= -\frac{\text{R}}{m}\cos\gamma\end{aligned}\right\};$$

$$\therefore\quad \frac{dx\,d^2x+dy\,d^2y+dz\,d^2z}{dt^2} = -\frac{\text{R}}{m}\{dx\cos a+dy\cos\beta+dz\cos\gamma\}$$

$$= 0;$$

$$\therefore\quad \frac{ds^2}{dt^2} = (\text{velocity})^2 = \text{a constant}.$$

Therefore $dt = k\,ds$, and therefore $s$ and $t$ are simultaneously equicrescent: and $\cos a$, $\cos\beta$, $\cos\gamma$ are severally proportional to $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, and are therefore the direction cosines of the principal normal.

R is, it will be observed, a pressure or a momentum; and is therefore compounded of two factors, of which one is mass and the other is velocity; and as R varies directly as $m$, $\frac{\text{R}}{m}$ is velocity. Thus equation (9) is homogeneous, and is formed in terms of velocity: equation (10), on the other hand, is formed in terms of momentum or pressure. It is convenient to have a distinctive name for the velocity, which is lost or gained by reason of the motion being constrained, and I shall call it the reaction of the curve: so that reaction is the pressure on the curve of an unit-mass; and the pressure of $m$ on the curve is the product of the mass and the reaction.

Equations (7) and (9) or (10) are the tangential and normal components of the velocity-increments, and the result is therefore equivalent to a tangential and normal resolution.

350.] If however the motion takes place wholly in one plane, we may take that to be the plane of $xy$, and the general formulæ become simplified. Thus the equations in terms of velocity-increments along the coordinate axes are,

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \text{X} \mp \frac{\text{R}}{m}\frac{dy}{ds}\\ \frac{d^2y}{dt^2} &= \text{Y} \pm \frac{\text{R}}{m}\frac{dx}{ds}\end{aligned}\right\}; \qquad (11)$$

and if the velocity-increments are resolved along the tangent and the normal,

$$\begin{aligned}\frac{d^2s}{dt^2} &= \frac{d^2x}{dt^2}\frac{dx}{ds} + \frac{d^2y}{dt^2}\frac{dy}{ds} \\ &= \mathrm{X}\frac{dx}{ds} + \mathrm{Y}\frac{dy}{ds};\end{aligned} \tag{12}$$

and integrating with limits corresponding to $t$ and $t_0$, we have

$$v^2 - v_0^2 = 2\int_{t_0}^{t} (\mathrm{X}\,dx + \mathrm{Y}\,dy). \tag{13}$$

Also if we multiply the two equations of (11) severally by $dy$ and $dx$ and subtract, we have

$$\begin{aligned}\mathrm{R} &= \pm m\left\{\mathrm{X}\frac{dy}{ds} - \mathrm{Y}\frac{dx}{ds} - \frac{dy\,d^2x - dx\,d^2y}{ds\,dt^2}\right\} \\ &= \pm m\left\{\mathrm{X}\frac{dy}{ds} - \mathrm{Y}\frac{dx}{ds} \pm \frac{v^2}{\rho}\right\},\end{aligned} \tag{14}$$

if $\rho$ is the radius of curvature of the curve at the point $(x, y)$; and the correct signs must, where an ambiguity exists, be determined by the circumstances of the problem.

And if the system of radial and transversal resolution is taken, and if P and Q are the radial and transversal impressed velocity-increments, as in Art. 256, then

$$\left.\begin{aligned}\frac{d^2r}{dt^2} - r\frac{d\theta^2}{dt^2} &= \mathrm{P} \pm \frac{\mathrm{R}}{m}\frac{r\,d\theta}{ds} \\ \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) &= \mathrm{Q} \mp \frac{\mathrm{R}}{m}\frac{dr}{ds}\end{aligned}\right\}; \tag{15}$$

and integrating these we have for the velocity,

$$v^2 - v_0^2 = 2\int_{t_0}^{t}\{\mathrm{P}\,dr + \mathrm{Q}\,r\,d\theta\}. \tag{16}$$

Also we have

$$\mathrm{R} = \pm m\left\{\mathrm{Q}\frac{dr}{ds} - \mathrm{P}\frac{r\,d\theta}{ds} \pm \frac{v^2}{\rho}\right\}. \tag{17}$$

We proceed now to give examples in which these principles and formulæ are applied.

The most simple is manifestly the motion of a heavy particle moving on a smooth inclined plane; the motion of it along the plane has been considered already: viz. in Section 2, Chapter X, wherein the tangential component has entered into computation: and for the normal component we have, if $a$ is the inclination of the plane to the horizon,

$$\mathrm{R} = mg\cos a;$$

that is, the pressure on the plane is the same at all points of it, and is independent of the velocity with which the particle moves down the plane.

351.] Ex. 1. From a given point on a parabola a particle is projected with a given velocity $u$ along the concave side of the curve, and is acted on by a force in the focus which is attractive and varies inversely as the square of the distance; it is required to determine the circumstances of motion.

Let the equation to the parabola be

$$r = \frac{2a}{1+\cos\theta},$$

which in terms of $r$ and $p$ becomes

$$p^2 = ar;$$

let $c$ be the initial distance of $m$ from the focus; then $\text{Q} = 0$, and

$$\text{P} = -\frac{\mu}{r^2};$$

therefore from (16),

$$v^2 = u^2 + 2\mu\left\{\frac{1}{r} - \frac{1}{c}\right\},$$

whereby the velocity is known at every point of the curve. And from (17),

$$\frac{\text{R}}{m} = \frac{v^2}{\rho} - \frac{\mu}{r^2}\frac{r\,d\theta}{ds}$$

$$= \frac{a^{\frac{1}{2}}}{2r^{\frac{3}{2}}}\left\{u^2 - \frac{2\mu}{b}\right\};$$

and thus the pressure is known at every point of the curve; and since $m$ is to move on the concave side of the curve, $u^2$ must be greater than $\frac{2\mu}{b}$; and if $u^2$ is less than $\frac{2\mu}{b}$, $m$ must move on the convex side; and if $u^2 = \frac{2\mu}{b}$ the pressure on the curve is zero at all its points; and rightly so; because $\left(\frac{2\mu}{b}\right)^{\frac{1}{2}}$ is, see Art. 319, the velocity acquired at the point of projection by $m$ moving from infinity under the action of the central force; and thus under these circumstances the parabola would be the unconstrained orbit.

Ex. 2. A particle moves on the convex side of a parabola, and is tied by an extensible string to a point in the focus; the unstretched length of the string is equal to the focal distance; the particle is placed at rest at the extremity of the latus rectum; it is required to determine the subsequent circumstances of motion.

$$r = \frac{2a}{1+\cos\theta}. \tag{18}$$

Let T = the tension of the string, so that T is the central force acting on $m$; then, if $r$ is the distance of $m$ from S at the time $t$, and if $e$ is the coefficient of elasticity,

$$r = a(1+e\text{T}), \qquad \therefore \quad \text{T} = \frac{r-a}{ea};$$

and therefore from (16),

$$\begin{aligned} \frac{ds^2}{dt^2} &= 2\int_{2a}^{r} \frac{a-r}{ae}\,dr \\ &= \frac{2}{ae}\left\{ar - \frac{r^2}{2}\right\}; \end{aligned} \tag{19}$$

and from (18), $\quad ds^2 = \dfrac{r\,dr^2}{r-a};$

and substituting this in (19) we have

$$dt = -\frac{(ae)^{\frac{1}{2}}\,dr}{\{(r-a)(2a-r)\}^{\frac{1}{2}}},$$

$$\therefore \quad t = (ae)^{\frac{1}{2}}\cos^{-1}\frac{2r-3a}{a};$$

therefore when $r = a$, $t = \pi(ae)^{\frac{1}{2}}$, and this is the time which the particle takes in passing from the extremity of the latus rectum to the vertex; and the velocity is a maximum when $r = a$, so that the particle passes on from the vertex to the other extremity of the latus rectum, at which it comes to rest; and this oscillatory motion continues, the period of an oscillation being $2\pi(ae)^{\frac{1}{2}}$.

Ex. 3. A particle moves on the convex side of an ellipse under the action of two forces parallel respectively to the axes of $x$ and $y$, and which vary respectively as the cubes of the corresponding coordinates: the particle is placed at rest at the point $(h, k)$; it is required to determine the pressure on the curve.

In this case, $\quad \text{X} = -\dfrac{\mu}{x^3}, \qquad \text{Y} = -\dfrac{\mu'}{y^3};$

therefore from (13),

$$\begin{aligned} \frac{ds^2}{dt^2} = v^2 &= -2\int\left(\frac{\mu\,dx}{x^3} + \frac{\mu'\,dy}{y^3}\right) \\ &= \frac{\mu}{x^2} + \frac{\mu'}{y^2} - \frac{\mu}{h^2} - \frac{\mu'}{k^2}. \end{aligned}$$

Let the equation to the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Then

$$\frac{\mathrm{R}}{m}=\left(\frac{\mu b^2}{x^2}+\frac{\mu' a^2}{y^2}\right)\frac{1}{(a^4y^2+b^4x^2)^{\frac{1}{2}}}-\left(\frac{\mu}{x^2}+\frac{\mu'}{y^2}-\frac{\mu}{h^2}-\frac{\mu'}{k^2}\right)\frac{a^4b^4}{(a^4y^2+b^4x^2)^{\frac{3}{2}}}.$$

Ex. 4. A particle moves on the convex side of an ellipse, and is under the action of (1) two central forces varying inversely as the square of the distance, and whose sources are at the foci, and (2) a central force varying as the distance and whose source is at the centre of the ellipse; it is required to determine the pressure on the curve.

Let $\mu_1, \mu_2, \mu$ be the absolute forces severally of the foci and of the centre; and let the centre of the ellipse be the origin. Let $r_1, r_2, r$ be the distances of $m$ at the time $t$ from the two foci and from the centre respectively, and let $r'_1, r'_2, r'$ be the initial values of these, that is, when $t=0$; and let $v_0$ be the initial velocity. Then

$$\mathrm{X}=-\mu x-\frac{\mu_1(x+ae)}{r_1^{\,3}}-\frac{\mu_2(x-ae)}{r_2^{\,3}},$$

$$\mathrm{Y}=-\mu y-\frac{\mu_1 y}{r_1^{\,3}}-\frac{\mu_2 y}{r_2^{\,3}};$$

$$\therefore\quad v^2-v_0^{\,2}=-\mu r^2+\frac{2\mu_1}{r_1}+\frac{2\mu_2}{r_2}+\mu r'^2-\frac{2\mu_1}{r'_1}-\frac{2\mu_2}{r'_2}.$$

Also $r^2=a^2+b^2-r_1r_2$, and $\rho=\dfrac{(r_1r_2)^{\frac{3}{2}}}{ab}$,

$$\therefore\quad \frac{\mathrm{R}}{m}\rho=\mu r'_1r'_2+\frac{\mu_1 r'_2}{ar'_1}+\frac{\mu_2 r'_1}{ar'_2}-v_0^{\,2}.$$

If $\mathrm{v}$, $\mathrm{v}_1$, $\mathrm{v}_2$ are the velocities with which a particle $m$ being projected from the given point would freely describe the preceding ellipse under the action of the preceding forces acting singly, then

$$\mathrm{v}^2=\mu r'_1r'_2,\qquad \mathrm{v}_1^{\,2}=\frac{\mu_1 r'_2}{ar'_1},\qquad \mathrm{v}_2^{\,2}=\frac{\mu_2 r'_1}{ar'_2}:$$

so that
$$\frac{\mathrm{R}\rho}{m}=\mathrm{v}^2+\mathrm{v}_1^{\,2}+\mathrm{v}_2^{\,2}-v_0^{\,2};$$

if therefore
$$v_0^{\,2}=\mathrm{v}^2+\mathrm{v}_1^{\,2}+\mathrm{v}_2^{\,2},$$

then $\mathrm{R}=0$, and the particle would describe the ellipse freely. This result is a particular application of the general Theorem of Article 347.

352.] Let us now consider another form of the same problem which is more simple than the preceding, and more general in application; that, viz., of a *heavy* particle moving on a given plane curve.

Let the axis of $x$ be horizontal, and the axis of $y$ vertical; so that the equations of motion are,

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= \pm\frac{\text{R}}{m}\frac{dy}{ds}\\ \frac{d^2y}{dt^2} &= g\mp\frac{\text{R}}{m}\frac{dx}{ds}\end{aligned}\right\}. \tag{20}$$

Multiplying these by $2\,dx$ and $2\,dy$ respectively, adding, and integrating, we have

$$v^2-v_0^2 = 2g\,(y-y_0); \tag{21}$$

that is, the excess of the square of the velocity at the time $t$ over that at the beginning of the time varies as the perpendicular distance through which the particle has moved, and is independent of the path which it has taken. This is a particular case of the general principle of vis viva which we shall discuss in the following Chapter.

Again, multiplying the first of (20) by $dy$ and the second by $dx$ and subtracting, we have

$$\pm\frac{\text{R}}{m} = g\frac{dx}{ds}+\frac{dy\,d^2x-dx\,d^2y}{dt^2\,ds}.$$

But if $\rho$ is the radius of curvature of the curve at the point $(x, y)$,

$$\begin{aligned} ds &= \rho\, d.\tan^{-1}\left(\frac{dx}{dy}\right)\\ &= \rho\,\frac{d^2x\,dy-d^2y\,dx}{ds^2},\end{aligned}$$

$$\therefore\quad d^2x\,dy-d^2y\,dx = \frac{ds^3}{\rho};$$

$$\therefore\quad \pm\frac{\text{R}}{m} = g\frac{dx}{ds}\pm\frac{v^2}{\rho}; \tag{22}$$

the ambiguity of the signs depends on the position and the direction of curvature of the curve; they are therefore for each particular problem to be determined by its circumstances.

Now $g\dfrac{dx}{ds}$ is the resolved part of gravity along the normal to the curve, and $\dfrac{v^2}{\rho}$ is the centrifugal force: the reaction therefore is equal to the algebraical sum of these two quantities.

We shall sometimes find it convenient to employ the system of tangential and normal resolution; the choice however of the method must depend on the particular application of it.

353.] The motion of a heavy particle on a cycloid.

Let the base of the cycloid be horizontal, and let the curve be in a vertical plane; and firstly let us suppose the vertex of the cycloid to be its lowest point, as in fig. 135; OM $= x$, MP $= y$; so that the equation of the curve is,

$$y = a \operatorname{versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}}. \tag{23}$$

Let the particle $m$ move on the concave side of the curve, and be at P $(x, y)$ at the time $t$, and at rest at K $(h, k)$ when $t = 0$. Then the normal PG is the line of action of R, and the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -g + \frac{\text{R}}{m}\frac{dy}{ds} \\ \frac{d^2y}{dt^2} &= -\frac{\text{R}}{m}\frac{dx}{ds} \end{aligned}\right\}; \tag{24}$$

whence, as heretofore, multiplying the first by $2\,dx$ and the second by $2\,dy$, adding and integrating, and observing that $v = 0$ when $x = h$, we have

$$\frac{ds^2}{dt^2} = v^2 = 2g(h-x); \tag{25}$$

similarly, multiplying the first of (24) by $dy$, and the second by $dx$, and subtracting, we have

$$\frac{\text{R}}{m} = g\frac{dy}{ds} + \frac{v^2}{\rho}. \tag{26}$$

From (23),
$$\frac{dy}{\{2a-x\}^{\frac{1}{2}}} = \frac{dx}{x^{\frac{1}{2}}} = \frac{ds}{(2a)^{\frac{1}{2}}},$$
$$\rho = \{8a(2a-x)\}^{\frac{1}{2}}.$$

If OP $= s$, and $m$ is descending to the lowest point of the curve, and if $t$ is the time of $m$ descending from K to O, then from (25) we have,

$$\begin{aligned} t &= -\frac{1}{(2g)^{\frac{1}{2}}}\int_h^0 \frac{ds}{(h-x)^{\frac{1}{2}}} \\ &= -\left(\frac{a}{g}\right)^{\frac{1}{2}}\int_h^0 \frac{dx}{(hx-x^2)^{\frac{1}{2}}} \\ &= \left(\frac{a}{g}\right)^{\frac{1}{2}}\left[\operatorname{versin}^{-1}\frac{2x}{h}\right]_0^h \\ &= \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}; \end{aligned} \tag{27}$$

the time of descent of a particle to the lowest point of the curve is thus independent of $h$, and is therefore the same for all particles starting from different points of the cycloid.

And since the velocity of $m$ at O is $(2gh)^{\frac{1}{2}}$, and has therefore its greatest value, and since by (25) $v=0$ when $x=h$, it follows that $m$ having passed through O ascends the other branch of the cycloid, and reaches a point K′ at the distance $h$ above the horizontal line through O; and the time of the ascent is $\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$: thus the time of a complete oscillation from K to K′ is $2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$, and is thus independent of the point on the curve at which the motion commences.

This property of a curve is called Tautochronism, and the cycloid is accordingly called a tautochronous curve.

The result (27) may be arrived at so shortly by the following process that I do not hesitate to insert it:

Let us resolve normally and tangentially: then we have

$$\frac{d^2s}{dt^2} = -g\frac{dx}{ds}.$$

From the equation to the cycloid, $s^2 = 8ax$,

$$\therefore \quad \frac{dx}{ds} = \frac{s}{4a},$$

$$\therefore \quad \frac{d^2s}{dt^2} = -\frac{g}{4a}s,$$

which is a linear differential equation; and of which the general integral is

$$s = \text{A}\cos\left\{\left(\frac{g}{4a}\right)^{\frac{1}{2}}t + \alpha\right\},$$

where A and $\alpha$ are undetermined constants of integration: let $l$ be the initial value of $s$; then since $\frac{ds}{dt}=0$, and $s=l$ when $t=0$; therefore $\alpha=0$, and $\text{A}=l$: so that the complete solution is

$$s = l\cos\left(\frac{g}{4a}\right)^{\frac{1}{2}}t. \tag{28}$$

Also from (26), substituting by means of the equation to the cycloid,

$$\frac{\text{R}}{m} = \frac{g(2a+h-2x)}{\{2a(2a-x)\}^{\frac{1}{2}}},$$

which assigns the pressure on the curve; and if $h=2a$, that is, if $m$ begins to move from the extremity of the base of the cycloid,

$$\frac{\text{R}}{m} = g\left(\frac{4a-2x}{a}\right)^{\frac{1}{2}};$$

and if $x = 0$, that is, at the lowest point of the curve,

$$\text{R} = 2mg,$$

that is, the pressure is twice the weight of the particle.

If the plane in which the cycloid is, and in which $m$ moves, is inclined to the horizon at an angle $\alpha$, and is smooth, the preceding results are applicable, if we replace $g$ by $g \sin \alpha$.

354.] A particle $m$ may have the cycloidal motion of the preceding Article, if we suppose it to move in a smooth tube, or an open canal, bent into the form of a cycloid. We may also obtain a path of the required kind by the following arrangement. Let the particle $m$ be fixed at the end of a perfectly flexible and inextensible string, which we shall assume to be without weight, of length $4a$; and let the upper end of this string be fastened at a point C, see fig. 135, in a vertical line through O, where $\text{OC} = 4a$; from C to B and from C to B′ let two cycloidal arcs be drawn, equal to OB or OB′; then BOB′ is the involute of CB and CB′; and therefore the end of a string of the length $4a$ fastened at C and wrapping round CB and CB′ will describe the cycloid BOB′.

Now when a heavy solid body oscillates about a fixed horizontal axis it is called a *pendulum;* and when a heavy particle is attached to the axis by means of a string or a rod, without weight and inextensible, this arrangement is called a *simple pendulum;* and although such a system never can be perfectly attained, yet approximations may be made to it which are near enough for practical purposes, and by means of it the incidents of pendulums may be compared.

Suppose then the particle $m$ to be fixed at one end of a string, whose length is $4a$, the other end being fastened at C, fig. 135; and suppose the plane of the curves to be vertical; then if CB, CB′ are such surfaces or cheeks that the string may be wrapped round them, $m$ will move in a cycloidal path, and we shall have a cycloidal pendulum. The time of the oscillations will be the same, whatever is the point whence $m$ begins to move; and if $l$ is the length of the string,

$$\text{the time of an oscillation} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}. \qquad (29)$$

Also the tension of the string corresponds to the pressure on the curve of the preceding Article, and we have

$$\text{the tension of the string} = mg\,\frac{2a+h-2x}{\{2a(2a-x)\}^{\frac{1}{2}}}; \qquad (30)$$

and therefore, if $x = 0$,

$$\text{the tension at the lowest point} = \frac{mg}{2a}(2a+h).$$

355.] As another example of cycloidal motion, let us suppose a cycloid to be placed in a vertical plane with its base horizontal, as in fig. 136; and let us suppose $m$ to be projected from the highest point with a given velocity along the convex side of the curve; it is required to determine the subsequent motion of $m$.

Let P, $(x, y)$, be the position of $m$ at the time $t$; let $u = (2gh)^{\frac{1}{2}}$ be the velocity of projection. The equations of motion are,

$$\frac{d^2x}{dt^2} = g - \frac{R}{m}\frac{dy}{ds},$$

$$\frac{d^2y}{dt^2} = \frac{R}{m}\frac{dx}{ds};$$

$$\therefore \quad \frac{ds^2}{dt^2} = v^2 = 2g(h+x),$$

$$\frac{R}{m} = g\frac{dy}{ds} - \frac{v^2}{\rho}$$

$$= \frac{g}{(2a)^{\frac{1}{2}}}\,\frac{2a-h-2x}{(2a-x)^{\frac{1}{2}}}.$$

So long as R has a positive sign, $m$ is in contact with the cycloid; but when $R = 0$, the particle leaves the curve; and being heavy and moving freely describes a parabolic path; this takes place when

$$x = a - \frac{h}{2};$$

in which case $v^2 = 2g\left(a+\frac{h}{2}\right)$; and the line of motion of $m$ at that time makes with a horizontal line an angle whose tangent is $\left(\frac{2a-h}{2a+h}\right)^{\frac{1}{2}}$; and the latus rectum of the subsequent parabolic path is

$$\frac{(2a+h)^2}{2a}.$$

356.] The motion of a heavy particle on a circle.

Let the plane of the circle be vertical, and let its radius be $a$; let it be placed, as in fig. 137, with the origin at the lowest point, and with its axis vertical; so that the equation of the circle is

$$y^2 = 2ax - x^2;$$

$$\therefore \quad \frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a};$$

and the equations of motion are

$$\frac{d^2x}{dt^2} = -g + \frac{\text{R}}{m}\frac{dy}{ds}, \qquad \frac{d^2y}{dt^2} = -\frac{\text{R}}{m}\frac{dx}{ds}.$$

Let the initial position of $m$ be $(h, k)$, and let the velocity of $m$ at that point be $v_0$; then

$$\frac{ds^2}{dt^2} = v^2 = 2g(h-x) + v_0^2. \tag{31}$$

Hence it appears that the velocity is a maximum when $x = 0$, that is, at the lowest point of the curve; and the velocity $= 0$, when

$$x = h + \frac{v_0^2}{2g},$$

subject to the condition that $h + \frac{v_0^2}{2g}$ is not greater than $2a$; and if this is not the case, then the velocity is a minimum when $x$ has its greatest value, that is, at the highest point of the circle.

Also
$$\frac{\text{R}}{m} = g\frac{dy}{ds} + \frac{v^2}{\rho} \tag{32}$$

$$= \frac{g}{a}\{a + 2h - 3x + v_0^2\}, \tag{33}$$

which gives the pressure on the curve. And if the circular motion of $m$ is caused by the fastening of $m$ to a string of length $a$, the other end of which is fixed at the centre of the circle, then R is the tension of the string, and we have,

$$\text{the tension} = \frac{mg}{a}\{a + 2h - 3x + v_0^2\}. \tag{34}$$

If $v_0 = 0$ and $h = 2a$, that is, if the particle moves from rest from A, the highest point of the circle, then, at the lowest point, when $x = 0$,

$$\text{R} = \text{T} = 5mg,$$

and is therefore equal to five times the weight of the particle.

Now to find the time taken by the particle in its descent to the lowest point. From (31), if OP $= s$, so that $s$ decreases as $t$ increases,

$$dt = -\frac{ds}{\{2g(h-x) + v_0^2\}^{\frac{1}{2}}}$$

$$= -\frac{a\,dx}{\{2ax - x^2\}^{\frac{1}{2}}\{2g(h-x) + v_0^2\}^{\frac{1}{2}}}. \tag{35}$$

This element-function does not generally admit of integration. In the case in which

$$h + \frac{v_0^2}{2g} = 2a,$$

that is, in which the particle comes to rest at the highest point of the circle, (35) becomes,

$$dt = -\frac{dx}{(2g)^{\frac{1}{2}}\, x^{\frac{1}{2}}\,(2a-x)}; \tag{36}$$

whence by integration, and observing that $t = 0$ when $x = h$, we have

$$t = -\frac{1}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}} \log \frac{(2a)^{\frac{1}{2}} + x^{\frac{1}{2}}}{(2a)^{\frac{1}{2}} - x^{\frac{1}{2}}} + \frac{1}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}} \log \frac{(2a)^{\frac{1}{2}} + h^{\frac{1}{2}}}{(2a)^{\frac{1}{2}} - h^{\frac{1}{2}}}.$$

357.] Let us however return to the general expression (35) for $dt$; and to simplify it, let us suppose the initial velocity to be zero, and $h$ to be the vertical abscissa to the point whence $m$ begins to move; then

$$dt = -\frac{a\,dx}{(2g)^{\frac{1}{2}}\,(h-x)^{\frac{1}{2}}\,(2ax-x^2)^{\frac{1}{2}}}; \tag{37}$$

this expression does not admit of integration. If however the radius of the circle is large, and the greatest amplitude to which $m$ moves is small, we may expand (37) in a series of terms in ascending powers of $\frac{x}{a}$, and thus approximately find the integral; and it is necessary to have recourse to this method, because the problem is that of a circular pendulum; and in the applications of it to the purposes of time-measuring it is desirable to know the extent of error which arises, if we assume one term of the series to be the measure of the time of descent of $m$ from its highest to its lowest point. Let $\mathrm{T}$ be the time of an oscillation, that is, from $x = h$, through $x = 0$, to $x = h$; then

$$\begin{aligned} \mathrm{T} &= -\left(\frac{a}{g}\right)^{\frac{1}{2}} \int_h^0 \frac{dx}{(hx-x^2)^{\frac{1}{2}}} \left(1 - \frac{x}{2a}\right)^{-\frac{1}{2}} \\ &= \left(\frac{a}{g}\right)^{\frac{1}{2}} \int_0^h \left\{ 1 + \frac{1}{2}\,\frac{x}{2a} + \frac{1.3}{2.4}\left(\frac{x}{2a}\right)^2 + \dots \right. \\ &\qquad \left. \dots + \frac{1.3\dots(2n-1)}{2.4\dots 2n}\left(\frac{x}{2a}\right)^n + \dots \right\} \frac{dx}{(hx-x^2)^{\frac{1}{2}}}. \end{aligned} \tag{38}$$

This series consists of terms each of which is of the form $\int_0^h \frac{x^m\,dx}{(hx-x^2)^{\frac{1}{2}}}$; and by Art. 51, Vol. II,

$$\int_0^h \frac{x^m\,dx}{(hx-x^2)^{\frac{1}{2}}} = \frac{2m-1}{2m}h\int_0^h \frac{x^{m-1}\,dx}{(hx-x^2)^{\frac{1}{2}}}$$

$$= \frac{(2m-1)(2m-3)\dots 5.3.1}{2m(2m-2)\dots 6.4.2}h^m\int_0^h \frac{dx}{(hx-x^2)^{\frac{1}{2}}}$$

$$= \frac{(2m-1)(2m-3)\dots 5.3.1}{2m(2m-2)\dots 6.4.2}h^m\pi;$$

$$\therefore \quad \text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}\left\{1+\left(\frac{1}{2}\right)^2\frac{h}{2a}+\left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2a}\right)^2+\dots \right.$$

$$\left. \dots+\left(\frac{1.3.5\dots(2n-1)}{2.4.6\dots 2n}\right)^2\left(\frac{h}{2a}\right)^n+\dots\right\}, \quad (39)$$

which is the complete expression for the time of an oscillation.

If $h$ is very small in comparison of $a$, the radius of the circle, and if we neglect all powers of $\frac{h}{2a}$,

$$\text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}, \quad (40)$$

which is an expression of the same form as that for the time of an oscillation in the motion of a heavy particle on a cycloid; see Art. 353; observing however that $a$ here is equal to $4a$ in that case. And if we include the first two terms of (39), we have,

$$\text{T} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}\left\{1+\frac{h}{8a}\right\}. \quad (41)$$

If $2\alpha$ is the angle at the centre which measures the oscillation,

$$h = a(1-\cos\alpha) = 2a\left(\sin\frac{\alpha}{2}\right)^2,$$

so that the difference between the expression for T given in (40) and the whole expression varies as the square of the angle of oscillation; and that between (41) and (39) varies as the fourth power of the same angle.

358.] This motion of a heavy particle in a circular arc is approximately realized by attaching the particle to the end of a very thin and light straight rod, which, turning about a fixed point at its other end, moves in a vertical plane. This arrangement in its perfect state, (which however can never be reached,) is called a simple circular pendulum. If $l$ is the length of the rod, the time of an oscillation is approximately given by the formula,

$$\text{T} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}, \quad (42)$$

when the angle of oscillation is very small.

The pressure on the curve becomes in this case the tension of the rod.

The formula (42) is applied to the determination of gravity at the different places of the earth's surface. Let L be the length of a pendulum which vibrates seconds at the place to which $g$ applies; then

$$1 = \pi \left(\frac{\text{L}}{g}\right)^{\frac{1}{2}},$$

$$\therefore \quad g = \pi^2 \text{L}; \tag{43}$$

from this formula $g$ has been calculated at many places on the earth. The method of determining L accurately will be explained in a future part of the treatise.

Equation (42) is also employed for the determination of (1) the height of a mountain, and (2) the depth of a mine.

(1) Let $r$ be the mean-radius of the earth's surface considered spherical; let $h$ be the altitude of the mountain above the surface, and $g$ and $g'$ the values of gravity on the earth's surface and the top of the mountain respectively: then, by Art. 179,

$$\frac{g}{g'} = \left(\frac{r+h}{r}\right)^2.$$

Let $n =$ the number of oscillations which the seconds' pendulum at the top of the mountain makes in 24 hours,

$$\therefore \quad 1 = \pi \left(\frac{\text{L}}{g}\right)^{\frac{1}{2}},$$

$$\frac{24 \times 60 \times 60}{n} = \pi \left(\frac{\text{L}}{g'}\right)^{\frac{1}{2}}$$

$$= \pi \frac{r+h}{r} \left(\frac{\text{L}}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad \frac{h}{r} = \frac{24 \times 60 \times 60}{n} - 1,$$

whereby $h$ is given in terms of $r$, the radius of the earth. Let us, for the sake of an example, suppose the pendulum to "lose 10 seconds in a day;" that is, to make 10 oscillations less than it would make on the surface of the earth. Then

$$n = 24 \times 60 \times 60 - 10,$$

and $r = 4000$ miles (approximately);

$$\therefore \quad h = 4000 \left\{\frac{24 \times 60 \times 60}{24 \times 60 \times 60 - 10} - 1\right\}$$

$$= 4000 \left\{\left(1 - \frac{1}{24 \times 6 \times 60}\right)^{-1} - 1\right\}$$

$$h = \frac{4000}{24 \times 6 \times 60} \text{ approximately}$$

$$= .4626 \text{ of a mile.}$$

(2) Let $r$ be the radius of the earth's surface, as in the last case: and let $h$ be the depth of the mine: let $g$ and $g'$ be the values of gravity on the earth's surface and at the bottom of the mine. Then, by Article 181,

$$\frac{g}{g'} = \frac{r}{r-h}.$$

Let $n =$ the number of oscillations which the seconds' pendulum at the bottom of the mine makes in 24 hours; therefore

$$1 = \pi \left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}},$$

$$\frac{24 \times 60 \times 60}{n} = \pi \left(\frac{\mathrm{L}}{g'}\right)^{\frac{1}{2}}$$

$$= \pi \left(\frac{r}{r-h}\right)^{\frac{1}{2}} \left(\frac{\mathrm{L}}{g}\right)^{\frac{1}{2}}$$

$$= \left(\frac{r}{r-h}\right)^{\frac{1}{2}};$$

$$\therefore \quad \frac{h}{r} = \left\{1 - \frac{n}{24 \times 60 \times 60}\right\};$$

whence $h$ may be found in terms of $r$, the earth's radius.

359.] If the arc through which the *bob* of a circular pendulum vibrates is very small, the time of an oscillation is determined by means of tangential resolution more easily than by the preceding process.

Let $2\alpha$ be the angle which the arc of oscillation subtends at the centre of the circle; let $\theta$ be the angular distance of $m$ from the lowest point at the time $t$; and let $s$ be the arc corresponding to $\theta$, so that, if $a$ is the radius of the circle,

$$s = a\theta; \tag{44}$$

then, if we suppose $s$ to decrease as $t$ increases, the equation of motion along the tangent is,

$$\frac{d^2 s}{dt^2} = -g \sin\theta; \tag{45}$$

and in terms of $\theta$,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{a} \sin\theta. \tag{46}$$

Multiplying by $2\,d\theta$, integrating, and taking the limits such that $\frac{d\theta}{dt} = 0$ when $\theta = a$,

$$\frac{d\theta^2}{dt^2} = \frac{2g}{a}(\cos\theta - \cos a). \tag{47}$$

This expression cannot be again integrated in the form in which it stands: $\cos\theta$ may however be expressed in a series, and the integral may be approximately found. If the oscillations are small, so that $\theta$ and $a$ are small, then expanding $\cos\theta$ and $\cos a$ in powers of $\theta$ and $a$ respectively, and omitting terms containing the fourth and higher powers of these quantities, we have

$$\frac{-d\theta}{(a^2-\theta^2)^{\frac{1}{2}}} = \left(\frac{g}{a}\right)^{\frac{1}{2}} dt;$$

and since $\theta = a$ when $t = 0$,

$$t = \left(\frac{a}{g}\right)^{\frac{1}{2}} \cos^{-1}\frac{\theta}{a};$$

and when $\theta = -a$, $\quad t = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}$,

which is the time of an oscillation.

Using the same notation, the equation of the normal components is, when the particle moves as the *bob* of a pendulum,

$$\begin{aligned} \text{R} &= \text{the tension of the rod} \\ &= m\left\{g\cos\theta + \frac{v^2}{\rho}\right\} \\ &= mg\{3\cos\theta - 2\cos a\}. \end{aligned}$$

If $m$ is projected from the highest point of the circle with a velocity due to the height $h$, and along the concave side of the circle; then if $\theta$ is the angle subtended at the centre of the circle by the arc which $m$ describes in the time $t$,

$$(\text{velocity})^2 = 2g\{h + a - a\cos\theta\},$$

$$\text{R} = mg\left\{\frac{2h}{a} + 2 - 3\cos\theta\right\}.$$

Hence it appears that the particle will leave the curve if $h$ and $\theta$ are such that the following condition is possible; viz.

$$\cos\theta = \frac{2}{3}\,\frac{h+a}{a}.$$

If $\text{R} = 0$ when the motion begins, then $h = \frac{a}{2}$; in which case at the lowest point of the circle, $\text{R} = 6mg$.

360.] Two or three problems are added in other circumstances of constrained motion on curves than those assumed above.

Ex. 1. A parabola is placed with its axis vertical, and its vertex the highest point; a heavy particle is projected from the vertex along the concave side of the curve with a given velocity: it is required to determine the subsequent circumstances of motion.

Let $h$ be the height to which the velocity of projection is due: and let the equation of the parabola be

$$y^2 = 4ax;$$

the equations of motion are,

$$\frac{d^2x}{dt^2} = g + \frac{\text{R}}{m}\frac{dy}{ds}, \qquad \frac{d^2y}{dt^2} = -\frac{\text{R}}{m}\frac{dx}{ds};$$

$$\therefore \quad (\text{velocity})^2 = 2g(x+h),$$

$$\text{R} = m\left\{\frac{v^2}{\rho} - g\frac{dy}{ds}\right\}$$

$$= mga^{\frac{1}{2}}\frac{h-a}{(a+x)^{\frac{3}{2}}}.$$

Therefore the pressure on the curve $= 0$, if $h = a$: that is, if the velocity of projection at the vertex is equal to that acquired in falling from the directrix; and in this case the pressure is zero at all points of the curve: the parabola is therefore the trajectory of $m$ moving freely. This is apparent from the investigations of Article 282.

Ex. 2. A heavy particle descends down a curve in a vertical plane: it is required to determine the equation of it, when the pressure on the curve is the same at all its points.

Let R be the constant pressure: $h$ the altitude to which the velocity of $m$ at the origin is due; $a =$ the angle which the tangent to the curve at the origin makes with the vertical line, which I shall take to be the axis of $x$: then from the equations of motion we have,

$$v^2 = 2g(h+x), \qquad \text{R} = m\left\{\frac{v^2}{\rho} + g\frac{dy}{ds}\right\}.$$

If $s$ is equicrescent, then from equation (18), Art. 236, Vol. I,

$$\frac{1}{\rho} = \frac{d^2y}{ds^2}\frac{ds}{dx};$$

and substituting this in the preceding equations, we have

$$\frac{\text{R}\,dx}{2(h+x)^{\frac{1}{2}}} = mg\left\{(h+x)^{\frac{1}{2}}\,d.\frac{dy}{ds} + \frac{dx}{2(h+x)^{\frac{1}{2}}}\frac{dy}{ds}\right\};$$

therefore integrating, and taking the assigned limits,

$$\text{R}\,\{(h+x)^{\frac{1}{2}} - h^{\frac{1}{2}}\} = mg\,\{(h+x)^{\frac{1}{2}}\frac{dy}{ds} - h^{\frac{1}{2}}\sin\alpha\};$$

$$\therefore\quad \frac{dy}{ds} = \frac{\text{R}}{mg} + \frac{h^{\frac{1}{2}}(mg\sin\alpha - \text{R})}{mg\,(h+x)^{\frac{1}{2}}}; \tag{48}$$

whence, if *ds* is replaced by its equivalent in terms of *dy* and *dx*, the equation to the required curve may be found. It is called the curve of equal pressure.

Ex. 3. A heavy particle *m* moves from rest on a curve, and the pressure on the curve varies as the *n*th power of the vertical distance through which the particle has moved; it is required to determine the nature of the curve.

In this case, $v^2 = 2gx,$

$$kx^n = \frac{2x}{\rho} + \frac{dy}{ds},$$

if *k* is conveniently assumed: and by a process similar to that of the last example, we have

$$\frac{k}{2n+1}\,x^{n+\frac{1}{2}} = x^{\frac{1}{2}}\frac{dy}{ds};$$

whence we have,

$$dy = x^n\left\{\left(\frac{2n+1}{k}\right)^2 - x^{2n}\right\}^{-\frac{1}{2}}dx, \tag{49}$$

which is integrable by rationalization, see Art. 44, Vol. II, whenever $\frac{n+1}{2n}$ or $\frac{1}{2n}$ is an integer.

Let $n=1$; and let $\frac{3}{k} = a$: then (49) becomes after integration,

$$(y \pm a)^2 + x^2 = a^2.$$

If $n = -1$, the resulting equation is that of the catenary.

Ex. 4. To find the equation of a curve, which is such that a heavy particle *m* moving on it may describe a given arc in the same time that it would describe the corresponding chord.

Let us take polar coordinates, and let the origin be the point at which the motion begins: let $(r, \theta)$ be the place of *m* at the time *t*, the prime radius vector being vertical. Then for the motion on the curve we have,

$$\frac{ds^2}{dt^2} = 2gr\cos\theta, \qquad \therefore \quad t = \frac{1}{(2g)^{\frac{1}{2}}}\int_0^r \frac{ds}{(r\cos\theta)^{\frac{1}{2}}};$$

and the time in which $m$ would describe the chord

$$= \left(\frac{2r}{g\cos\theta}\right)^{\frac{1}{2}};$$

$$\therefore \quad \int_0^r \frac{(dr^2+r^2\,d\theta^2)^{\frac{1}{2}}}{(r\cos\theta)^{\frac{1}{2}}} = 2\left(\frac{r}{\cos\theta}\right)^{\frac{1}{2}};$$

and differentiating,

$$\frac{(dr^2+r^2\,d\theta^2)^{\frac{1}{2}}}{(r\cos\theta)^{\frac{1}{2}}} = \left(\frac{\cos\theta}{r}\right)^{\frac{1}{2}}\frac{dr\cos\theta + r\sin\theta\,d\theta}{(\cos\theta)^2},$$

$$\therefore \quad \frac{\cos 2\theta}{\sin 2\theta}\,d\theta = \frac{dr}{r},$$

$$r^2 = a^2\sin 2\theta;$$

which is the equation to a lemniscata, the axis of which is inclined at an angle of 45° to the vertical line through the origin.

361.] I proceed now to the investigation of curves which possess certain general properties, and which are suggested by the preceding inquiry. And first of the general equation of tautochronous curves.

It appears from Article 353 that the cycloid, in either a vertical or an inclined plane, with its base horizontal, is such that the time taken by a heavy particle in moving down the curve to its lowest point is the same whatever is the point on the curve from which the particle begins to move. Our object now is to inquire whether any other curve besides the cycloid possesses this property of tautochronism for heavy particles in vacuo; and we shall also extend the inquiry to forces of other kinds.

Let the impressed forces be resolved tangentially; and let $\mathrm{T}$ be the tangential component at the time $t$ of the impressed velocity-increments, and tending to diminish $s$ as $t$ increases; let $s$ be the arc measured from a certain point on the curve chosen as the origin, and let $a$ be the initial value of $s$: then

$$\frac{ds^2}{dt^2} = 2\int_s^a \mathrm{T}\,ds, \tag{50}$$

so that the time of passage from $a$ to $s$ is given by the following:

$$t = \frac{1}{2^{\frac{1}{2}}}\int_s^a \left\{\int_s^a \mathrm{T}\,ds\right\}^{-\frac{1}{2}} ds. \tag{51}$$

Now this definite integral must be independent of $a$ when $s = a$; it must therefore be of the following form:

$$\int_s^a \left\{\int_s^a \tau \, ds\right\}^{-\frac{1}{2}} ds = \phi\left(\frac{s}{a}\right),$$

$$\therefore \quad -\left\{\int_s^a \tau \, ds\right\}^{-\frac{1}{2}} = \frac{1}{a}\phi'\left(\frac{s}{a}\right);$$

$$\int_s^a \tau \, ds = \left\{\frac{1}{a}\phi'\left(\frac{s}{a}\right)\right\}^{-2};$$

and taking the $s$-differential,

$$-\tau = \frac{d}{ds}\left\{\frac{1}{a}\phi'\left(\frac{s}{a}\right)\right\}^{-2}, \tag{52}$$

which is the relation between $\tau$ and the equation to the tautochronous curve. But the equation to the curve must be independent of $a$; and therefore

$$a^2\left\{\phi'\left(\frac{s}{a}\right)\right\}^{-2}$$

must be independent of $a$; whence we infer that

$$\left\{\phi'\left(\frac{s}{a}\right)\right\}^2 = -\frac{2}{k}\frac{a^2}{s^2},$$

where $k$ is a constant. Substituting this in (52), we have

$$\tau = ks; \tag{53}$$

whence we infer that the tangential force which acts on the particle must vary directly as the length of the path to be described by the particle to the origin. Of the action of a force varying according to this law we have already had examples in Art. 232, where the particle $m$ describes a rectilinear path to the origin, in Art. 239, and in Art. 353.

From (53) and (50) we have

$$\frac{ds^2}{dt^2} = k(a^2 - s^2); \tag{54}$$

let us suppose that the particle is moving towards the origin, and that $T$ is the time from $s = a$ to $s = 0$, then

$$k^{\frac{1}{2}} dt = \frac{-ds}{(a^2 - s^2)^{\frac{1}{2}}},$$

$$\therefore \quad k^{\frac{1}{2}} T = \left[\cos^{-1}\frac{s}{a}\right]_a^0$$

$$= \frac{\pi}{2};$$

$$\therefore \quad \tau = \frac{\pi}{2k^{\frac{1}{2}}}; \quad \text{and} \quad k = \frac{\pi^2}{4\tau^2};$$

so that from (53), $$\mathrm{T} = \frac{\pi^2}{4\tau^2} s. \tag{55}$$

362.] Some examples of tautochronism are subjoined.

Ex. 1. Let the impressed velocity-increment be that of gravity; and let us suppose the axes of coordinates and origin to be taken so that the axis of $x$ is vertical, and $s = 0$ when $x = 0$; then

$$\mathrm{T} = g\frac{dx}{ds};$$

therefore from (55), $$g\frac{dx}{ds} = \frac{\pi^2}{4\tau^2} s,$$

therefore integrating, and replacing the constants by another constant $8a$, which is chosen conveniently,

$$s^2 = 8ax, \qquad s = 2(2ax)^{\frac{1}{2}}, \qquad ds = \frac{(2a)^{\frac{1}{2}} dx}{x^{\frac{1}{2}}};$$

whence $$y = a \operatorname{versin}^{-1}\frac{x}{a} + (2ax - x^2)^{\frac{1}{2}},$$

which is the equation to the cycloid; and which is therefore the only tautochronous curve of a heavy particle in vacuo.

Also since only the length-element of the curve and the *vertical* distance of the initial above the terminal point of the particle's motion are involved in the preceding investigations, the result will be the same if the vertical plane, in which the cycloid is, is wrapped round a vertical cylinder.

Ex. 2. Let the origin and axes be taken as in Ex. 1, and let the force be parallel to the axis of $x$, and $= \mu x$;

$$\therefore \quad \mathrm{T} = \mu x\frac{dx}{ds} = \frac{\pi^2}{4\tau^2} s, \qquad \mu x^2 = \frac{\pi^2}{4\tau^2} s^2;$$

whence we have ultimately,

$$y = \beta x,$$

where $\beta$ is an arbitrary constant: this is the equation of a straight line.

Ex. 3. Let $\mathrm{T} = \mu x^n \dfrac{dx}{ds}$; and let the origin and axes be the same as before;

$$\therefore \quad \mu x^n dx = \frac{\pi^2}{4\tau^2} s\, ds, \qquad x^{n+1} = \frac{(n+1)\pi^2}{8\mu\tau^2} s^2;$$

$$\therefore \quad dy = (-1 + k^2 x^{n-1})^{\frac{1}{2}}\, dx,$$

where $k^2$ is substituted for other constants. Now if $x$ is small, $y$ will be imaginary, unless $n-1$ is negative. Let $n-1 = -m$; then

$$dy = x^{-\frac{m}{2}}(k^2 - x^m)^{\frac{1}{2}}\, dx,$$

which is integrable by rationalization, whenever either $\frac{2-m}{2m}$ or $\frac{1}{m}$ is an integer. Thus let $\frac{2-m}{2m} = 1$; or let $m = \frac{2}{3}$, then

$$dy = x^{-\frac{1}{3}}(k^2 - x^{\frac{2}{3}})^{\frac{1}{2}}\, dx, \qquad \therefore \quad y = -(k^2 - x^{\frac{2}{3}})^{\frac{3}{2}},$$

$$y^{\frac{2}{3}} + x^{\frac{2}{3}} = k^2,$$

which is the equation to an hypocycloid; see Art. 179, Vol. I.

Ex. 4. Suppose the force to be an attracting central force and to vary inversely as the square of the distance; it is required to find the equation to the tautochronous curve.

Let $r$ and $p$ be the radius vector and the perpendicular on the tangent of the path of $m$ at the time $t$: and let the central attracting force be

$$-\frac{\mu}{r^2};$$

$$\therefore \quad \mathrm{T} = \frac{\mu}{r^3}(r^2 - p^2)^{\frac{1}{2}};$$

and since

$$ds = \frac{r\, dr}{(r^2 - p^2)^{\frac{1}{2}}},$$

therefore from (55),

$$d.\frac{\mu}{r^3}(r^2 - p^2)^{\frac{1}{2}} = \frac{\pi^2}{4\tau^2}\,\frac{r\, dr}{(r^2 - p^2)^{\frac{1}{2}}};$$

$$\therefore \quad \frac{\mu}{r^3}(r^2 - p^2)^{\frac{1}{2}}\, d.\frac{\mu}{r^3}(r^2 - p^2)^{\frac{1}{2}} = \frac{\pi^2 \mu}{4\tau^2}\,\frac{dr}{r^2};$$

whence integrating, and introducing the arbitrary constant $c$,

$$\frac{\mu^2}{r^6}(r^2 - p^2) = \frac{\pi^2 \mu}{2\tau^2}\left\{\frac{1}{c} - \frac{1}{r}\right\},$$

$$\therefore \quad p^2 = r^2 - \frac{\pi^2 r^5}{2\mu c \tau^2}(r - c),$$

which is the equation to the tautochronous curve.

363.] In the simple problem of tautochronism under the action of gravity in vacuo, the following process is more easy.

From (25) we have

$$\frac{ds^2}{dt^2} = 2g(h-x); \qquad \therefore \quad t = \frac{1}{(2g)^{\frac{1}{2}}}\int_0^h \frac{ds}{(h-x)^{\frac{1}{2}}};$$

therefore expanding the denominator,

$$t = \frac{1}{(2g)^{\frac{1}{2}}}\int_0^h \Big\{ h^{-\frac{1}{2}} + \frac{1}{2}\, x\, h^{-\frac{3}{2}} + \frac{1.3}{2.4}\, x^2 h^{-\frac{5}{2}} + \dots$$

$$\dots + \frac{1.3\dots(2n-1)}{2.4\dots 2n}\, x^n h^{-(n+\frac{1}{2})} + \dots \Big\}\, ds.$$

But this series must after integration be homogeneous in terms of $x$ and $h$, and of no dimensions, because under this circumstance only will $h$ disappear in the definite integral. Therefore taking the general term,

$$\int x^n\, ds = k\, x^{n+\frac{1}{2}};$$

$$\therefore \quad x^n\, ds = \frac{2n+1}{2}\, k\, x^{n-\frac{1}{2}}\, dx;$$

$$\therefore \quad s = (2n+1)\, k x^{\frac{1}{2}};$$

which is the equation to the cycloid with its base horizontal. The cycloid therefore is the only tautochronous curve for gravity in vacuo.

364.] If there is a family of curves similar and similarly placed, all of which originate at a common point, and if heavy particles move down these curves from the common point, the locus of the points, at which all of them are at the end of a given time, is called the *synchronous curve* of the family of curves. One and the most simple case of this class of curves we have had already in Art. 271, where the circle is shewn to be the synchronous curve of a series of straight lines in either a vertical or an inclined plane originating at a given point.

If the given point at which the curves originate is taken as the origin, and the axis of $x$ is vertical, and that of $y$ horizontal, then $(x, y)$ being the place of $m$ at the time $t$, and $\frac{ds}{dt}$ being its velocity,

$$\frac{ds^2}{dt^2} = 2gx;$$

$$\therefore \quad t = \int_0^x \frac{ds}{(2gx)^{\frac{1}{2}}}; \tag{56}$$

whence may the time be found, when $ds$ is expressed in terms of $x$ by means of the equation of the curve. Now the equation of any one of the curves contains an arbitrary parameter, by the variation of which the several individuals of the family are determined; and by the elimination of this variable parameter between the equation of the curves and (56) will the equation to the synchronous curve be found.

For an example let us take a series of cycloids, placed in a vertical plane with their vertices downwards, a common starting point, and their bases along the same horizontal line: then the equation to them is, $a$ being variable,

$$y = a \operatorname{versin}^{-1} \frac{x}{a} - (2ax - x^2)^{\frac{1}{2}}; \tag{57}$$

and if $\tau$ is the common time down all to the synchronous curve,

$$\tau = \left(\frac{a}{g}\right)^{\frac{1}{2}} \operatorname{versin}^{-1} \frac{x}{a};$$

between which and (57), if $a$ is eliminated, the resulting equation in terms of $x$ and $y$ will represent the required synchronous curve.

365.] We proceed now to a problem of a more interesting character: viz., the general problem of Brachistochronism: the object of which is to determine the nature of the path which a particle under the action of certain given forces takes, when the time of that passage is a minimum: or if the particle moves in a tube, it is required to determine the form of the tube, when the time along it is less than that along any other tube.

In this section I shall suppose the motion to be along a tube, and not on a surface, and if $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are the coordinate-components of the impressed velocity-increments, I shall suppose $\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz$ to be an exact differential: so that $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are functions of $x, y, z$ only. The meaning of this restriction will be explained in the following Chapter. By equation (6), Art. 349, if the velocity at the initial point is zero,

$$v^2 = \frac{ds^2}{dt^2} = 2\int_0^1 (\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz); \tag{58}$$

using the notation of Vol. II to indicate the limits of the definite integral.

The problem evidently requires the Calculus of Variations; and since,

$$\frac{ds}{dt} = v; \qquad \therefore \quad dt = \frac{ds}{v};$$

$$\therefore \quad t = \int_0^1 \frac{ds}{v}; \tag{59}$$

and $t$ is the function of $x, y, z$ (according to the assumption above made) which is to be a minimum: and $v$ is given by the equation (58); and no other general condition is given. Some conditions must be given at the limits; for the curve may be drawn either between given points, or from one given curve to another given curve.

Let $(x_1, y_1, z_1)$ $(x_0, y_0, z_0)$ be the terminal and initial positions of $m$; which, if the points are fixed, do not admit of variation: but if they are on given curves, the variations to which they are subject must be consistent with the equations to the curves. Taking the variation of (59), and equating it to zero, because $t$ is to be a minimum, we have

$$\begin{aligned}\delta t = 0 &= \delta \int_0^1 \frac{ds}{v} \\ &= \int_0^1 \left\{ \frac{\delta . ds}{v} - \frac{ds}{v^2}\,\delta v \right\} \\ &= \int_0^1 \left\{ \frac{1}{v}\left( \frac{dx}{ds}\, d.\delta x + \frac{dy}{ds}\, d.\delta y + \frac{dz}{ds}\, d.\delta z \right) \right. \\ &\qquad\qquad \left. - \frac{ds}{v^3} (\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z) \right\}, \qquad (60)\end{aligned}$$

because from (58),

$$v\,\delta v = \mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z; \qquad (61)$$

and integrating by parts the former part of (60), we have

$$\begin{aligned}0 = &\left[ \frac{1}{v}\left( \frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z \right) \right]_0^1 \\ &- \int_0^1 \left\{ \left( d.\frac{dx}{v\,ds} + \frac{\mathrm{X}\,ds}{v^3} \right) \delta x + \left( d.\frac{dy}{v\,ds} + \frac{\mathrm{Y}\,ds}{v^3} \right) \delta y + \left( d.\frac{dz}{v\,ds} + \frac{\mathrm{Z}\,ds}{v^3} \right) \delta z \right\}; \quad (62)\end{aligned}$$

and each of these, the integrated and the unintegrated parts, must separately be equal to zero. As to the integrated parts, if the limits are fixed points, they admit of no variation, and the expression vanishes identically. If the limits are on given lines, the expression shews that the brachistochron cuts both of them orthogonally.

As to the unintegrated part; since no relation is given between $x$ $y$ and $z$, the coefficients of $\delta x$, $\delta y$, $\delta z$ must separately vanish: and therefore

$$\left.\begin{aligned} d.\frac{dx}{v\,ds} + \frac{\mathrm{X}\,ds}{v^3} &= 0 \\ d.\frac{dy}{v\,ds} + \frac{\mathrm{Y}\,ds}{v^3} &= 0 \\ d.\frac{dz}{v\,ds} + \frac{\mathrm{Z}\,ds}{v^3} &= 0 \end{aligned}\right\}; \qquad (63)$$

and expanding the first terms of these equations, and multiplying by $v$, we have,

$$\left.\begin{aligned} d.\frac{dx}{ds} - \frac{dx}{ds}\frac{dv}{v} + \frac{\text{x}\,ds}{v^2} &= 0 \\ d.\frac{dy}{ds} - \frac{dy}{ds}\frac{dv}{v} + \frac{\text{y}\,ds}{v^2} &= 0 \\ d.\frac{dz}{ds} - \frac{dz}{ds}\frac{dv}{v} + \frac{\text{z}\,ds}{v^2} &= 0 \end{aligned}\right\}. \tag{64}$$

Let $\rho$ be the radius of absolute curvature of the path: then by Art. 325, Vol. I,

$$\frac{ds^2}{\rho^2} = \left(d.\frac{dx}{ds}\right)^2 + \left(d.\frac{dy}{ds}\right)^2 + \left(d.\frac{dz}{ds}\right)^2; \tag{65}$$

also,
$$\frac{dx}{ds}\,d.\frac{dx}{ds} + \frac{dy}{ds}\,d.\frac{dy}{ds} + \frac{dz}{ds}\,d.\frac{dz}{ds} = 0.$$

Therefore multiplying the equations (64) severally by $d.\frac{dx}{ds}$, $d.\frac{dy}{ds}$, $d.\frac{dz}{ds}$, adding and substituting, we have

$$\frac{ds^2}{\rho^2} + \frac{ds}{v^2}\left\{\text{x}\,d.\frac{dx}{ds} + \text{y}\,d.\frac{dy}{ds} + \text{z}\,d.\frac{dz}{ds}\right\} = 0. \tag{66}$$

But if $\lambda, \mu, \nu$ are the direction-angles of the radius of absolute curvature, or, which is the same thing, of the principal normal, Art. 326, Vol. I,

$$\cos\lambda = \frac{\rho}{ds}\,d.\frac{dx}{ds}, \quad \cos\mu = \frac{\rho}{ds}\,d.\frac{dy}{ds}, \quad \cos\nu = \frac{\rho}{ds}\,d.\frac{dz}{ds}:$$

so that (66) becomes

$$\frac{v^2}{\rho} + (\text{x}\cos\lambda + \text{y}\cos\mu + \text{z}\cos\nu) = 0; \tag{67}$$

and therefore, in absolute magnitude, the centrifugal force at every point on the brachistochron, is equal to the resolved part of the impressed velocity-increment along the radius of absolute curvature. This is a general property of brachistochronous curves, and is one by which the path may in many cases be found when the laws of the acting forces are given.

If the brachistochron is a plane curve, the centrifugal force is equal to the normal component of the impressed forces.

This general property of the unconstrained brachistochron was discovered by Euler.

366.] We will now consider some particular cases of brachistochronous curves.

(1) Suppose the velocity of the particle to be constant; so that $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$: then also $dv = 0$, and from (64) we have,

$$d.\frac{dx}{ds} = 0, \qquad d.\frac{dy}{ds} = 0, \qquad d.\frac{dz}{ds} = 0;$$

$$\therefore \quad \frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} = \frac{z-z_0}{z_1-z_0};$$

which are the equations to a straight line.

(2) Let gravity be the only acting force; and let its line of action be parallel to the axis of $z$; then $\mathrm{X} = \mathrm{Y} = 0$, $\mathrm{Z} = g$; therefore also $v^2 = 2g(z-z_0)$.

And following a process parallel to that of the last Article, equations (63) become,

$$d.\frac{dx}{v\,ds} = 0, \qquad d.\frac{dy}{v\,ds} = 0; \qquad d.\frac{dz}{v\,ds} + \frac{g\,ds}{v^3} = 0;$$

and integrating the first two,

$$\frac{dx}{v\,ds} = \alpha, \qquad \frac{dy}{v\,ds} = \beta;$$

$$\therefore \quad \frac{dx}{\alpha} = \frac{dy}{\beta};$$

and therefore the motion takes place in a plane perpendicular to that of $xy$; let the plane be that of $xz$; so that all the terms involving $y$ disappear, and we have

$$d.\frac{dx}{v\,ds} = 0, \qquad d.\frac{dz}{v\,ds} + g\frac{ds}{v^3} = 0;$$

from the first of these

$$\frac{dx}{v\,ds} = \text{a constant} = \frac{1}{(4ga)^{\frac{1}{2}}}\ \text{(say)},$$

where $a$ is an arbitrary constant:

$$\therefore \quad \frac{dx}{ds} = \left(\frac{z-z_0}{2a}\right)^{\frac{1}{2}}; \tag{68}$$

so that $\frac{dx}{ds} = 0$, when $z = z_0$; that is, the curve is vertical at the point $(x_0, z_0)$ of departure of $m$.

And since $\qquad ds^2 = dx^2 + dz^2,$

we have

$$dx = \frac{(z-z_0)\,dz}{\{2a(z-z_0)-(z-z_0)^2\}^{\frac{1}{2}}};$$

$$x - x_0 = a \operatorname{versin}^{-1}\frac{z-z_0}{a} - \{2a(z-z_0)-(z_0-z_0)^2\}^{\frac{1}{2}};$$

which is the equation to a cycloid, whose base is horizontal, and starting point is $(x_0, z_0)$. Let $(x_0, z_0)$ be the origin; then the equation becomes,

$$x = a \operatorname{versin}^{-1}\frac{z}{a} - (2az - z^2)^{\frac{1}{2}}; \qquad (69)$$

$a$ is still undetermined. It is however to be such that the point $(x_n, z_n)$ may be on the curve, so that for its determination we have,

$$x_n = a \operatorname{versin}^{-1}\frac{z_n}{a} - (2az_n - z_n^2)^{\frac{1}{2}}.$$

(3) To find the brachistochron when the force is a central attracting force, and varies inversely as the square of the distance.

In this case
$$v^2 = 2\mu\left(\frac{1}{r} - \frac{1}{r_0}\right),$$
$$\rho = \frac{r\,dr}{dp};$$

and the normal component of the impressed force $= \frac{\mu}{r^2}\frac{p}{r}$.

So that by the general property (67) we have

$$\frac{2\,dp}{p} + \frac{r_0\,dr}{r(r_0 - r)} = 0;$$

$$\therefore \quad \log\frac{p^2}{c^2} + \log\frac{r}{r_0 - r} = 0;$$

$$\therefore \quad p^2 = c^2\frac{r_0 - r}{r};$$

where $c^2$ is an undetermined constant.

367.] Thus far the tubes or curves on which the particle has been constrained to move have been fixed; the tube however may move in the time during which the particle moves in it, so that the actual motion of the particle in space will be compounded of the motion of the tube, and also of its own motion in the tube. I proceed to consider some simple cases of motion of this complex kind. And I shall suppose the magnitudes of the particle and the tube to be such that the particle just fills the smooth tube.

The principle of solution is the same in all cases; the reaction of the tube will be along the normal, and the motion along the tube will be the effect of the impressed velocity-increments resolved along the tangent to the tube.

Ex. 1. A tube bent into the form of a plane curve is attached to a vertical axis which it meets, and about which it revolves with an uniform angular velocity: it is required to determine the form of the tube, when a heavy particle placed in it remains at rest in all parts of the tube.

Let the vertical line be the axis of $z$, and let the point where the tube meets it be the origin. Let $(x, y, z)$ be the place of $m$ at the time $t$; let $x^2+y^2=r^2$, $\theta=$ the angle between $r$ and the plane of $xz$; $\omega=$ the constant angular velocity; so that

$$\frac{d\theta}{dt} = 0, \qquad \theta = \omega t,$$

if $\theta$ and $t$ simultaneously are equal to zero.

The impressed velocity-increments on $m$ are (1) gravity, (2) the centrifugal force due to the rotation of the tube about the vertical axis: resolving these along the tangent to the tube at the point $(r, z)$, we have

$$\frac{d^2 s}{dt^2} = \omega^2 r \frac{dr}{ds} - g \frac{dz}{ds}; \tag{70}$$

and as the particle is to be at rest, $\frac{d^2 s}{dt^2} = 0$; therefore

$$\omega^2 r\, dr = g\, dz, \qquad r^2 = \frac{2g}{\omega^2} z,$$

since $r=0$ when $z=0$; and this is the equation of a parabola, of which the latus rectum is $\frac{2g}{\omega^2}$.

Since also $\frac{d^2 s}{dt^2} = 0$, when the particle moves with a constant velocity, the preceding solution is applicable, when the particle moves along the tube with a constant velocity.

Also if the velocity of the particle is a function of the path which it has passed over, equation (70) may be integrated. Thus suppose $v^2 = k^2(s^2-a^2)$, then

$$k^2(s^2-a^2) = \omega^2 r^2 - 2gz + \mathrm{C};$$

where C is undetermined.

Ex. 2. To determine the motion of a heavy particle placed in a rectilinear tube which is attached to a vertical axis about which it revolves with a given angular velocity.

Let $\alpha$ be the angle at which the tube is inclined to the vertical axis, and let $\omega$ be the constant angular velocity: let the vertex of the cone described by the tube be the origin; $(x, y, z)$ the place of $m$ at the time $t$; let $r$ be the distance of $m$ from the vertex of the cone: so that the centrifugal force at the time $t$ is $\omega^2 r \sin\alpha$; then taking the components of the velocity-increments along the tube, we have

$$\frac{d^2 r}{dt^2} = \omega^2 r (\sin\alpha)^2 - g\cos\alpha.$$

Multiplying these by $2\frac{dr}{dt}$, and integrating on the supposition that $\frac{dr}{dt} = u$ when $r = 0$, we have

$$\frac{dr^2}{dt^2} - u^2 = \omega^2 r^2 (\sin a)^2 - 2 g r \cos a;$$

whence the final integral may easily be found.

If the tube revolves with an uniform velocity $\omega$ in a horizontal plane, we have

$$\frac{d^2 r}{dt^2} = \omega^2 r, \qquad \frac{dr^2}{dt^2} = \omega^2 (r^2 - a^2),$$

if $\frac{dr}{dt} = 0$, when $r = a$. Hence we have

$$r = \frac{a}{2}\{e^{\omega t} + e^{-\omega t}\}.$$

Ex. 3. To determine the motion of a particle placed inside a circular tube which revolves uniformly in a horizontal plane about a vertical axis.

Let us suppose the particle to be initially at the extremity of the diameter of the circle, the other end of which is the fixed point of the circle through which the axis passes. Let $\omega$ be the angular velocity of the circle. Let, fig. 138, OC $=$ CA $= a$; OA being the diameter which is, when $t = 0$, coincident with OX: thus XOA $= \omega t$. Let P be the place of $m$ at the time $t$; OP $= r$, POC $=$ CPO $= \theta$; therefore PCA $= 2\theta$. Now this is a case of relative motion, and thus we may, as no other forces act, consider the motion of the ring with reference to $m$: and the angular velocity of the ring about O being $\omega$, the centrifugal force at P along OP is $\omega^2 r$; the component of this along the curve $= -\omega^2 r \sin\theta$; therefore, if AP $= s$,

$$\frac{d^2 s}{dt^2} = -\omega^2 r \sin\theta;$$

But $s = 2a\theta$, and $r = 2a\cos\theta$,

$$\therefore \quad 2a\frac{d^2\theta}{dt^2} = -2a\omega^2 \sin\theta\cos\theta,$$

$$d.\frac{d\theta^2}{dt^2} = -\omega^2 \sin 2\theta \, d\theta;$$

therefore integrating, and observing that $\theta = 0$, $\frac{d\theta}{dt} = \omega$, when $t = 0$, we have

$$\frac{d\theta^2}{dt^2} - \omega^2 = \frac{\omega^2}{2}\cos 2\theta - \frac{\omega^2}{2},$$

$$\therefore \quad \frac{d\theta^2}{dt^2} = \frac{\omega^2}{2}(1+\cos 2\theta)$$
$$= \omega^2 (\cos\theta)^2,$$
$$\frac{d\theta}{\cos\theta} = \omega\, dt,$$
$$\log\frac{1+\sin\theta}{1-\sin\theta} = 2\omega t, \qquad \therefore \quad \sin\theta = \frac{e^{\omega t}-e^{-\omega t}}{e^{\omega t}+e^{-\omega t}}.$$

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## Section 2.—*Motion of particles constrained to move on a given curved surface.*

368.] The particle which has been the subject of motion in the preceding Section has been constrained to move in a given tube: we proceed now to investigate the equations of motion and the results of these equations when the particle is constrained to be in contact with a given surface, but is free to describe on the surface such a path as is compatible with the forces to which it is subject.

Let the equation to the surface be

$$F(x,y,z) = c; \tag{71}$$

and let $(x,y,z)$ be the place of $m$ at the time $t$. Let $U, V, W$ express the several partial derived-functions of $F$, as in Article 346, Vol. I; and let $Q^2 = U^2+V^2+W^2$, as in the same Article; let $X, Y, Z$ be the components along the coordinate axes of the impressed velocity-increments; let $R =$ the normal pressure of the surface on $m$ at the time $t$: the direction-cosines of the line of action of $R$ are

$$\frac{U}{Q}, \quad \frac{V}{Q}, \quad \frac{W}{Q}; \tag{72}$$

so that the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= X \pm \frac{R}{m}\frac{U}{Q} \\ \frac{d^2y}{dt^2} &= Y \pm \frac{R}{m}\frac{V}{Q} \\ \frac{d^2z}{dt^2} &= Z \pm \frac{R}{m}\frac{W}{Q} \end{aligned}\right\}; \tag{73}$$

if we eliminate $R$ from these equations taken two and two together, three equations will result, which together with (71) will determine $x, y, z$ in terms of $t$.

Let us, as heretofore, suppose in all cases that $\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz$ is an exact differential; and let us multiply equations (73) severally by $2\,dx$, $2\,dy$, $2\,dz$, and add: then, remembering that $\mathrm{U}\,dx+\mathrm{V}\,dy+\mathrm{W}\,dz=0$, we have

$$\frac{2\,dx\,d^2x+2\,dy\,d^2y+2\,dz\,d^2z}{dt^2}=2\,\{\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz\};$$

and integrating between limits corresponding to $t=t$ and to $t=t_0$, we have

$$v^2-v_0^2=2\int_{t_0}^{t}\{\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz\}, \tag{74}$$

if $v_0$ is the velocity of $m$ when $t=t_0$.

From this equation the time taken by $m$ in passing from one to another point on the surface may be found, as in equation (8), Art. 349.

Again, multiplying equations (73) severally by $\frac{\mathrm{U}}{\mathrm{Q}}, \frac{\mathrm{V}}{\mathrm{Q}}, \frac{\mathrm{W}}{\mathrm{Q}}$, and adding, we have

$$\frac{\mathrm{U}\,d^2x+\mathrm{V}\,d^2y+\mathrm{W}\,d^2z}{\mathrm{Q}\,dt^2}=\frac{\mathrm{XU}+\mathrm{YV}+\mathrm{ZW}}{\mathrm{Q}}\pm\frac{\mathrm{R}}{m}, \tag{75}$$

whereby the pressure on the surface is known.

If the path which the particle is taking at $(x, y, z)$ is a principal normal section of the surface at that point; and if $\rho$ is the length of the corresponding principal radius of curvature, then by (12), Article 347, Vol. I,

$$\rho=\frac{\mathrm{Q}\,ds^2}{\mathrm{U}\,d^2x+\mathrm{V}\,d^2y+\mathrm{W}\,d^2z}; \tag{76}$$

so that (75) becomes

$$\pm\,\mathrm{R}=m\left\{\frac{\mathrm{XU}+\mathrm{YV}+\mathrm{ZW}}{\mathrm{Q}}\pm\frac{v^2}{\rho}\right\}, \tag{77}$$

that is, the pressure on the curve is the algebraic sum of the normal components at that point of the impressed momentum-increments and of the centrifugal force.

If the particle moves on the surface, and is under the action of no impressed momentum-increments, so that $\mathrm{X}=\mathrm{Y}=\mathrm{Z}=0$, then equations (73) give,

$$\frac{\;\frac{d^2x}{dt^2}\;}{\mathrm{U}}=\frac{\;\frac{d^2y}{dt^2}\;}{\mathrm{V}}=\frac{\;\frac{d^2z}{dt^2}\;}{\mathrm{W}}=\pm\frac{\mathrm{R}}{m\mathrm{Q}}; \tag{78}$$

but the velocity of $m$ is constant; and therefore the numerators of (78) are proportional to $\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}$; and therefore we have, if $s$ is not equicrescent,

$$\frac{d.\frac{dx}{ds}}{\text{U}} = \frac{d.\frac{dy}{ds}}{\text{V}} = \frac{d.\frac{dz}{ds}}{\text{W}}, \tag{79}$$

which are the equations to a geodesic line on the surface; see Art. 226, Vol. II; the path therefore of $m$ is a geodesic line.

369.] In this and the following Articles I propose to apply the preceding principles to the motion of a heavy particle on a sphere; and, to fix our thoughts, I shall suppose the particle to move on the *inside* of the sphere, and I shall take, as in fig. 139, the horizontal plane through the centre of the sphere to be the plane of $xy$, and the axis of $z$ to be vertical downwards through the centre. Then if $a$ is the radius of the sphere,

$$x^2+y^2+z^2 = a^2; \tag{80}$$

so that the equations of motion become,

$$\frac{d^2x}{dt^2} = -\frac{\text{R}}{m}\frac{x}{a}, \quad \frac{d^2y}{dt^2} = -\frac{\text{R}}{m}\frac{y}{a}, \quad \frac{d^2z}{dt^2} = g-\frac{\text{R}}{m}\frac{z}{a}. \tag{81}$$

Multiplying these respectively by $2\,dx$, $2\,dy$, $2\,dz$, adding and integrating,

$$\begin{aligned} \frac{ds^2}{dt^2} = v^2 &= v_0{}^2+2g\,(z-z_0) \\ &= c+2g\,z \text{ (say)}, \end{aligned} \tag{82}$$

where $c = v_0{}^2-2g\,z_0$; $v_0$ and $z_0$ being the values of $v$ and of $z$, when $t=0$.

Also from the first two of (81),

$$x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2} = 0;$$

$$\therefore \quad x\,dy-y\,dx = k\,dt, \tag{83}$$

where $k$ is an arbitrary constant. Let the angle $\text{NOM} = \phi$, $\text{ON} = \rho$, so that $x = \rho\cos\phi$, $y = \rho\sin\phi$; therefore, as in Art. 307,

$$\rho^2\,d\phi = k\,dt; \tag{84}$$

and therefore $\rho$ describes equal sectorial areas in equal times.

To find an expression for the time in terms of $z$; from the equation to the sphere we have

$$\begin{aligned} x\,dx+y\,dy &= -z\,dz; \\ \text{also} \qquad x\,dy-y\,dx &= k\,dt; \\ \therefore \quad (x^2+y^2)\,dx &= -xz\,dz-ky\,dt, \\ (x^2+y^2)\,dy &= -yz\,dz+kx\,dt; \\ \therefore \quad (dx^2+dy^2)(x^2+y^2) &= z^2dz^2+k^2dt^2; \end{aligned}$$

$$\therefore \quad \frac{ds^2}{dt^2} = c + 2gz = \frac{dx^2 + dy^2 + dz^2}{dt^2}$$

$$= \frac{a^2\,dz^2 + k^2\,dt^2}{(a^2 - z^2)\,dt^2};$$

$$\therefore \quad dt = \frac{a\,dz}{\{(a^2 - z^2)(c + 2gz) - k^2\}^{\frac{1}{2}}}; \tag{85}$$

whence might the time be found in terms of $z$, if the expression were integrable.

Also since $\rho^2 = a^2 - z^2$, from (84) we have,

$$d\phi = \frac{ka\,dz}{(a^2 - z^2)\{(a^2 - z^2)(c + 2gz) - k^2\}^{\frac{1}{2}}}; \tag{86}$$

which expression does not admit of integration in a finite form. (85) will give the time taken by the particle in passing from $z = z_0$ to $z = z$; and (86) will give the curve described by $m$ on the spherical surface, which will be a kind of spherical spiral. If we equate to zero $\frac{dz}{d\phi}$, we shall have the values of $z$, which render $z$ a maximum or a minimum if there is a change of sign, and to which in all cases corresponds a horizontal motion of $m$. And since $(a^2 - z^2)(c + 2gz) - k^2 = 0$ is a cubic equation, it has always one real root; and as a factor of the first degree will correspond to this, so will the curve always be such that $z$ will have a maximum or minimum value.

Equations (85) and (86) may also be reduced to elliptic functions, and their properties may be studied in that relation; but it is beside our purpose to proceed further with the inquiry in that direction.

The constant $c$ is known in terms of the initial velocity and of the $z$-ordinate of the initial position of $m$. As to $k$; let $\rho_0, v_0$ be the initial values of $\rho, v$; and suppose the line of $v_0$ to make an angle $a$ with the parallel of latitude at the initial point; then the component of the velocity along that parallel of latitude is $v_0 \cos a$; and $\rho_0$ is the radius of that parallel of latitude; therefore $\rho_0 v_0 \cos a$ is twice the sectorial area described in one unit of time by $\rho_0$ on the horizontal plane: and from (84) this quantity is equal to $k$, therefore

$$k = \rho_0 v_0 \cos a.$$

As to the normal pressure on the surface; multiplying equations (81) severally by $x, y, z$, and adding, we have

$$\frac{x\,d^2x+y\,d^2y+z\,d^2z}{dt^2} = gz-\frac{\mathrm{R}\,a}{m};$$

and from the equation to the sphere,

$$x\,dx+y\,dy+z\,dz = 0,$$

$$\begin{aligned}\therefore\quad x\,d^2x+y\,d^2y+z\,d^2z &= -(dx^2+dy^2+dz^2)\\ &= -ds^2\\ &= -v^2\,dt^2;\end{aligned}$$

$$\therefore\quad \mathrm{R} = \frac{m}{a}(v^2+gz). \tag{87}$$

$\frac{v^2}{a}$ is the centrifugal force of the path described by $m$, and as $\frac{z}{a}$ is the cosine of the angle between the radius of the sphere and the line of action of $g$, $\frac{mgz}{a}$ is the normal component of the weight of $m$; so that the pressure on the surface is equal to the centrifugal force, and to the normal component of the weight of $m$.

370.] The motion on a sphere can of course be effected by means of a heavy *bob* or mass $m$ attached by a string or thin rod of a given length $a$ to a point about which it can turn in all directions; and thus the preceding investigations become of importance, because they are those of the motion of a spherical pendulum: and although the expressions do not generally admit of integration, yet when the distances of $m$ from the vertical line do not exceed a small quantity, we can expand in ascending powers of that small quantity, and obtain results which are approximately exact.

Let $\mathrm{COP} = \theta$, and let $\alpha$ be the initial value of $\theta$; $\alpha$ and $\theta$ being always so small that we shall omit all powers of them above the second; and let us suppose the initial velocity $v_0$ to be in a horizontal line, so that $k = \rho_0 v_0$; therefore

$$\begin{aligned} z &= a\cos\theta & z_0 &= a\cos\alpha\\ &= a-\frac{a\theta^2}{2} & &= a-\frac{a\alpha^2}{2},\end{aligned}$$

$$c = v_0^2-2ga+ga\alpha^2,$$

$$\begin{aligned}k^2 &= \rho_0^2 v_0^2 = (a^2-z_0^2)\,v_0^2\\ &= a^2\alpha^2 v_0^2;\end{aligned}$$

so that (85) and (86) become, after simplification by putting

$$\frac{v_0^2}{ga} = \beta^2, \tag{88}$$

$$dt = \left(\frac{a}{g}\right)^{\frac{1}{2}} \frac{-\theta\, d\theta}{\{(\alpha^2-\theta^2)(\theta^2-\beta^2)\}^{\frac{1}{2}}}, \tag{89}$$

$$d\phi = \frac{-\alpha\beta\, d\theta}{\theta\{(\alpha^2-\theta^2)(\theta^2-\beta^2)\}^{\frac{1}{2}}}; \tag{90}$$

from these it appears that $\theta$ must always be intermediate to $\alpha$ and $\beta$; and therefore if $\beta = \alpha$, $\theta$ is always equal to $\alpha$; the pendulum, that is, describes a circular cone, of which $\alpha$ is the semi-vertical angle, and the bob moves in a circle; and dividing (89) by (90), we have in this case,

$$dt = \left(\frac{a}{g}\right)^{\frac{1}{2}} d\phi;$$

and if $\phi = 0$ when $t = 0$, the time of a revolution is

$$2\pi\left(\frac{a}{g}\right)^{\frac{1}{2}}, \tag{91}$$

which is twice the time, see (40), Art. 357, of an oscillation, when the pendulum vibrates in one vertical plane. If therefore two pendulums of the same length $a$ start simultaneously from the same line OA, which is inclined to the vertical at the angle $\alpha$, the one from rest, the other with a velocity equal to $\alpha(ga)^{\frac{1}{2}}$ in a line perpendicular to the vertical plane containing OA, both will return again simultaneously to the same line OA, but the former will have already reached it once in the mean time.

Let however $\alpha$ and $\beta$ be unequal: then from (89) we have

$$dt = \left(\frac{a}{g}\right)^{\frac{1}{2}} \frac{-\theta\, d\theta}{\left\{\left(\frac{\alpha^2-\beta^2}{2}\right)^2 - \left(\theta^2 - \frac{\alpha^2+\beta^2}{2}\right)^2\right\}^{\frac{1}{2}}};$$

whence integrating, and observing that $\theta = \alpha$ when $t = 0$,

$$t = \frac{1}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}} \cos^{-1}\frac{2\theta^2-\alpha^2-\beta^2}{\alpha^2-\beta^2}; \tag{92}$$

$$\therefore\quad \theta^2 = \frac{\alpha^2+\beta^2}{2} + \frac{\alpha^2-\beta^2}{2}\cos 2t\left(\frac{g}{a}\right)^{\frac{1}{2}} \tag{93}$$

$$= \alpha^2\left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2 + \beta^2\left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2. \tag{94}$$

From (93) it appears that $\theta^2$ is periodic, and that its greatest and least values are $\alpha^2$ and $\beta^2$; the time elapsing between $\theta = \alpha$ and $\theta = \beta$ is

$$\frac{\pi}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

For the azimuthal motion of the vertical plane which contains the pendulum we have, from (89) and (90),

$$\begin{aligned} d\phi &= \left(\frac{g}{a}\right)^{\frac{1}{2}} \frac{\alpha\beta\, dt}{\theta^2} \\ &= \left(\frac{g}{a}\right)^{\frac{1}{2}} \frac{\alpha\beta\, dt}{\alpha^2 \left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2 + \beta^2 \left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}; \end{aligned}$$

integrating which, and supposing that $\phi = 0$ when $t = 0$, we have

$$\alpha \tan\phi = \beta \tan t\left(\frac{g}{a}\right)^{\frac{1}{2}}; \tag{95}$$

$\phi$ therefore does not vary directly as the time, as is the case when $\beta = \alpha$; but the plane revolves through 90°, during a time which is equal to

$$\frac{\pi}{2}\left(\frac{a}{g}\right)^{\frac{1}{2}}.$$

From (95) we have

$$(\cos\phi)^2 = \frac{\alpha^2 \left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}{\alpha^2 \left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2 + \beta^2 \left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}, \tag{96}$$

$$(\sin\phi)^2 = \frac{\beta^2 \left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}{\alpha^2 \left\{\cos t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2 + \beta^2 \left\{\sin t\left(\frac{g}{a}\right)^{\frac{1}{2}}\right\}^2}. \tag{97}$$

Also if $x$ and $y$ refer to the place of $m$ at the time $t$, we have

$$\left.\begin{aligned} x^2 &= (a^2-z^2)(\cos\phi)^2 = a^2\theta^2(\cos\phi)^2 \\ y^2 &= (a^2-z^2)(\sin\phi)^2 = a^2\theta^2(\sin\phi)^2 \end{aligned}\right\}; \tag{98}$$

therefore from (96) (97) and (98),

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = a^2; \tag{99}$$

which is the equation of an ellipse, of which the semi-axes are $a\alpha$ and $a\beta$. Hence we infer that the projection on a horizontal plane of the path described by the bob of the pendulum is an ellipse, whose centre is in a vertical line drawn through the centre of suspension, and one of whose principal axes lies in the vertical plane perpendicular to the line along which $m$ is at first projected. Now an ellipse may be described by a particle moving under the attraction of a central force situated in the centre of the ellipse and which varies directly as the distance; we may therefore suppose the bob of the pendulum to move under the action of such a central force. Let $\mu$ be the absolute central force; then by Art. 310,

$$\text{the periodic time} = \frac{2\pi}{\mu^{\frac{1}{2}}};$$

but from above, the periodic time $= 2\pi \left(\frac{a}{g}\right)^{\frac{1}{2}}$,

$$\therefore \quad \mu = \frac{g}{a};$$

so that if $P$ is the central force, $P = \frac{g}{a}\rho$.

And this is the resolved part of the tension of the rod along the direction of $\rho$; as may thus be shewn: if $R =$ the tension, then the resolved part of $R$ along $\rho$

$$\begin{aligned} &= \frac{R}{m}\sin\theta \\ &= \frac{v^2+gz}{a}\sin\theta, \text{ by reason of (87),} \\ &= \left(1+\beta^2+\alpha^2-\frac{3\theta^2}{2}\right) g\theta; \end{aligned}$$

and omitting cubes of small quantities, if $P =$ this resolved part,

$$P = g\theta;$$

but from (98), $x^2+y^2 = \rho^2 = a^2\theta^2$,

$$\therefore \quad P = \frac{g}{a}\rho;$$

that is, varies directly as $\rho$.

In the Mécanique Analytique, second part, Sect. VIII, Ch. II, § I, the approximations are carried on so as to involve powers of $\alpha$ and $\beta$ higher than those which we have taken account of; and M. Bravais, after correcting some errors of M. Lagrange, shews that the angle between two successive points corresponding to $\theta = \alpha$ and to $\theta = \beta$ is

$$\frac{\pi}{2}\left(1+\frac{3\alpha\beta}{8}\right),$$

instead of being $\frac{\pi}{2}$, as it is in the preceding investigations.

371.] As another example let us consider the motion of a material particle on a surface of revolution, and acted on by forces in a plane passing through the axis. Let the axis of revolution of the surface be the coordinate axis of $z$; and let $x^2+y^2 = \rho^2$; so that the equation to the surface is

$$\rho^2 = x^2+y^2 = f(z);$$

$$\therefore \quad \text{F}(x, y, z) = x^2 + y^2 - f(z) = 0; \tag{100}$$

$$\therefore \quad \text{U} = 2x, \qquad \text{V} = 2y, \qquad \text{W} = -f'(z);$$

$$\therefore \quad \text{Q}^2 = 4\rho^2 + \{f'(z)\}^2.$$

Since the impressed forces act in a meridian plane only, let them be resolved parallel to the axis of $z$, and perpendicular to that axis; of which let the components be severally Z and P; then of P's line of action the $x$- and $y$-direction-cosines are $\frac{x}{\rho}$, $\frac{y}{\rho}$; and let R, the pressure on the surface, be resolved in a similar way: so that its components are

$$\text{R}\frac{dz}{ds}\frac{x}{\rho}, \qquad \text{R}\frac{dz}{ds}\frac{y}{\rho}, \qquad \text{R}\frac{d\rho}{ds},$$

where $ds^2 = d\rho^2 + dz^2$; and the equations of motion are,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \text{P}\frac{x}{\rho} + \frac{\text{R}}{m}\frac{dz}{ds}\frac{x}{\rho} = \left(\text{P} + \frac{\text{R}}{m}\frac{dz}{ds}\right)\frac{x}{\rho} \\ \frac{d^2y}{dt^2} &= \text{P}\frac{y}{\rho} + \frac{\text{R}}{m}\frac{dz}{ds}\frac{y}{\rho} = \left(\text{P} + \frac{\text{R}}{m}\frac{dz}{ds}\right)\frac{y}{\rho} \\ \frac{d^2z}{dt^2} &= \text{Z} + \frac{\text{R}}{m}\frac{d\rho}{ds} \end{aligned}\right\}. \tag{101}$$

From the first two we have

$$x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2} = 0,$$

$$\therefore \quad x\,dy - y\,dx = h\,dt; \tag{102}$$

so that the projections on a plane perpendicular to the axis of revolution of the sectorial areas described by $\rho$ vary as the times in which they are described.

Again,

$$\begin{aligned} \frac{2\,dx\,d^2x + 2\,dy\,d^2y + 2\,dz\,d^2z}{dt^2} &= 2\text{P}\frac{x\,dx + y\,dy}{\rho} + 2\text{Z}\,dz \\ &= 2\text{P}\,d\rho + 2\text{Z}\,dz; \end{aligned}$$

$$\therefore \quad \frac{ds^2}{dt^2} = v^2 = c - 2\int_{t_0}^{t}\{\text{P}\,d\rho + \text{Z}\,dz\}; \tag{103}$$

where $c$ is a constant depending on the initial velocity and the initial coordinates of $m$. Without carrying these general investigations farther, I shall apply them to a few examples.

**372.]** To determine the circumstances under which a heavy particle will describe a parallel of latitude when the axis of revolution of the surface is vertical, and the velocity of projection is a function of the coordinates of the point of projection.

In this case $\mathrm{P}=0$, $\mathrm{Z}=-g$; the particle is projected in a horizontal plane, and along the tangent to the parallel of latitude. Then the equations become

$$\frac{d^2x}{dt^2} = \frac{\mathrm{R}}{m}\frac{dz}{ds}\frac{x}{\rho}, \qquad \frac{d^2y}{dt^2} = \frac{\mathrm{R}}{m}\frac{dz}{ds}\frac{y}{\rho},$$

$$\frac{d^2z}{dt^2} = -g + \frac{\mathrm{R}}{m}\frac{d\rho}{ds};$$

$$\therefore \quad v^2 = v_0^2 + 2g(z_0 - z);$$

where $v_0$ and $z_0$ are the initial values of $v$ and $z$; but $z = z_0$, because the particle describes a parallel of latitude; therefore the velocity is constant.

Let us consider the equation to the meridian curve as it is in the plane of $xz$: and let the initial velocity at the point $(x, z)$ be $\{2gf(x,z)\}^{\frac{1}{2}}$; then resolving along the meridian curve the centrifugal force and gravity, we have

$$\frac{2gf(x,z)}{x}\frac{dx}{ds} = g\frac{dz}{ds};$$

$$\therefore \quad f(x,z) = \frac{x}{2}\frac{dz}{dx}. \tag{104}$$

(1) Let the velocity be constant; and let $f(x,z) = a$,

$$\therefore \quad \frac{2\,dx}{x} = \frac{dz}{a}, \qquad x^2 = ce^{\frac{z}{a}}.$$

(2) Let $f(x,z) = \dfrac{x^2}{a}$;

$$\therefore \quad \frac{2\,x\,dx}{a} = dz, \qquad x^2 = az.$$

373.] To complete the subject of brachistochronous curves, we must consider the properties of those paths which a particle moving on a curved surface under the action of given forces takes, when the time of passing from one point to another, or from one curve to another, is a minimum.

Now on referring to Article 365, the investigation is the same as far as equation (62); and as to the integrated part, if the initial and terminal points are given, it vanishes identically; if the curves on which they are to be on the surface are given, the equation shews that the brachistochron cuts both curves at right angles: this result is evident from general reasoning.

As to the unintegrated part of (62), $\delta x, \delta y, \delta z$ are no longer independent of each other; but if the equation to the surface is

$$\mathrm{F}(x, y, z) = 0, \tag{105}$$

and if U, V, W are the partial derived-functions of it, we have

$$\text{U}\,\delta x + \text{V}\,\delta y + \text{W}\,\delta z = 0; \tag{106}$$

and as this relation exists at *all* points of the brachistochron, we have from the comparison of it with the unintegrated part of (62),

$$\frac{d.\dfrac{dx}{v\,ds} + \dfrac{\text{X}\,ds}{v^3}}{\text{U}} = \frac{d.\dfrac{dy}{v\,ds} + \dfrac{\text{Y}\,ds}{v^3}}{\text{V}} = \frac{d.\dfrac{dz}{v\,ds} + \dfrac{\text{Z}\,ds}{v^3}}{\text{W}}; \tag{107}$$

which are the general equations to the brachistochron; because the integrals of these equations will give two surfaces, the intersection of which is the brachistochronous curve.

If no forces act, $\text{X} = \text{Y} = \text{Z} = 0$, and the velocity is constant: so that (107) become,

$$\frac{d.\dfrac{dx}{ds}}{\text{U}} = \frac{d.\dfrac{dy}{ds}}{\text{V}} = \frac{d.\dfrac{dz}{ds}}{\text{W}}, \tag{108}$$

which are the equations to a geodesic line; in this case therefore the geodesic joining the two given points, or that which is orthogonal to the two given curves, is the brachistochron.

In these investigations I shall suppose $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz$ to be an exact differential; then since, see Art. 365,

$$d.v^2 = 2\,\{\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz\}; \tag{109}$$

$$\therefore \quad dv = \frac{\text{X}}{v}\,dx + \frac{\text{Y}}{v}\,dy + \frac{\text{Z}}{v}\,dz; \tag{110}$$

$$\therefore \quad \text{X} = v\left(\frac{dv}{dx}\right), \qquad \text{Y} = v\left(\frac{dv}{dy}\right), \qquad \text{Z} = v\left(\frac{dv}{dz}\right). \tag{111}$$

Now $v$ is a function of $x, y, z$; therefore $v = c$ is the equation of a surface; and since, when $v = 0$, $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz = 0$, I shall, as in Art. 190, call $v = c$ the equilibrium-surface: it is evidently such that at all points of the brachistochronous path, the resultant line of action of the impressed forces is normal to the equilibrium-surface.

374.] It is also convenient to consider the brachistochronous lines with reference to lines of another property which can be drawn on a surface. Suppose the particle $m$, which is the subject of motion, to be at rest at a given point on a surface, and to be under the action of given impressed forces; then, if the particle is constrained to move on the surface, the first element of its path will be that length-element on the surface which makes the least angle with the line of action of the resultant of the impressed forces; and at the end of this first length-element another element will have the same property; and so on: thus for the system of forces there will be a series of such elements on the surface, and we shall have a curve, which I propose to call *a line of easy motion* on the surface. Such a line will at all its points be normal to the curve of intersection of the equilibrium-surface and of the given surface. Its equations are found in the following manner:

Let $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ be the direction-cosines of the length-element of the line of easy motion; then if $\text{R}^2 = \text{X}^2 + \text{Y}^2 + \text{Z}^2$, $\frac{\text{X}}{\text{R}}, \frac{\text{Y}}{\text{R}}, \frac{\text{Z}}{\text{R}}$ are the direction-cosines of the line of action of R; and if $\theta$ is the angle which is to be a minimum,

$$\cos\theta = \frac{1}{\text{R}}\left\{\text{X}\frac{dx}{ds} + \text{Y}\frac{dy}{ds} + \text{Z}\frac{dz}{ds}\right\};$$

also

$$1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2,$$

$$0 = \text{U}\frac{dx}{ds} + \text{V}\frac{dy}{ds} + \text{W}\frac{dz}{ds};$$

therefore differentiating these, and equating to zero $d.\cos\theta$, we have,

$$d.\cos\theta = 0 = \frac{1}{\text{R}}\left\{\text{X}\,d.\frac{dx}{ds} + \text{Y}\,d.\frac{dy}{ds} + \text{Z}\,d.\frac{dz}{ds}\right\},$$

$$0 = \frac{dx}{ds}\,d.\frac{dx}{ds} + \frac{dy}{ds}\,d.\frac{dy}{ds} + \frac{dz}{ds}\,d.\frac{dz}{ds},$$

$$0 = \text{U}\,d.\frac{dx}{ds} + \text{V}\,d.\frac{dy}{ds} + \text{W}\,d.\frac{dz}{ds};$$

multiplying the second and third by indeterminate multipliers $\lambda$ and $\mu$, and adding, we have

$$\left(\text{X} + \lambda\frac{dx}{ds} + \mu\text{U}\right)d.\frac{dx}{ds} + \left(\text{Y} + \lambda\frac{dy}{ds} + \mu\text{V}\right)d.\frac{dy}{ds} + \left(\text{Z} + \lambda\frac{dz}{ds} + \mu\text{W}\right)d.\frac{dz}{ds} = 0;$$

In accordance with the method of Art. 142, Vol. I, let

$$\left.\begin{aligned} \text{X} + \lambda\frac{dx}{ds} + \mu\text{U} &= 0 \\ \text{Y} + \lambda\frac{dy}{ds} + \mu\text{V} &= 0 \\ \text{Z} + \lambda\frac{dz}{ds} + \mu\text{W} &= 0 \end{aligned}\right\}; \qquad (112)$$

then multiplying these severally by $dx, dy, dz$, and adding, we have, by reason of (110),

$$v\,dv + \lambda\,ds = 0;$$

$$\therefore \quad \lambda = -\frac{v\,dv}{ds};$$

so that (112) become,

$$\left.\begin{aligned} \text{X} - \frac{v\,dv}{ds}\frac{dx}{ds} + \mu\text{U} &= 0 \\ \text{Y} - \frac{v\,dv}{ds}\frac{dy}{ds} + \mu\text{V} &= 0 \\ \text{Z} - \frac{v\,dv}{ds}\frac{dz}{ds} + \mu\text{W} &= 0 \end{aligned}\right\}; \qquad (113)$$

therefore replacing X, Y, Z by their values from (111), we have,

$$\frac{\left(\frac{dv}{dx}\right)ds - \frac{dx}{ds}dv}{\left(\frac{d\text{F}}{dx}\right)} = \frac{\left(\frac{dv}{dy}\right)ds - \frac{dy}{ds}dv}{\left(\frac{d\text{F}}{dy}\right)} = \frac{\left(\frac{dv}{dz}\right)ds - \frac{dz}{ds}dv}{\left(\frac{d\text{F}}{dz}\right)}; \quad (114)$$

from which, by integration, two functions of $x, y, z$ may (theoretically) be found: the line of intersection of the surfaces represented by which are the lines of easy motion.

The paths taken by water in its descent towards the lowest level which it can reach, by avalanches in their descents, by balls in their descent through bent tubes, are all cases of lines of easy motion.

When gravity is the only force which acts on $m$, the line of easy motion is called *the line of greatest slope.* In this case let the axis of $z$ be vertical, so that

$$\text{X} = v\left(\frac{dv}{dx}\right) = 0, \qquad \text{Y} = v\left(\frac{dv}{dy}\right) = 0, \qquad \text{Z} = v\left(\frac{dv}{dz}\right) = g;$$

whereby (114) become,

$$\frac{dx}{\left(\frac{d\text{F}}{dx}\right)} = \frac{dy}{\left(\frac{d\text{F}}{dy}\right)} = -\frac{dx^2 + dy^2}{\left(\frac{d\text{F}}{dz}\right)dz}; \qquad (115)$$

which are the equations to the line of greatest slope.

Thus on the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

we have, from the first two of (115),

$$\frac{a^2\,dx}{x} = \frac{b^2\,dy}{y};$$

$$\therefore \quad a^2 \log \frac{x}{x_0} = b^2 \log \frac{y}{y_0}; \qquad \therefore \quad \left(\frac{x}{x_0}\right)^{a^2} = \left(\frac{y}{y_0}\right)^{b^2};$$

which is the equation to a cylinder perpendicular to the plane of $xy$; the line of intersection of which with the ellipsoid is the line of greatest slope. In the sphere, $a = b$, and the meridian line is the line of greatest slope.

Also in surfaces of revolution, whose equations are comprised in the form,

$$x^2 + y^2 = f(z),$$

and where the axis of revolution is vertical, we have

$$\left(\frac{dF}{dx}\right) = 2x, \qquad \left(\frac{dF}{dy}\right) = 2y;$$

and thus the first two of (115) become,

$$\frac{dx}{x} = \frac{dy}{y}; \qquad \therefore \quad \frac{x}{x_0} = \frac{y}{y_0};$$

which is the equation to a meridian plane: the meridian therefore which passes through a given point on a surface of revolution is the line of greatest slope at that point.

375.] The general equations of the brachistochron (107) may be expressed in the following form by means of (111),

$$\frac{d.\frac{dx}{v\,ds} - ds\frac{d}{dx}\frac{1}{v}}{\left(\frac{dF}{dx}\right)} = \frac{d.\frac{dy}{v\,ds} - ds\frac{d}{dy}\frac{1}{v}}{\left(\frac{dF}{dy}\right)} = \frac{d.\frac{dz}{v\,ds} - ds\frac{d}{dz}\frac{1}{v}}{\left(\frac{dF}{dz}\right)}; \qquad (116)$$

which may again be expressed in the form

$$\frac{v\,d.\frac{dx}{ds}}{\left(\frac{dF}{dx}\right)} + \frac{\left(\frac{dv}{dx}\right) ds - \frac{dx}{ds}\,dv}{\left(\frac{dF}{dx}\right)} = \frac{v\,d.\frac{dy}{ds}}{\left(\frac{dF}{dy}\right)} + \frac{\left(\frac{dv}{dy}\right) ds - \frac{dy}{ds}\,dv}{\left(\frac{dF}{dy}\right)}$$

$$= \frac{v\,d.\frac{dz}{ds}}{\left(\frac{dF}{dz}\right)} + \frac{\left(\frac{dv}{dz}\right) ds - \frac{dz}{ds}\,dv}{\left(\frac{dF}{dz}\right)}; \qquad (117)$$

now these equations are satisfied, if we have simultaneously,

$$\frac{d.\frac{dx}{ds}}{\left(\frac{dF}{dx}\right)} = \frac{d.\frac{dy}{ds}}{\left(\frac{dF}{dy}\right)} = \frac{d.\frac{dz}{ds}}{\left(\frac{dF}{dz}\right)}, \qquad (118)$$

and

$$\frac{\left(\frac{dv}{dx}\right) ds - \frac{dx}{ds}\, dv}{\left(\frac{d\text{F}}{dx}\right)} = \frac{\left(\frac{dv}{dy}\right) ds - \frac{dy}{ds}\, dv}{\left(\frac{d\text{F}}{dy}\right)} = \frac{\left(\frac{dv}{dz}\right) dv - \frac{dz}{ds}\, dv}{\left(\frac{d\text{F}}{dz}\right)}; \qquad (119)$$

but these are severally the equations to the geodesic, and to the line of easy motion; therefore a curve which is at the same time a geodesic and a line of easy motion is also a brachistochron. Hence also we conclude that of curves which have the properties of being geodesic, lines of easy motion, and brachistochronous, any line which possesses two of these properties possesses also the third.

The brachistochron at the point whence $m$ starts, and generally at all points where the velocity of $m$ is zero, touches the line of easy motion.

For if we take equations (117), if $v = 0$, we have only the second members of each, so that at the points where $v = 0$, the brachistochron is identical with the line of easy motion.

376.] I proceed, lastly, to consider the case where gravity is the only acting force, and particularly when $m$ moves on a surface of revolution, the axis of which is vertical.

In this case $\text{X} = \text{Y} = 0$, $\text{Z} = g$;

$$\therefore \quad v^2 = 2g(z - z_0);$$

so that the first two of (107) become,

$$\frac{d.\dfrac{dx}{ds\,(z-z_0)^{\frac{1}{2}}}}{\left(\frac{d\text{F}}{dx}\right)} = \frac{d.\dfrac{dy}{ds\,(z-z_0)^{\frac{1}{2}}}}{\left(\frac{d\text{F}}{dy}\right)}. \qquad (120)$$

(1) Let the surface be the vertical plane of $xz$: so that

$$\left(\frac{d\text{F}}{dx}\right) = 1, \qquad \left(\frac{d\text{F}}{dy}\right) = 0;$$

$$\therefore \quad d.\frac{dx}{ds\,(z-z_0)^{\frac{1}{2}}} = 0,$$

$$\frac{dx}{ds\,(z-z_0)^{\frac{1}{2}}} = \text{a constant},$$

which is the differential equation to a cycloid whose base is horizontal; see Article 366.

(2) Let the surface be the inclined plane, whose equation is

$$\frac{y}{b} = \frac{z}{c},$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = 0, \qquad \left(\frac{d\text{F}}{dy}\right) = \frac{1}{b}, \qquad \left(\frac{d\text{F}}{dz}\right) = -\frac{1}{c};$$

so that from the first of (120),

$$\frac{dx}{ds\,(z-z_0)^{\frac{1}{2}}} = c \text{ (a constant)},$$

which will be the equation to a cycloid on the inclined plane.

(3) Let the surface be one of revolution, of which let the equation be

$$x^2+y^2 = f(z);$$

$$\therefore \quad \left(\frac{d\text{F}}{dx}\right) = 2x, \qquad \left(\frac{d\text{F}}{dy}\right) = 2y, \qquad \left(\frac{d\text{F}}{dz}\right) = -f'(z);$$

so that from (120),

$$y\,d.\frac{dx}{ds\,(z-z_0)^{\frac{1}{2}}} - x\,d.\frac{dy}{ds\,(z-z_0)^{\frac{1}{2}}} = 0;$$

whence integrating,

$$y\,dx - x\,dy = c\,ds\,(z-z_0)^{\frac{1}{2}}, \tag{121}$$

where $c$ is an arbitrary constant. Let $x^2+y^2 = r^2$, $x = r\cos\theta$, $y = r\sin\theta$; then (121) becomes

$$r^2\,d\theta = c\,ds\,(z-z_0)^{\frac{1}{2}}.$$

If $v$ is the velocity of $m$ at the time $t$,

$$v^2 = 2g\,(z-z_0) = \frac{ds^2}{dt^2}; \qquad \therefore \quad r^2\,d\theta = \text{A}\,v^2\,dt;$$

so that the sectorial area described by $r$ in a horizontal plane varies as the square of the velocity of the particle.

If the equation to the generating curve of the solid is

$$r = f(z),$$

$$\text{then} \qquad dr = f'(z)\,dz;$$

$$\begin{aligned} \therefore \quad ds^2 &= dz^2+dr^2+r^2\,d\theta^2 \\ &= dz^2\,\{1+(f'(z))^2\}+(f(z))^2\,d\theta^2; \end{aligned}$$

so that

$$d\theta = \frac{c\,dz}{f(z)}\left\{\frac{\{1+(f'(z))^2\}\,(z-z_0)}{(\text{F}(z))^2-c^2\,(z-z_0)}\right\}^{\frac{1}{2}}; \tag{122}$$

whence the equation to the brachistochron may be found in terms of $\theta$ and $z$.

## SECTION 3.—*Constrained motion of material particles in resisting media.*

377.] In the present Section we shall find it convenient to employ the method of tangential resolution; and as the principles which have been investigated in the preceding pages are sufficient for the solution of the problem of a particle moving in a resisting medium, it is necessary for us only to give examples.

The most simple case is that of a heavy material particle moving down a smooth inclined plane. Let $\alpha$ be the angle of inclination of the plane to the horizon; and let $s$ be measured along the plane and be the distance of $m$ from the origin at the time $t$; then, if the resistance varies as the velocity, the equation of motion is

$$\frac{d^2 s}{dt^2} = g \cos \alpha - k \frac{ds}{dt},$$

which is of the same form as (122) in Art. 243; and the results therein deduced are also applicable to this case.

If the resistance varies as the square of the velocity, the equation of motion is

$$\frac{d^2 s}{dt^2} = g \cos \alpha - k \left(\frac{ds}{dt}\right)^2,$$

which is of the same form as (113) in Art. 242; and therefore similar results may be deduced.

We proceed next to the consideration of curvilinear motion, and in the first place that along the cycloid.

Ex. 1. A heavy particle moves on a cycloid in a vertical plane with its base horizontal and vertex downwards, and in a medium of which the resistance varies as the volocity; it is required to determine the circumstances of motion.

Let the origin be taken at the lowest point of the cycloid; let the axis of $x$ be vertical; let $a$ be the radius of the generating circle; $s =$ the arc of the cycloid measured from its lowest point; $\alpha =$ the initial value of $s$; $2k =$ the resistance of the medium for an unit-mass. Therefore the equation of motion along the curve is, as in Art. 353,

$$\frac{d^2 s}{dt^2} = -\frac{g}{4a} s - 2k \frac{ds}{dt},$$

$$\frac{d^2 s}{dt^2} + 2k \frac{ds}{dt} + \frac{g}{4a} s = 0; \tag{123}$$

which is a linear differential equation of the second order with constant coefficients; and of which the general integral is, see Article 350, Vol. II,

$$s = e^{-kt} \{c_1 e^{\left(k^2 - \frac{g}{4a}\right)^{\frac{1}{2}} t} + c_2 e^{-\left(k^2 - \frac{g}{4a}\right)^{\frac{1}{2}} t}\}, \tag{124}$$

where $c_1$ and $c_2$ are undetermined constants.

Here we have two cases; viz. according as the radical in the exponent is imaginary or possible.

(1) Let the radical be imaginary, that is, let $k^2 - \frac{g}{4a} = -h^2$,

$$\begin{aligned} \therefore \quad s &= e^{-kt}\{c_1 e^{ht\sqrt{-1}} + c_2 e^{-ht\sqrt{-1}}\} \\ &= e^{-kt}(a_1 \cos ht + a_2 \sin ht), \end{aligned}$$

where $a_1$ and $a_2$ are undetermined constants: to determine them we have the following data: when $t = 0$, $s = a$, $\frac{ds}{dt} = 0$; therefore

$$\left.\begin{aligned} a &= a_1 \\ 0 &= ha_2 - ka_1 \end{aligned}\right\}, \qquad \therefore \quad \begin{aligned} a_1 &= a, \\ a_2 &= \frac{ka}{h}; \end{aligned}$$

$$\therefore \quad s = e^{-kt}\frac{a}{h}\{h \cos ht + k \sin ht\}. \tag{125}$$

Let $\tau$ = the time from the point where $s = a$ to the lowest point; therefore

$$\tan h\tau = -\frac{h}{k}; \tag{126}$$

which is independent of $a$, and is therefore the same whatever is the point on the cycloid whence $m$ begins to descend. The cycloid therefore is tautochronous in a medium wherein the resistance varies as the velocity, as well as in vacuo.

On comparing the results of this problem with observed facts, they are found so nearly to coincide, that the resistance of the air seems to vary nearly as the velocity, when the arc through which $m$ moves is small, and when consequently the velocity is small: in this case also $k$ is small, and if $a$ is not very large, $h$, which is equal to $\left(\frac{g}{4a}\right)^{\frac{1}{2}}$, is not small;

$$\begin{aligned} \therefore \quad h\tau &= \tan^{-1}\left(-\frac{h}{k}\right) \\ &= \frac{\pi}{2} + \tan^{-1}\frac{k}{h} \\ &= \frac{\pi}{2} + \frac{k}{h} \text{ (approximately)}; \end{aligned}$$

$$\therefore \quad \tau = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}} + \frac{4a}{g}k; \tag{127}$$

on comparing which with (27), Art. 353, it appears that the time of descent to the lowest point is greater than it is in vacuo, by a quantity which varies as the coefficient of resistance.

Again, from (125) we have,

$$\frac{ds}{dt} = -\frac{a(h^2+k^2)}{h} e^{-kt} \sin ht; \tag{128}$$

therefore $\frac{ds}{dt} = 0$, when $t = 0$, $t = \frac{\pi}{h}$, $= \ldots$; and the time of a complete oscillation is $\frac{\pi}{h}$. And substituting these values of $t$ successively in (125), the amplitudes of vibration successively become,

$$a, \qquad a e^{-\frac{k}{h}\pi}, \qquad a e^{-\frac{2k}{h}\pi}, \ldots\ldots;$$

so that the amplitudes decrease in geometrical progression.

(2) Let $k^2 - \frac{g}{4a} = h^2$; and (124) becomes,

$$s = e^{-kt}\{c_1 e^{ht} + c_2 e^{-ht}\}; \tag{129}$$

and since when $t = 0$, $s = a$, and $\frac{ds}{dt} = 0$, we have

$$s = \frac{a}{2h}\{(h+k)e^{-(k-h)t} - (k-h)e^{-(h+k)t}\}; \tag{130}$$

in which expression $s = 0$, only when $t = \infty$; that is, the particle never reaches the lowest point of the cycloid.

And differentiating twice (130), it will be found that $\frac{d^2 s}{dt^2} = 0$, that is, that $\frac{ds}{dt}$ is a maximum, when

$$t = \frac{1}{h}\log\frac{k+h}{k-h}.$$

(3) If $k^2 = \frac{4a}{g}$, that is, if two roots of the auxiliary equation of (123) are equal to each other, then the integral of (123) is,

$$s = a e^{-kt}(1+kt); \tag{131}$$

which formula shews that $s = 0$, only when $t = \infty$.

378.] Let us in the next place consider the motion of a circular pendulum in a resisting medium, when the resistance varies as the velocity, and when the amplitudes of vibration are small.

Let $a$ be the radius of the circle, and thus the length of the pendulum; $\theta =$ the angle between the pendulum and the vertical line at the time $t$; $a =$ the greatest value of $\theta$; then the equation of motion is,

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{a}\sin\theta = 0; \tag{132}$$

and as $\theta$ is always small, $\sin\theta$ may be replaced by $\theta$, so that we have,

$$\frac{d^2\theta}{dt^2} + 2k\frac{d\theta}{dt} + \frac{g}{a}\theta = 0, \tag{133}$$

which is of the same form as (123). Now $k$ is in this case small, so that $k^2$ is less than $\frac{g}{a}$; therefore let

$$\frac{g}{a} - k^2 = h^2;$$

so that the integral is the same as (125), and we have

$$\theta = \frac{\alpha}{h}e^{-kt}\{h\cos ht + k\sin ht\}, \tag{134}$$

$$\text{and} \quad \frac{d\theta}{dt} = -\frac{\alpha(h^2+k^2)}{h}e^{-kt}\sin ht; \tag{135}$$

whereby the position and the velocity of the pendulum at any time are known.

$\frac{d\theta}{dt} = 0$, whenever $t = \frac{n\pi}{h}$; so that the time of an oscillation

$$= \frac{\pi}{h} = \pi\left(\frac{a}{g}\right)^{\frac{1}{2}}\left(1 - \frac{k^2a}{g}\right)^{-\frac{1}{2}},$$

and is independent of $\alpha$. Thus, see Art. 357, (40), the time of an oscillation in vacuo is to that in the resisting medium as 1 to $\left(1 - \frac{k^2a}{g}\right)^{-\frac{1}{2}}$. The amplitudes of the oscillation, as it has been shewn in the last Article, diminish successively in geometrical progression.

On these results M. Poisson remarks (in Art. 187 of his Traité de Mécanique, Vol. I, 2nd edition), that experiments in air shew how the amplitudes of the vibrations (approximately) decrease in a geometrical progression. In an experiment made by Borda, where $\alpha$ was one-third of a degree, the amplitudes were evidently reduced in geometrical progression, and the greatest amplitude was reduced by about two-thirds after 1800 oscillations.

379.] Let us assume that the resistance of the air varies as the square of the velocity; so that if $k$ is the coefficient of resistance for an unit-mass, the equation of motion is,

$$\frac{d^2\theta}{dt^2} - ak\left(\frac{d\theta}{dt}\right)^2 + \frac{g}{a}\sin\theta = 0;$$

$$\therefore \quad d\left(\frac{d\theta}{dt}\right)^2 - 2ak\left(\frac{d\theta}{dt}\right)^2 d\theta = -\frac{2g}{a}\sin\theta\, d\theta,$$

which is a linear differential equation of the first order, see Art. 278, Vol. II; and of which the general integral is,

$$\left(\frac{d\theta}{dt}\right)^2 = c\,e^{2ak\theta} + \frac{2g}{a}\,\frac{\cos\theta + 2ak\sin\theta}{1+4a^2k^2};$$

but $\theta = a$, when $\frac{d\theta}{dt} = 0$; therefore

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a(1+4a^2k^2)}\left\{\cos\theta + 2ak\sin\theta - (\cos a + 2ak\sin a)\,e^{-2ak(a-\theta)}\right\}. \quad (136)$$

Let $-a_1$ be the value of $\theta$, when the pendulum comes to rest on the other side of the vertical;

$$\therefore \quad (\cos a_1 - 2ak\sin a_1)\,e^{2aka_1} = (\cos a + 2ak\sin a)\,e^{-2aka}. \quad (137)$$

As $k$ is small, let us expand the exponential and omit terms involving the square and higher powers of $k$; so that we have,

$$\cos a_1 - 2ak(\sin a_1 - a_1\cos a_1) = \cos a + 2ak(\sin a - a\cos a).$$

Let $\Delta a =$ the decrement of the amplitude; so that

$$a = a_1 + \Delta a;$$

then neglecting the square and higher powers of $\Delta a$, and also the product $k\Delta a$, we have,

$$\Delta a \sin a = 4ak(\sin a - a\cos a)$$

$$\Delta a = \frac{4ak(\sin a - a\cos a)}{\sin a};$$

and if we suppose the arcs of oscillation to be small, then neglecting the cubes and higher powers of $a$, we have

$$\Delta a = \frac{4aka^2}{3}; \qquad \therefore \quad a_1 = a - \frac{4aka^2}{3};$$

and $m$ having come to rest will again descend and ascend until $\theta = a_2$ (say); where, as a process similar to the preceding one will shew,

$$a_2 = a_1 - \frac{4aka_1^2}{3};$$

and so on: until finally the oscillations will cease, and the pendulum will be in a vertical position.

For the determination of the time of an oscillation, and the successive decrements of it, I must refer the reader to M. Poisson's Traité de Mécanique, Vol. I, second edition, p. 356. And for an inquiry into various other circumstances connected with the motion of a pendulum in air to (1) a memoir of M. Poisson entitled "Mémoire sur les mouvements simultanés d'un pendule et de l'air environnant," and contained in the Mém. de l'Académie des Sciences de Paris, tome XI; (2) a paper by Professor Stokes of Cambridge, and contained in the Transactions of the Cambridge Philosophical Society, Vol. IX, Part II, 1851, in the introduction to which will be found a succinct account of all the investigations which had been previously made on the subject. These inquiries however are hydrodynamical, and properly belong to a future part of our treatise.

380.] It remains for us still to investigate the general equation of tautochronous curves in a resisting medium; and with this object I shall inquire into the most general expression for the tangential component of tautochronous curves.

Let o be the common extremity of the arcs, which are to be described in the same time $\tau$; let $s =$ the distance along the curve of $m$ from o at the time $t$, and let $\sigma$ be the initial value of $s$. And let $\mathrm{T}$ be the tangential force for which the curve is tautochronous; let $v$ be the velocity at the time $t$; then

$$\tau = -\int_0^\sigma \frac{ds}{v}; \tag{138}$$

also $$\mathrm{T} = \frac{d^2 s}{dt^2} = \frac{d}{dt}\frac{ds}{dt} = v\frac{dv}{ds},$$

which is to be determined.

By reason of the property of tautochronism, $\tau$ is to be independent of $\sigma$; therefore

$$\frac{d\tau}{d\sigma} = 0. \tag{139}$$

Let us in the first place so transform the right-hand member of (138) that the limits of integration may be constant. For this purpose let $s = \psi(z)$, where $z$ is a new variable, and where $\psi(z) = 0$, when $z = 0$; and let $z = u\mathrm{z}$, where $u$ is another new variable, and where $\mathrm{z}$ is the value of $z$, when $s = \sigma$; so that

$$s = \psi(z) = \psi(u\mathrm{z}); \tag{140}$$

therefore $u = 1$ when $s = \sigma$, or when $z = \mathrm{z}$; also $\sigma = \psi(\mathrm{z})$.

From these we have, $\mathrm{z}$ being independent of $s$,

$$ds = \psi'(u\mathrm{z})\,\mathrm{z}\,du;$$

$$\therefore\quad \tau = -\int_0^1 \frac{\psi'(u\mathrm{z})\,\mathrm{z}\,du}{v} = -\int_0^1 \frac{\psi'(z)\,\mathrm{z}\,du}{v}; \tag{141}$$

$$\therefore\ \frac{d\tau}{d\sigma} = -\int_0^1 \frac{\left\{\psi''(z)\dfrac{dz}{d\sigma}\mathrm{z} + \psi'(z)\dfrac{d\mathrm{z}}{d\sigma}\right\} v - \mathrm{z}\,\psi'(z)\left\{\dfrac{dv}{d\sigma} + \dfrac{dv}{ds}\dfrac{ds}{d\sigma}\right\}}{v^2}\,du. \tag{142}$$

Now $$\frac{d\mathrm{z}}{d\sigma} = u\frac{d\mathrm{z}}{d\sigma} = u\,\mathrm{z}' \text{ (say)},$$

$$\frac{ds}{d\sigma} = \psi'(u\,\mathrm{z})\,u\frac{d\mathrm{z}}{d\sigma} = \psi'(z)\,u\,\mathrm{z}';$$

so that (142) becomes

$$\frac{d\tau}{d\sigma} = -\frac{1}{\mathrm{z}}\int_0^\sigma \frac{\{\psi''(z)\,z\,\mathrm{z}' + \psi'(z)\,\mathrm{z}'\}\,v - \left\{\frac{dv}{d\sigma}\,\mathrm{z} + \frac{dv}{ds}\psi'(z)\,z\,\mathrm{z}'\right\}\psi'(z)}{v^2}\,dz. \quad (143)$$

Let the quantity under the sign of integration be represented by

$$d.f(s,\sigma); \quad (144)$$

also from (140),

$$\frac{dz}{ds} = \frac{1}{\psi'(z)}, \qquad \therefore \quad \frac{d^2z}{ds^2} = -\frac{\psi''(z)}{(\psi'(z))^3};$$

and let $\frac{dz}{ds}$, $\frac{d^2z}{ds^2}$ be represented by $z'$, $z''$, respectively: so that

$$\psi'(z) = \frac{1}{z'}, \qquad \psi''(z) = -\frac{z''}{z'^3}; \quad (145)$$

therefore, since $\frac{d\tau}{d\sigma} = 0$, we have, substituting in (143),

$$\mathrm{z}\frac{dv}{d\sigma} + \frac{z\,\mathrm{z}'}{z'}\frac{dv}{ds} - \mathrm{z}'v\left(1 - \frac{zz''}{z'^2}\right) + \frac{d.f(s,\sigma)}{ds}v^2 = 0; \quad (146)$$

which is a partial differential equation of the first order, where $\frac{dv}{ds}$ and $\frac{dv}{d\sigma}$ are partial derived-functions of $v$; let it be divided through by $\mathrm{z}'$; then by (84), Art. 281, Vol. II, we have,

$$\frac{\mathrm{z}'}{\mathrm{z}}\,d\sigma = \frac{z'}{z}\,ds = \frac{dv}{v\left(1 - \frac{zz''}{z'^2}\right) - \frac{v^2}{\mathrm{z}'}\frac{d.f(s,\sigma)}{ds}}; \quad (147)$$

and our object is to find two integrals of these equations. Let us take the first two of these three equalities; and let

$$\frac{z}{z'} = \phi(s); \qquad \therefore \quad \frac{\mathrm{z}}{\mathrm{z}'} = \phi(\sigma); \quad (148)$$

so that we have $$\frac{d\sigma}{\phi(\sigma)} = \frac{ds}{\phi(s)} \quad (149)$$

Let $$\int\frac{d\sigma}{\phi(\sigma)} = \omega(\sigma); \qquad \therefore \quad \int\frac{ds}{\phi(s)} = \omega(s); \quad (150)$$

and (149) becomes

$$d.\omega(\sigma) - d.\omega(s) = 0;$$

$$\therefore \quad \omega(\sigma) - \omega(s) = c_1; \quad (151)$$

where $c_1$ is an arbitrary constant of integration.

Now let us take the last two of (147): from (148) we have

$$1-\frac{zz''}{z'^2} = \phi'(s,), \qquad \frac{1}{z'} = \phi(\sigma)\, e^{-\int \frac{d\sigma}{\phi(\sigma)}}$$

$$= \phi(\sigma)\, e^{-\omega(\sigma)};$$

so that we have $\dfrac{ds}{\phi(s)} = \dfrac{dv}{v\phi'(s)-v^2\phi(\sigma)\, e^{-\omega(\sigma)}\dfrac{d.f(s,\sigma)}{ds}}$;

$$\therefore \quad \frac{v\phi'(s)\,ds-\phi(s)\,dv}{v^2} = \phi(\sigma)\, e^{-\omega(\sigma)}\, d.f(s,\sigma); \qquad (152)$$

in the right-hand member of this equation let $\sigma$ be replaced by its value in terms of $s$ from (151), and let the indefinite integral of the quantity after the substitution be $\chi(s)$: then integrating (152) we have,

$$\frac{\phi(s)}{v} = \chi(s)+c_2; \qquad (153)$$

where $c_2$ is another arbitrary constant; by the general theory of the integration of partial differential equations, $c_1 = \mathrm{F}_1(c_2)$, where $\mathrm{F}_1$ is the symbol of an arbitrary function: therefore in this case we have,

$$\omega(\sigma)-\omega(s) = \mathrm{F}_1\left(\frac{\phi(s)}{v}-\chi(s)\right); \qquad (154)$$

$\mathrm{F}_1$ being such that $v=0$, when $s=\sigma$; and this is the general integral of the differential equation (146).

Let us take the $s$-differential of it, and replacing $d.\omega(s)$ from (150), we have

$$-\frac{1}{\phi(s)} = d.\mathrm{F}_1\left(\frac{\phi(s)}{v}-\chi(s)\right)\left\{\frac{v\phi'(s)-\phi(s)\dfrac{dv}{ds}}{v^2}-\chi'(s)\right\}.$$

$$\text{Let} \quad d.\mathrm{F}_1\left(\frac{\phi(s)}{v}-\chi(s)\right) = \left\{\mathrm{F}\left(\frac{\phi(s)}{v}-\chi(s)\right)\right\}^{-1}; \qquad (155)$$

$$\therefore \quad v\frac{dv}{ds} = \mathrm{T} = \frac{v^3}{\{\phi(s)\}^2}\left\{\mathrm{F}\left(\frac{\phi(s)}{v}-\chi(s)\right)+\frac{\phi(s)\,\phi'(s)}{v}-\phi(s)\,\chi'(s)\right\}; \quad (156)$$

which is the most general expression for the tangential velocity-increment, where $\mathrm{F}$, $\phi$ and $\chi$ are symbols of arbitrary functions.

As a particular case of (156), let us suppose $\chi(s)=0$; then

$$\mathrm{T} = \frac{v^3}{\{\phi(s)\}^2}\left\{\mathrm{F}\left(\frac{\phi(s)}{v}\right)+\frac{\phi(s)\,\phi'(s)}{v}\right\}. \qquad (157)$$

$$\text{Let} \quad \phi(s) = \xi; \qquad \therefore \quad \phi'(s) = \frac{d\xi}{ds};$$

let $\dfrac{v}{\phi(s)}\,\mathrm{F}\left(\dfrac{\phi(s)}{v}\right) = \Pi\left(\dfrac{v}{\phi(s)}\right) = \Pi\left(\dfrac{v}{\xi}\right)$, where $\Pi$ is the symbol of an arbitrary function: then (157) becomes,

$$\text{T} = \frac{v^2}{\xi}\left\{\Pi\left(\frac{v}{\xi}\right) + \frac{d\xi}{ds}\right\}; \tag{158}$$

which formula was given by Lagrange in the Mém. de l'Académie de Berlin, 1765, 1770. The general formula (156) is due to M. Brioschi, and is given by him in "Annali di Scienze Matematiche e Fisiche compilati da B. Tortolini, Roma, 1852, p. 362." For the preceding references I am indebted to M. Jullien, "Problèmes de Mécanique rationelle," Vol. I. Page 398. Mallet-Bachelier, Paris, 1855.

381.] As an application of formula (158), let us investigate the cases of tautochronism relative to a heavy particle, which moves on a rough curve in a medium, of which the resistance varies as any function of the velocity.

Let the axis of $x$ be vertical, let $f(v)$ be the resistance of the medium, and let $\mu$ be the coefficient of friction; so that the tangential and normal components of the velocity-increments are respectively,

$$g\frac{dx}{ds} - f(v), \qquad \text{and} \qquad g\frac{dy}{ds} + \frac{v^2}{\rho};$$

and as the friction varies as the pressure on the curve, we have

$$\text{T} = -g\frac{dx}{ds} + \mu g\frac{dy}{ds} + \mu\frac{v^2}{\rho} + f(v); \tag{159}$$

and this expression is to be identical with the right-hand member of (158).

If in (159) T is differentiated thrice with respect to $v$, and once with respect to $s$, the result $= 0$; so that

$$\frac{d^4\text{T}}{dv^3\, ds} = 0; \tag{160}$$

the particular form of (158) must therefore be consistent with this condition.

Let $\frac{v}{\xi}\Pi\left(\frac{v}{\xi}\right) = \pi\left(\frac{v}{\xi}\right)$, and $\frac{1}{\xi}\frac{d\xi}{ds} = \chi(\xi)$;

so that (158) becomes,

$$\text{T} = v\pi\left(\frac{v}{\xi}\right) + v^2\chi(\xi); \tag{161}$$

and as $s$ is a function of $\xi$, (160) becomes,

$$\frac{d^4\text{T}}{dv^3\, d\xi} = 0: \tag{162}$$

applying this to (161), we have

$$\frac{v^2}{\xi^5}\pi''''\left(\frac{v}{\xi}\right) + \frac{6v}{\xi^4}\pi'''\left(\frac{v}{\xi}\right) + \frac{6}{\xi^3}\pi''\left(\frac{v}{\xi}\right) = 0; \tag{163}$$

and if $\frac{v}{\xi} = \zeta$, we have

$$\zeta^2 \pi''''(\zeta) + 6\zeta \pi'''(\zeta) + 6\pi''(\zeta) = 0;$$

whence integrating we have

$$\zeta^2 \pi'(\zeta) = \mathrm{A}\zeta^2 + \mathrm{B}\zeta + \mathrm{C};$$

$$\therefore \quad \pi(\zeta) = \mathrm{A}\zeta + \mathrm{B}\log\zeta - \frac{\mathrm{C}}{\zeta} + \mathrm{D};$$

where A, B, C, D are arbitrary constants. Substituting this in (161) we have,

$$\mathrm{T} = -\mathrm{B}v\log\frac{v}{\xi} + \mathrm{D}v + \frac{v^2}{\xi}\left(\mathrm{A} + \frac{d\xi}{ds}\right) - \mathrm{C}\xi. \tag{164}$$

That this and (159) may be identical, we must have,

(1) $\mathrm{B} = 0$,

(2) $\mathrm{D}v = f(v)$,

(3) $\frac{1}{\xi}\left(\mathrm{A} + \frac{d\xi}{ds}\right) = \frac{\mu}{\rho}$,

(4) $-\mathrm{C}\xi = g\left(\mu\frac{dy}{ds} - \frac{dx}{ds}\right)$;

therefore from (2) the resistance of the medium varies directly as the velocity; and from (3) and (4), after all reductions, we have,

$$\frac{d^2x}{ds^2} = -\frac{\mathrm{AC}}{g(1+\mu^2)} = \text{a constant}; \tag{165}$$

and this is the equation to a cycloid.

Thus the cycloid is tautochronous both in vacuo and in a medium, of which the resistance is proportional to the velocity, and with friction; it is also the only case of tautochronism with friction which is given by (158).

If $a$ is the radius of the generating circle of the cycloid, and $\alpha$ is the distance from the lowest point of the cycloid of the common extremity of all the tautochronous arcs, then from the equation to the curve we have

$$\frac{dx}{ds} = \frac{s+\alpha}{4a}.$$

Since all the arcs are tautochronous, $\alpha$ is evidently the distance along the arc from the lowest point of that point on the rough curve at which $m$ being placed will remain at rest.

# CHAPTER XIII.

## GENERAL PROPERTIES AND PRINCIPLES OF THE MOTION OF A PARTICLE UNDER THE ACTION OF GIVEN FORCES.

### SECTION 1.—*The principles of Vis Viva, and of Least Action.*

382.] In the present Chapter I propose to investigate certain principles and properties which are either generally, or under certain conditions, true of the motion of a material particle: and also to explain some methods of investigation, which, when direct processes fail, enable us to obtain approximate solutions of certain physical problems. And, firstly, I propose to consider the principle of Vis Viva in its relation to one moving particle; this condition is made, because it will hereafter be shewn that the principle is also true of a system of moving particles.

Let the equations of motion of a particle moving freely be,

$$m\frac{d^2x}{dt^2} = m\mathrm{X}, \qquad m\frac{d^2y}{dt^2} = m\mathrm{Y}, \qquad m\frac{d^2z}{dt^2} = m\mathrm{Z}; \tag{1}$$

and let us consider the circumstances of motion when $t = t_0$, and when $t = t$: and let the corresponding places of $m$ be $(x_0, y_0, z_0)$, $(x, y, z)$: and let the corresponding velocities be $v_0$ and $v$; then multiplying the equations (1) severally by $2dx$, $2dy$, $2dz$, and adding,

$$m\frac{2\,dx\,d^2x + 2\,dy\,d^2y + 2\,dz\,d^2z}{dt^2} = 2m\,\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\}; \tag{2}$$

therefore integrating

$$mv^2 - mv_0^2 = 2m\int_{t_0}^{t}\{\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz\}. \tag{3}$$

Suppose X, Y, Z to be functions of $x, y, z$ only; and to be such that

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz \tag{4}$$

is an exact differential; and let us also suppose it to be the differential of $\mathrm{F}(x, y, z)$, so that

$$\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz = \mathrm{D}.\mathrm{F}(x, y, z); \tag{5}$$

therefore from (3),

$$mv^2 - mv_0^2 = 2m\{\mathrm{F}(x, y, z) - \mathrm{F}(x_0, y_0, z_0)\}. \tag{6}$$

Therefore the increase of the vis viva of $m$ in moving from one point to another depends on the coordinates of those points only; and is independent of the path which the particle takes, and of the time which it occupies, in the passage. This principle, which is expressed in equation (6), is called "*The principle of vis viva.*"

As to the expression $\text{F}(x,y,z)$, let us suppose that $\text{F}(x,y,z) = c$ is the equation to a surface; then since by (5),

$$\text{X} = \left(\frac{d\text{F}}{dx}\right), \qquad \text{Y} = \left(\frac{d\text{F}}{dy}\right), \qquad \text{Z} = \left(\frac{d\text{F}}{dz}\right);$$

and also since

$$\left(\frac{d\text{F}}{dx}\right) dx + \left(\frac{d\text{F}}{dy}\right) dy + \left(\frac{d\text{F}}{dz}\right) dz = dc = 0,$$

$$\therefore \quad \text{X}\, dx + \text{Y}\, dy + \text{Z}\, dz = 0;$$

and the surface $\text{F}(x, y, z) = c$, is at all its points normal to the line of action of the impressed velocity-increments; so that $\text{F}(x,y,z) = c$ is an equilibrium-surface, and is of course such that, if it were capable of bearing the pressure, $m$ being placed on it, and under the action of the forces, would be at rest. And suppose that we have another equilibrium-surface, $\text{F}(x_0, y_0, z_0) = c'$; then from (6),

$$mv^2 - mv_0^2 = 2m(c-c'). \tag{7}$$

The loss of vis viva therefore of a particle moving from any point of the first to any point of the second equilibrium-surface is the same, whatever are the initial and final points on the two respectively, and whatever is the number of times the particle has passed through either one or other of the surfaces.

Hence also if a moving particle in its path is several times on the same equilibrium-surface, the vis viva is the same at all points on the surface.

Also if no forces act on a particle, its vis viva is unaltered.

383.] As to the nature of the forces which satisfy the condition (5).

(1) It is evident that it is satisfied whenever the forces are central, and their intensities depend on the distance between the centre and the particle.

Thus let there be a series of central forces $\text{P}_1, \text{P}_2, \ldots \text{P}_n$, whose centres are at $(a_1, b_1, c_1)$, $(a_2, b_2, c_2)$, … $(a_n, b_n, c_n)$; and let the distances of their centres from $m$ respectively be $r_1, r_2, \ldots r_n$; and let the forces be functions of these distances. Then

$$\left.\begin{aligned}
\text{X} &= \text{P}_1\frac{x-a_1}{r_1}+\text{P}_2\frac{x-a_2}{r_2}+\ldots+\text{P}_n\frac{x-a_n}{r_n}\\
\text{Y} &= \text{P}_1\frac{y-b_1}{r_1}+\text{P}_2\frac{y-b_2}{r_2}+\ldots+\text{P}_n\frac{y-b_n}{r_n}\\
\text{Z} &= \text{P}_1\frac{z-c_1}{r_1}+\text{P}_2\frac{z-c_2}{r_2}+\ldots+\text{P}_n\frac{z-c_n}{r_n}
\end{aligned}\right\};\tag{8}$$

$$\begin{aligned}
\therefore\quad \text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz &= \text{P}_1dr_1+\text{P}_2dr_2+\ldots+\text{P}_ndr_n\\
&= \Sigma.\text{P}\,dr;
\end{aligned}$$

which is evidently an exact differential; so that (5) becomes,

$$mv^2-mv_0^2 = 2m\int_{r_0}^{r}\Sigma.\text{P}\,dr. \tag{9}$$

(2) If $m$ is acted on by a force whose line of action is always perpendicular to a given plane, and which is a function of the perpendicular distance of $m$ from the plane, the condition (5) is also satisfied. Thus let the equation to the plane be

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p = 0; \tag{10}$$

then if $u$ is the perpendicular distance from $(x, y, z)$ on (10),

$$u = x\cos\alpha+y\cos\beta+z\cos\gamma-p. \tag{11}$$

Let U represent the function of $u$, which expresses the force; then

$$\text{X} = \text{U}\cos\alpha, \qquad \text{Y} = \text{U}\cos\beta, \qquad \text{Z} = \text{U}\cos\gamma,$$

$$\begin{aligned}
\therefore\quad \text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz &= \text{U}\{dx\cos\alpha+dy\cos\beta+dz\cos\gamma\}\\
&= \text{U}\,du,
\end{aligned}$$

$$\therefore\quad mv^2-mv_0^2 = 2m\int_{u_0}^{u}\text{U}\,du. \tag{12}$$

(3) In the case of gravity, $\text{X}=0$, $\text{Y}=0$, $\text{Z}=g$;

$$\therefore\quad mv^2-mv_0^2 = 2mg(z-z_0). \tag{13}$$

Of this result we have had instances in (32) Art. 220; in (46) Art. 227; in (50) Art. 270; (98) Art. 282; in all which cases it appears that the vis viva acquired by a heavy particle moving under the action of gravity depends only on the vertical distance between the initial and final points, and on neither the path nor the time of its passage.

(4) If X, Y, Z consist severally of many terms, and if any parts of them, say $\text{X}_1$, $\text{Y}_1$, $\text{Z}_1$, are such that

$$\text{X}_1\,dx+\text{Y}_1\,dy+\text{Z}_1\,dz = 0,$$

these terms may be omitted in the right-hand member of (3).

This is the case if $m$ moves on a smooth surface; for if R is the pressure on the surface, and if U, V, W, as in Art. 368, are proportional to the direction-cosines of its line of action,

$$\mathrm{R}\{\mathrm{U}\,dx+\mathrm{V}\,dy+\mathrm{W}\,dz\} = 0, \tag{14}$$

so that the principle of vis viva is true also for a particle moving on a smooth surface. It is similarly true also for a particle moving in a smooth tube. But it is not necessarily true when the surface or the tube in which the particle moves is rough; for in these cases X, Y, Z may not be functions of $x, y, z$.

When the condition (5) is satisfied, we have the following relations; see Vol. II, Art. 268;

$$\left(\frac{d\mathrm{Y}}{dz}\right)=\left(\frac{d\mathrm{Z}}{dy}\right),\quad \left(\frac{d\mathrm{Z}}{dx}\right)=\left(\frac{d\mathrm{X}}{dz}\right),\quad \left(\frac{d\mathrm{X}}{dy}\right)=\left(\frac{d\mathrm{Y}}{dx}\right). \tag{15}$$

In the equations (8) and (14) they are evidently satisfied.

If however the particle moves on a surface, then all the points in its path are on the surface, and the coordinates to these points satisfy the equation to the surface: and thus if U, V, W are the partial derived-functions of the equation to the surface,

$$\mathrm{U}\,dx+\mathrm{V}\,dy+\mathrm{W}\,dz = 0; \tag{16}$$

where $dx$, $dy$, $dz$ are the projections of the elements of the path of the particles. Let this be multiplied by an undetermined quantity $\mu$; then it is sufficient that

$$(\mathrm{X}+\mu\mathrm{U})\,dx+(\mathrm{Y}+\mu\mathrm{V})\,dy+(\mathrm{Z}+\mu\mathrm{W})\,dz \tag{17}$$

should be an exact differential. For this end it is necessary that

$$\left.\begin{aligned}\frac{d}{dz}(\mathrm{Y}+\mu\mathrm{V}) &= \frac{d}{dy}(\mathrm{Z}+\mu\mathrm{W})\\ \frac{d}{dx}(\mathrm{Z}+\mu\mathrm{W}) &= \frac{d}{dz}(\mathrm{X}+\mu\mathrm{U})\\ \frac{d}{dy}(\mathrm{X}+\mu\mathrm{U}) &= \frac{d}{dx}(\mathrm{Y}+\mu\mathrm{V})\end{aligned}\right\}, \tag{18}$$

that is,

$$\left.\begin{aligned}\ast\quad+\mathrm{W}\left(\frac{d\mu}{dy}\right)-\mathrm{V}\left(\frac{d\mu}{dz}\right) &= \frac{1}{\mu}\left\{\left(\frac{d\mathrm{Y}}{dz}\right)-\left(\frac{d\mathrm{Z}}{dy}\right)\right\}\\ -\mathrm{W}\left(\frac{d\mu}{dx}\right)+\quad\ast\quad+\mathrm{U}\left(\frac{d\mu}{dz}\right) &= \frac{1}{\mu}\left\{\left(\frac{d\mathrm{Z}}{dx}\right)-\left(\frac{d\mathrm{X}}{dz}\right)\right\}\\ \mathrm{V}\left(\frac{d\mu}{dx}\right)-\mathrm{U}\left(\frac{d\mu}{dy}\right)+\quad\ast\quad &= \frac{1}{\mu}\left\{\left(\frac{d\mathrm{X}}{dy}\right)-\left(\frac{d\mathrm{Y}}{dx}\right)\right\}\end{aligned}\right\}. \tag{19}$$

Now multiplying these respectively by U, V, W, and adding, we have

$$\text{U}\left\{\left(\frac{d\text{Y}}{dz}\right)-\left(\frac{d\text{Z}}{dy}\right)\right\}+\text{V}\left\{\left(\frac{d\text{Z}}{dx}\right)-\left(\frac{d\text{X}}{dz}\right)\right\}+\text{W}\left\{\left(\frac{d\text{X}}{dy}\right)-\left(\frac{d\text{Y}}{dx}\right)\right\}=0, \tag{20}$$

which is the condition requisite that the principle of vis viva may be true of a particle moving on a smooth surface.

If therefore a particle moves freely in space, three conditions, viz. those contained in (15), are to be fulfilled: but if the particle moves on a surface then only one condition, viz. (20).

If X, Y, Z are functions of $t$ the time, $\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz$ cannot be an exact differential; and therefore the principle cannot be applicable.

The great advantage of the principle is that it always enables us to obtain one integral of the equations of motion; so that the time may be determined by means of a single integration.

384.] Closely connected with the preceding is another principle known as that of Least Action, and which is applicable when the equation of vis viva is satisfied.

Let us suppose that a particle $m$ is moving either freely or on a surface, under the action of forces X, Y, Z along the coordinate axes, which are such that,

$$\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz = \text{D.F}(x,y,z), \tag{21}$$

that is, is an exact differential; then the vis viva acquired by $m$ in an unit of time (the increase of velocity being constant during that unit) is called the *action* of the particle; and if $mv^2$ is the vis viva acquired in $dt$, then $mv^2\,dt$ is the *action* acquired in $dt$; so that,

$$m\int_{t_0}^{t} v^2\,dt \tag{22}$$

is the action acquired during the time of motion. The principle of least action consists in this. The definite integral (22) is under the action of the given forces less for the path which the particle takes than it would be for any other path infinitesimally near to it, in space when $m$ moves freely, or on the surface when the motion of $m$ is constrained.

Equation (22) may be expressed in the following form: since

$$v=\frac{ds}{dt},$$

$$\therefore\quad v^2\,dt = v\,ds; \tag{23}$$

let $u$ represent the definite integral; and let the limits be expressed as in Vol. II; then

$$u = m\int_0^1 v\,ds. \tag{24}$$

Let us investigate the variation of $u$ according to the Calculus of Variations; then

$$\begin{aligned}\delta u &= m\delta.\int_0^1 v\,ds \\ &= m\int_0^1 \{\delta v\,ds + v\delta.ds\}. \qquad (25)\end{aligned}$$

Now $$\delta.ds = \frac{dx}{ds}\delta.dx + \frac{dy}{ds}\delta.dy + \frac{dz}{ds}\delta.dz, \qquad (26)$$

and $$\delta v = \frac{\mathrm{X}}{v}\delta x + \frac{\mathrm{Y}}{v}\delta y + \frac{\mathrm{Z}}{v}\delta z; \qquad (27)$$

so that (25) becomes,

$$\begin{aligned}\delta u &= m\int_0^1 \left\{\frac{ds}{v}(\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z) + \frac{v}{ds}(dx\,d.\delta x + dy\,d.\delta y + dz\,d.\delta z)\right\} \\ &= m\left[v\left(\frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\right)\right]_0^1 \\ &\quad + m\int_0^1 \left\{\left(\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds}\right)\delta x + \left(\frac{\mathrm{Y}\,ds}{v} - d.\frac{v\,dy}{ds}\right)\delta y + \left(\frac{\mathrm{Z}\,ds}{v} - d.\frac{v\,dz}{ds}\right)\delta z\right\}. \qquad (28)\end{aligned}$$

Since the limits of the integral are given, the first part vanishes of itself. Also

$$\begin{aligned}\mathrm{X}\frac{ds}{v} - d.\frac{v\,dx}{ds} &= \mathrm{X}\,dt - d.\frac{dx}{dt} \\ &= dt\left(\mathrm{X} - \frac{d^2x}{dt^2}\right) \\ &= 0;\end{aligned}$$

similarly each of the other parts in (28) vanishes; therefore $\delta u = 0$; and therefore $u$ is either a maximum or a minimum or a constant. And either of these it may be: generally however it will be a minimum; although we shall presently have an example wherein $u$ is a maximum.

Since $u$ is the sum of all the vires vivæ which are in successive elements of time generated in the moving particle, the principle may also be called that of the greatest or least vis viva.

385.] Now assuming the truth of the principle of least action, let us apply it to the motion of a particle under the action of given forces X, Y, Z, and moving (1) freely, (2) on a given smooth surface.

(1) Pursuing exactly the same course as in the last Article, and equating $\delta u$ to zero, we have

$$0 = \Big[v\Big(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y + \frac{dz}{ds}\,\delta z\Big)\Big]_0^1$$

$$+\int_0^1 \Big\{\Big(\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds}\Big)\delta x + \Big(\frac{\mathrm{Y}\,ds}{v} - d.\frac{v\,dy}{ds}\Big)\delta y + \Big(\frac{\mathrm{Z}\,ds}{v} - d.\frac{v\,dz}{ds}\Big)\delta z\Big\}; \quad (29)$$

and as no relation is given between $\delta x, \delta y, \delta z$, we have

$$\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds} = 0, \quad \frac{\mathrm{Y}\,ds}{v} - d.\frac{v\,dy}{ds} = 0, \quad \frac{\mathrm{Z}\,ds}{v} - d.\frac{v\,dz}{ds} = 0; \quad (30)$$

$$\therefore \quad \mathrm{X} = \frac{d}{dt}\frac{dx}{dt}, \qquad \mathrm{Y} = \frac{d}{dt}\frac{dy}{dt}, \qquad \mathrm{Z} = \frac{d}{dt}\frac{dz}{dt}; \qquad (31)$$

and if $t$ is equicrescent,

$$\mathrm{X} = \frac{d^2x}{dt^2}, \qquad \mathrm{Y} = \frac{d^2y}{dt^2}, \qquad \mathrm{Z} = \frac{d^2z}{dt^2}; \qquad (32)$$

which are the three ordinary equations of motion.

(2) Let the motion of $m$ be constrained to a surface whose equation is $\mathrm{F}(x, y, z) = c$, and of which the partial derived-functions are $\mathrm{U}, \mathrm{V}, \mathrm{W}$; so that $\delta x, \delta y, \delta z$ are connected by the equation

$$\mathrm{U}\,\delta x + \mathrm{V}\,\delta y + \mathrm{W}\,\delta z = 0; \qquad (33)$$

$$\therefore \quad \frac{\mathrm{X}\,dt - d.\dfrac{dx}{dt}}{\mathrm{U}} = \frac{\mathrm{Y}\,dt - d.\dfrac{dy}{dt}}{\mathrm{V}} = \frac{\mathrm{Z}\,dt - d.\dfrac{dz}{dt}}{\mathrm{W}} = \lambda\,dt \text{ (say)}; \quad (34)$$

$$\therefore \quad \left.\begin{aligned} \mathrm{X} &= \lambda\mathrm{U} + \frac{d}{dt}\frac{dx}{dt} \\ \mathrm{Y} &= \lambda\mathrm{V} + \frac{d}{dt}\frac{dy}{dt} \\ \mathrm{Z} &= \lambda\mathrm{W} + \frac{d}{dt}\frac{dz}{dt} \end{aligned}\right\}; \qquad (35)$$

let, as heretofore, $\mathrm{Q}^2 = \mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2$; then multiplying these equations severally by $\frac{\mathrm{U}}{\mathrm{Q}}, \frac{\mathrm{V}}{\mathrm{Q}}, \frac{\mathrm{W}}{\mathrm{Q}}$, and adding, we have, if $t$ is equicrescent,

$$\lambda = \frac{\mathrm{UX} + \mathrm{VY} + \mathrm{WZ}}{\mathrm{Q}^2} - \frac{\mathrm{U}\,d^2x + \mathrm{V}\,d^2y + \mathrm{W}\,d^2z}{\mathrm{Q}^2\,dt^2}; \qquad (36)$$

so that $\lambda$ is determined. And $\lambda\mathrm{Q}$ is evidently the normal reaction of the surface: for if the motion of $m$ were unconstrained, then a comparison of (32) and (35) shews that $\lambda = 0$; so that $\lambda$ is a force which is introduced by the surface: and as $\mathrm{U}, \mathrm{V}, \mathrm{W}$ are proportional to the direction-cosines of the normal to the surface at the point $(x, y, z)$, where $m$ is at the time $t$, the line of action of $\lambda$ is the normal to the surface: so that if R is the pressure of $m$ on the surface,

$$\frac{\mathrm{R}}{m} = \frac{\mathrm{UX}+\mathrm{VY}+\mathrm{WZ}}{\mathrm{Q}} - \frac{\mathrm{U}\,d^2x+\mathrm{V}\,d^2y+\mathrm{W}\,d^2z}{\mathrm{Q}\,dt^2}; \tag{37}$$

and (35) become the three common equations of motion of a particle moving on a surface. See Article 368, equations (73) and (75).

386.] In this and the following Articles I propose to apply the principle of least action to two problems; (1) that of a heavy projectile in vacuo; (2) that of the trajectory of a free particle under the action of a central force, which varies inversely as the square of the distance.

In the former example let the axis of $y$ be vertical, and let the axis of $x$ be horizontal; and let the initial and final positions of the curve be given; let the initial position be the origin; at which point let the velocity be $(2gh)^{\frac{1}{2}}$, and let the line of motion of $m$ make an angle $a$ with the horizontal line: then at the point $(x, y)$,

$$v^2 = 2g(h-y), \tag{38}$$

and therefore

$$u = m\int_0^1 \{2g(h-y)\}^{\frac{1}{2}}\,ds; \tag{39}$$

and to simplify the calculation let us assume the motion to be wholly in the plane of $xy$. The variation of $u$, being equated to zero, gives

$$\delta u = 0 = m(2g)^{\frac{1}{2}}\int_0^1 \left\{(h-y)^{\frac{1}{2}}\,\delta.ds - \frac{ds\,\delta y}{2(h-y)^{\frac{1}{2}}}\right\}; \tag{40}$$

but

$$\delta.ds = \frac{dx}{ds}\,\delta.dx + \frac{dy}{ds}\,\delta.dy;$$

$$\therefore\quad 0 = \int_0^1 \left\{(h-y)^{\frac{1}{2}}\left(\frac{dx}{ds}\,d.\delta x + \frac{dy}{ds}\,d.\delta y\right) - \frac{ds\,\delta y}{2(h-y)^{\frac{1}{2}}}\right\}$$

$$= \left[(h-y)^{\frac{1}{2}}\left(\frac{dx}{ds}\,\delta x + \frac{dy}{ds}\,\delta y\right)\right]_0^1$$

$$-\int_0^1 \left\{d.\frac{dx}{ds}(h-y)^{\frac{1}{2}}\,\delta x + \left(d.\frac{dy}{ds}(h-y)^{\frac{1}{2}} + \frac{ds}{2(h-y)^{\frac{1}{2}}}\right)\delta y\right\}: \tag{41}$$

of this expression the first part vanishes, because the points at which the motion begins and ends are given. And, as no relation is given between $x$ and $y$, we have,

$$d.\frac{dx\,(h-y)^{\frac{1}{2}}}{ds} = 0, \tag{42}$$

$$d.\frac{dy\,(h-y)^{\frac{1}{2}}}{ds} + \frac{ds}{2\,(h-y)^{\frac{1}{2}}} = 0. \tag{43}$$

Taking (42), and integrating,

$$\frac{dx}{ds}\,(h-y)^{\frac{1}{2}} = h^{\frac{1}{2}}\cos a, \tag{44}$$

if $\frac{dx}{ds} = \cos a$, when $y = 0$; whence we have

$$dx = \frac{\cos a\, h^{\frac{1}{2}}\, dy}{\{h\,(\sin a)^2 - y\}^{\frac{1}{2}}}; \tag{45}$$

integrating which, and supposing $x = 0$, when $y = 0$, we have

$$y = x\tan a - \frac{x^2}{4h\,(\cos a)^2}; \tag{46}$$

which is the equation to a parabola, and is the same as equation (87), Art. 281, if $u^2 = 2gh$. (43) is also satisfied by (46).

387.] In the next place it is required to investigate, by means of the principle of least action, the path of a particle moving freely in space under the action of a central force which varies inversely as the square of the distance.

In this problem we will use polar coordinates; let $v_0$ be the initial value of $v$ when $r = a$, so that

$$\begin{aligned} v^2 &= v_0^2 + 2\mu\left(\frac{1}{r} - \frac{1}{a}\right) \\ &= c + \frac{2\mu}{r}, \end{aligned}$$

if $$c = v_0^2 - \frac{2\mu}{a};$$

$$\therefore \quad u = m\int_0^1 \left(c + \frac{2\mu}{r}\right)^{\frac{1}{2}} ds, \tag{47}$$

where $$ds^2 = dr^2 + r^2\,d\theta^2; \tag{48}$$

so that $$\delta.ds = \frac{dr}{ds}\,\delta.dr + \frac{r^2\,d\theta}{ds}\,\delta.d\theta + \frac{r\,d\theta^2}{ds}\,\delta r;$$

and taking the variation of (47), and equating it to zero, we have,

$$\begin{aligned} 0 = \Big[\left(c + \frac{2\mu}{r}\right)^{\frac{1}{2}}\left(\frac{dr}{ds}\,\delta r + \frac{r^2\,d\theta}{ds}\,\delta\theta\right)\Big]_0^1 & \\ + \int_0^1 \left\{\frac{r\,d\theta^2}{ds} - \frac{\mu}{r^2}\left(c + \frac{2\mu}{r}\right)^{-\frac{1}{2}} - d.\left(c + \frac{2\mu}{r}\right)^{\frac{1}{2}}\frac{dr}{ds}\right\}\delta r & \\ - \int_0^1 d\left\{\left(c + \frac{2\mu}{r}\right)^{\frac{1}{2}}\frac{r^2\,d\theta}{ds}\right\}\delta\theta &: \quad (49) \end{aligned}$$

and equating to zero, the coefficient of $\delta\theta$ under the sign of integration, we have

$$\left(c+\frac{2\mu}{r}\right)^{\frac{1}{2}} \frac{r^2\, d\theta}{ds} = \text{a constant}$$
$$= k \text{ (say)};$$

therefore if $u = \frac{1}{r}$, we have

$$u = \frac{\mu}{k^2} + \frac{(c k^2 + \mu^2)^{\frac{1}{2}}}{k^2} \cos(\theta - \gamma), \tag{50}$$

which is the equation to a conic section.

388.] If the velocity $v$ of the moving particle $m$ is constant, then

$$u = m v \int_0^1 ds$$
$$= m v s, \tag{51}$$

and therefore is proportional to the length of the path. The path therefore of an unconstrained particle is in this case a straight line. If however the motion is constrained to take place on a given surface, the path of least action is the geodesic on the surface which joins the two given positions, viz. the initial and the final places of the particle. If the surface is closed, as that of a sphere, there will be at least two geodesic paths joining the initial and the terminal places of $m$; one of which will be a maximum, and the other a minimum; thus in one case the *action* also will be a maximum, and in the other a minimum; and the action also may be constant, whatever is the path, provided that in this case $\delta u = 0$. For suppose the two points to be on a sphere: the great circle of the sphere passing through these two points will be the geodesic; one segment of which will be a maximum, and the other will be a minimum; and if the two points are opposite poles of the sphere, there will be an infinite number of great circles passing through them, and the lengths of all the arcs joining the two points will be the semi-circumference of a great circle, and thus the same for all. In this case $u$ is constant, and thus $\delta u = 0$.

If the velocity is constant,

$$\text{X}\, dx + \text{Y}\, dy + \text{Z}\, dz = 0: \tag{52}$$

this is satisfied by $\text{X} = \text{Y} = \text{Z} = 0$, that is, when the particle $m$ is acted on by no force. Also when all the impressed forces act along lines, which are perpendicular to the path of $m$ at every point of it. Thus if a particle moves on a surface, and is subject to no force except the normal reaction of the surface, (52) is satisfied, and the path of the particle is a geodesic line.

## SECTION 2.—*The method of variation of parameters.*

389.] In some problems which have been investigated in the preceding Articles (and there are many of the kind), the integrations have been performed without difficulty in the more simple forms of the questions; whereas the integrations have been impossible in finite terms, when another term, which has generally expressed an additional impressed momentum-increment, has been introduced. The most salient instance of this circumstance occurs in equations (162) of Art. 295. When the expressions involving the disturbing function R are omitted, the equations become those marked (138), and admit of integration; and the complete integral is a conic, say, an ellipse; but equations (162) cannot be integrated in their complete form. A method of dealing generally with such questions has been discovered by Lagrange, and largely applied in Physical Astronomy. It is called the method of variation of parameters, and will be most conveniently explained by means of an example; and for this purpose I shall take the case of Art. 295, and shall for the sake of simplicity assume all the bodies to be in the plane of $x$, $y$; so that the equations of motion are, when obvious substitutions are made, of the form,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X}+\mathrm{X}' \\ \frac{d^2y}{dt^2} &= \mathrm{Y}+\mathrm{Y}' \end{aligned}\right\}. \tag{53}$$

Let us suppose that these equations admit of complete integration, when $\mathrm{X}'$ and $\mathrm{Y}'$ are omitted: and that the integrals of the equations,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \mathrm{X} \\ \frac{d^2y}{dt^2} &= \mathrm{Y} \end{aligned}\right\}, \tag{54}$$

are $\qquad x = f(a, \alpha, t), \qquad y = \phi(b, \beta, t);$ $\qquad$ (55)

where $a, \alpha, b, \beta$ are four constants, as yet undetermined, introduced in the process of integration. Let us suppose the solutions of (53) to be of the form (55), in which $a, \alpha, b, \beta$ are not constant, but functions of $t$; and let them be determined, so that not only shall the coordinates of the particle, whose motion is represented by (53), be the same at the time $t$ as that whose motion is expressed by (54), but also let the component velocities be the same in both cases: so that

$$\frac{dx}{dt} = \frac{df}{dt}, \qquad \frac{dy}{dt} = \frac{d\phi}{dt}. \tag{56}$$

Now from (55) we have

$$\left.\begin{aligned} \frac{dx}{dt} &= \left(\frac{df}{dt}\right) + \left(\frac{df}{da}\right)\frac{da}{dt} + \left(\frac{df}{d\alpha}\right)\frac{d\alpha}{dt} \\ \frac{dy}{dt} &= \left(\frac{d\phi}{dt}\right) + \left(\frac{d\phi}{db}\right)\frac{db}{dt} + \left(\frac{d\phi}{d\beta}\right)\frac{d\beta}{dt} \end{aligned}\right\}; \tag{57}$$

therefore

$$\left.\begin{aligned} \left(\frac{df}{da}\right)\frac{da}{dt} + \left(\frac{df}{d\alpha}\right)\frac{d\alpha}{dt} &= 0 \\ \left(\frac{d\phi}{db}\right)\frac{db}{dt} + \left(\frac{d\phi}{d\beta}\right)\frac{d\beta}{dt} &= 0 \end{aligned}\right\}. \tag{58}$$

Also from (56) we have

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \left(\frac{d^2f}{dt^2}\right) + \left(\frac{d^2f}{da\,dt}\right)\frac{da}{dt} + \left(\frac{d^2f}{d\alpha\,dt}\right)\frac{d\alpha}{dt} \\ \frac{d^2y}{dt^2} &= \left(\frac{d^2\phi}{dt^2}\right) + \left(\frac{d^2\phi}{db\,dt}\right)\frac{db}{dt} + \left(\frac{d^2\phi}{d\beta\,dt}\right)\frac{d\beta}{dt} \end{aligned}\right\}; \tag{59}$$

and substituting these in (53), we have

$$\left.\begin{aligned} \left(\frac{d^2f}{da\,dt}\right)\frac{da}{dt} + \left(\frac{d^2f}{d\alpha\,dt}\right)\frac{d\alpha}{dt} &= \mathrm{X}' \\ \left(\frac{d^2\phi}{db\,dt}\right)\frac{db}{dt} + \left(\frac{d^2\phi}{d\beta\,dt}\right)\frac{d\beta}{dt} &= \mathrm{Y}' \end{aligned}\right\}: \tag{60}$$

by (58) and (60) may the four quantities $a, \alpha, b, \beta$ be determined.

Since the components of the velocity are the same in both curves at their common point at the given instant, it is evident that the curves at that point touch each other, and thus have a common tangent. And as the parameters, $a, \alpha, b, \beta$, which determine the orbit, vary with the time, so does the form of the curve continually undergo change. From this point of view, the curve in which the particle moves has received the name of the instantaneous orbit, and the forces which produce the changes in the instantaneous orbit are called disturbing forces. The actual orbit therefore is the envelope of all these instantaneous orbits, the change of the instantaneous orbit being due to the disturbing forces, and the law of that change being assigned by their variations. I propose to illustrate the method by one or two simple examples; but the most important application of it is beside the scope of our present work.

390.] A heavy particle falls from rest in a medium the resistance of which varies as the square of the velocity; it is required to determine the circumstances of motion.

In this case the equation of motion is

$$\frac{d^2x}{dt^2} = g-k\left(\frac{dx}{dt}\right)^2: \tag{61}$$

and omitting the last term, the equation becomes,

$$\frac{d^2x}{dt^2} = g. \tag{62}$$

The solution of which is

$$x = a+at+\frac{gt^2}{2}, \tag{63}$$

$$\therefore \quad \frac{dx}{dt} = a+gt+\frac{da}{dt}+t\,\frac{da}{dt}: \tag{64}$$

if the velocity is the same in both the orbits represented by (61) and (62), then

$$\frac{da}{dt}+t\,\frac{da}{dt} = 0: \tag{65}$$

and from the former part of (64) we have,

$$\frac{d^2x}{dt^2} = g+\frac{da}{dt}: \tag{66}$$

substituting which in (61), we have

$$\begin{aligned}\frac{da}{dt} &= -k\left(\frac{dx}{dt}\right)^2\\ &= -k(a+gt)^2;\end{aligned}$$

and adding $g$ to both sides of the equation, and dividing by $g-k(a+gt)^2$, we have

$$\frac{da+g\,dt}{g-k(a+gt)^2} = dt;$$

integrating, and taking limits such that $a$ and $t$ are simultaneously zero, we have

$$\log\frac{g^{\frac{1}{2}}+k^{\frac{1}{2}}(a+gt)}{g^{\frac{1}{2}}-k^{\frac{1}{2}}(a+gt)} = 2(kg)^{\frac{1}{2}}t; \tag{67}$$

$$\therefore \quad a = -gt+\left(\frac{g}{k}\right)^{\frac{1}{2}}\frac{e^{2(kg)^{\frac{1}{2}}t}-1}{e^{2(kg)^{\frac{1}{2}}t}+1}. \tag{68}$$

Also from (65), we have, integrating by parts,

$$\begin{aligned} a &= -\alpha t + \int \alpha \, dt \\ &= \frac{g t^2}{2} - \left(\frac{g}{k}\right)^{\frac{1}{2}} \frac{e^{2(kg)^{\frac{1}{2}} t} - 1}{e^{2(kg)^{\frac{1}{2}} t} + 1} t + \frac{1}{k} \log \frac{e^{(kg)^{\frac{1}{2}} t} + e^{-(kg)^{\frac{1}{2}} t}}{2}; \qquad (69) \end{aligned}$$

so that $a$ and $\alpha$ are both known in terms of $t$; and substituting in (63), we have a result the same as (111) in Article 241.

391.] Another problem again, on account of its importance in the theory of gunnery, suggests itself for solution by the preceding process.

To determine the path of a particle projected with a given velocity in a line inclined at a given angle to the horizon, and moving in a medium the resistance of which varies as the square of the velocity.

In this case the equations of motion are, as in Art. 302,

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= -k \frac{ds}{dt} \frac{dx}{dt} \\ \frac{d^2y}{dt^2} &= -g - k \frac{ds}{dt} \frac{dy}{dt} \end{aligned}\right\}. \qquad (70)$$

If there is no resisting medium, the equations of motion are

$$\frac{d^2x}{dt^2} = 0, \qquad \frac{d^2y}{dt^2} = -g; \qquad (71)$$

of which let the solutions be,

$$x = a + \alpha t, \qquad y = b + \beta t - \frac{g t^2}{2}. \qquad (72)$$

As the velocity is to be the same in the disturbed and in the undisturbed paths, we have

$$\frac{dx}{dt} = \alpha, \qquad \frac{dy}{dt} = \beta - g t, \qquad (73)$$

$$\frac{da}{dt} + t \frac{d\alpha}{dt} = 0, \qquad \frac{db}{dt} + t \frac{d\beta}{dt} = 0; \qquad (74)$$

therefore from (73), $$\frac{ds}{dt} = \{\alpha^2 + (\beta - g t)^2\}^{\frac{1}{2}}. \qquad (75)$$

Also from (73), $$\frac{d^2x}{dt^2} = \frac{d\alpha}{dt}, \qquad \frac{d^2y}{dt^2} = \frac{d\beta}{dt} - g;$$

and substituting these in (70), we have

$$\frac{d\alpha}{dt} = -k \alpha \{\alpha^2 + (\beta - g t)^2\}^{\frac{1}{2}}, \qquad (76)$$

$$\frac{d\beta}{dt} = -k (\beta - g t) \{\alpha^2 + (\beta - g t)^2\}^{\frac{1}{2}}: \qquad (77)$$

and therefore from (74),

$$\frac{da}{dt} = k\alpha t\,\{\alpha^2+(\beta-gt)^2\}^{\frac{1}{2}}, \tag{78}$$

$$\frac{db}{dt} = kt\,(\beta-gt)\,\{\alpha^2+(\beta-gt)^2\}^{\frac{1}{2}}; \tag{79}$$

from which four equations $a, \alpha, b, \beta$ are to be found in terms of $t$.

Eliminating $t$ from (72), we have

$$\left(x-a-\frac{\alpha\beta}{g}\right)^2 = \frac{2\alpha^2}{g}\left(\frac{\beta^2}{2g}+b-y\right), \tag{80}$$

which is the equation to the instantaneous path, and is a parabola; of which, if the latus rectum is $4p$, and if $(h, l)$ is the focus,

$$2p = \frac{\alpha^2}{g}, \tag{81}$$

$$h = a+\frac{\alpha\beta}{g}, \qquad l = \frac{\beta^2-\alpha^2}{2g}+b; \tag{82}$$

therefore, if the velocity at $(x, y) = v$,

$$\begin{aligned}\frac{dp}{dt} &= \frac{\alpha}{g}\frac{d\alpha}{dt} = -k\frac{\alpha^2}{g}v\\ &= -2pkv,\end{aligned} \tag{83}$$

$$\left.\begin{aligned}\frac{dh}{dt} &= -\frac{2k\alpha}{g}(\beta-gt)\,v\\ \frac{dl}{dt} &= -\frac{kv}{g}(v^2-2\alpha^2)\end{aligned}\right\}. \tag{84}$$

From (83) it appears that the latus rectum of the parabola continually diminishes; and since $v = \dfrac{ds}{dt}$, we have from it by integration,

$$\log\frac{p}{p_0} = 2k\,(s_0-s);$$

so that the logarithm of the ratio of any two latera recta varies as the length of the arc between the points to which they correspond.

Also from (84), the focus of the parabola moves upwards or downwards according as $v^2$ is less than or greater than $2\alpha^2$; that is, according as the angle at which the line of the particle's motion is inclined to the horizon is less or greater than 45°; and from the first of (84) it appears that as the particle is ascending to the highest point of the orbit, the axis of the instantaneous parabola is moving backwards; when the particle is at the highest point, the axis is stationary; and when the particle descends, the axis moves forward.

392.] The method of variation of parameters has also been applied (for the first time, as far as I know, by Mr. Airy, the present Astronomer Royal*) to the calculation of the alterations produced in the amplitudes and the time of oscillation of a cycloidal pendulum, when a small disturbing force acts on it.

Let the forces be resolved normally and tangentially; and let the disturbing force be $\mathrm{s}$, and act along the tangent to the cycloid, and diminish the velocity of the pendulum in its descent. Let $a$ be the radius of the generating circle of the cycloid; then the equation of undisturbed motion is, see Art. 353,

$$\frac{d^2 s}{dt^2} = -\frac{g}{4a}s.$$

Let $\frac{g}{4a} = n^2$, where $4a$, be it observed, is the length of the pendulum, see Article 354, and we have,

$$\frac{d^2 s}{dt^2} + n^2 s = 0; \tag{85}$$

and the equation for the disturbed motion is,

$$\frac{d^2 s}{dt^2} + n^2 s = \mathrm{s}. \tag{86}$$

The general integral of (85) is,

$$s = c \sin (nt + \alpha), \tag{87}$$

where $c$ and $\alpha$ are arbitrary constants; and where $c$ is the amplitude measured along the cycloidal arc; and where $\frac{\pi}{2n} - \frac{\alpha}{n}$ is the time at which $s = c$.

From (87) the expression for the velocity in the undisturbed path is,

$$\frac{ds}{dt} = cn \cos (nt + \alpha); \tag{88}$$

and as the velocity is the same in the disturbed path, when $c$ and $\alpha$ vary, we have

$$\frac{dc}{dt} \sin (nt + \alpha) + c \cos (nt + \alpha) \frac{d\alpha}{dt} = 0; \tag{89}$$

and differentiating again (88) for the disturbed motion, and substituting in (86), we have

$$n \frac{dc}{dt} \cos (nt + \alpha) - nc \sin (nt + \alpha) \frac{d\alpha}{dt} = \mathrm{s}; \tag{90}$$

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* See the Transactions of the Cambridge Philosophical Society, Vol. III, Part I.

so that
$$\frac{dc}{dt} = \frac{\text{s}}{n}\cos(nt+a), \tag{91}$$
$$\frac{da}{dt} = -\frac{\text{s}}{cn}\sin(nt+a); \tag{92}$$
which give the variations of $c$ and $a$ in terms of $t$; and if these equations were susceptible of integration, the problem would be completely solved. In only a few cases is the solution possible.

If it is required to find the alteration of $c$ due to one vibration, it is necessary to integrate
$$\frac{\text{s}}{n}\cos(nt+a)\,dt,$$
from a value of $t$ which makes $nt+a$ equal to $\theta$, to a value which makes it equal to $\theta+\pi$; so that the increase in the amplitude of vibration
$$= \int\frac{\text{s}}{n}\cos(nt+a)\,dt \tag{93}$$
between the proper limits; as $\theta$ is undetermined, the result is general.

If it is required to find the alteration in the time of vibration during one oscillation, we proceed as follows. Let $a_1$ and $t_1$ be the values of $a$ and $t$ when the pendulum comes to rest, that is, when $\cos(nt+a)$, see (88), $=0$; and let $a_2$ and $t_2$ be the values when the pendulum comes to rest the next time; so that, say,
$$nt_1+a_1 = \frac{\pi}{2}, \qquad nt_2+a_2 = \frac{3\pi}{2};$$
$$\therefore\quad n(t_2-t_1)+a_2-a_1 = \pi; \tag{94}$$
$$\therefore\quad t_2-t_1 = \frac{\pi}{n}-\frac{a_2-a_1}{n}$$
$$= \frac{\pi}{n}+\frac{1}{cn^2}\int\text{s}\sin(nt+a)\,dt, \tag{95}$$
by (92); the integral being definite, and taken between limits corresponding to the extreme values of the arc of vibration. If $\text{s}=0$, the time of vibration $=\frac{\pi}{n}$; so that
$$t_2-t_1 = \frac{\pi}{n}\left\{1+\frac{1}{cn\pi}\int\text{s}\sin(nt+a)\,dt\right\}; \tag{96}$$
and therefore the proportionate increase of the time of vibration is
$$\frac{1}{cn\pi}\int\text{s}\sin(nt+a)\,dt. \tag{97}$$
If s is expressed in terms of $t$, (93) and (97) may be used; but if s is a function of $s$, then from (87) and (88) we have,

$$\sin(nt+\alpha) = \frac{s}{c}, \qquad \cos(nt+\alpha) = \frac{(c^2-s^2)^{\frac{1}{2}}}{c},$$

and $$dt = \frac{ds}{n\,(c^2-s^2)^{\frac{1}{2}}};$$

$$\therefore \quad \text{the increase of amplitude} = \frac{1}{cn^2}\int \mathrm{s}\,ds, \tag{98}$$

$$\text{the prop. increase of time of vibration} = \frac{1}{c^2n^2\pi}\int \frac{\mathrm{s}\,s\,ds}{(c^2-s^2)^{\frac{1}{2}}}. \tag{99}$$

393.] One or two examples are subjoined.

(1) Let the pendulum make small vibrations in a circular arc; then the tangential impressed velocity-increment is

$$-g\sin\frac{s}{a},$$

which is equal to $$-g\left\{\frac{s}{a}-\frac{s^3}{6a^3}+\dots\right\};$$

and omitting powers of $\frac{s}{a}$ above the cube, we have

$$\mathrm{s} = \frac{gs^3}{6a^3}; \quad \text{and} \quad n^2 = \frac{g}{a}.$$

Therefore the proportionate increase in the time of vibration

$$= \frac{g}{3\pi c^2n^2a^3}\int_0^c \frac{s^4\,ds}{(c^2-s^2)^{\frac{1}{2}}}$$

$$= \frac{c^2}{16a^2},$$

which result is the same as (41), Art. 357. Also

$$\text{increase of amplitude} = \frac{1}{cn^2}\int_{-c}^{c} \frac{gs^3}{6a^3}\,ds$$

$$= \frac{g}{24cn^2a^3}\Big[s^4\Big]_{-c}^{c}$$

$$= 0.$$

(2) Let the friction at the point of suspension be such as to cause a constant tangential retardation; and suppose therefore $\mathrm{s} = -f$;

$$\therefore \quad \text{increase of amplitude} = -\frac{f}{cn^2}\int_{-c}^{c} ds$$

$$= \frac{-2f}{n^2}.$$

Proportionate increase in the time of vibration is

$$-\frac{f}{c^2n^2\pi}\int_{-c}^{+c} \frac{s\,ds}{(c^2-s^2)^{\frac{1}{2}}} = 0.$$

Other examples will be found in the memoir of Mr. Airy, which is referred to in the beginning of the preceding Article.

# CHAPTER XIV.

## ON VIRTUAL VELOCITIES.

394.] In Article 103 of the present volume it has been shewn that when a system of statical pressures acting on a body in space is in equilibrium, the differential of the central moment of that system is zero; and the mathematical expression of that property is

$$\Sigma . P \, dp = 0. \qquad (1)$$

This equation is usually called the equation of Virtual Velocities; and expresses a theorem of which the enuntiation is the following:

If a system of statical forces is in equilibrium, and the system receives an infinitesimal displacement of the most general kind possible, whereby the points of application of the forces are displaced; but the forces still act along lines parallel to, and infinitesimally distant from, their former lines of action; and if the relative distances of their points of application are unaltered; then the sum of the products of each force and the projection on its line of action of the displacement of its point of application is equal to zero.

From this theorem, as it is observed in Article 104, the six equations of statical equilibrium are at once deducible; and thus it contains in an abridged form all the principles and results of Statics. It is my object now to prove that it is deducible from the general principles of the science of motion: and that the equations of Dynamics take the form (1) when the motions of the particles on which the dynamical forces act are not actual, but only potential or virtual. And that thus it is the proposition intermediate to Statics and Dynamics, and by means of which Statics becomes a branch of the science of motion.

In this inquiry I propose to follow the method which seems most convenient for a didactic treatise. The principle of virtual velocities has already been enuntiated, and its mathematical expression is given in (1): a principle of affixing signs to the forces according to certain modifications of them must be conventionally settled; and conditions as to their action must be assumed, which may be either in act or in hypothesis realized. Then the principle will be applied to some well known problems, so that it may be more fully grasped: and lastly it will be proved in its completeness from Dynamical considerations; and some corollaries will be drawn from it.

395.] From the point of application which a given force has after the displacement, let a perpendicular be drawn on the original line of action of the force. Thus in fig. 140, let AMP be the line of action of P, ere the displacement takes place: let the system be infinitesimally displaced, so that the point of application of the force is shifted from A to A′; AA′ being of infinitesimal length; let us suppose the line of action of the force after the displacement to be parallel to its line of action before the displacement, so that A′P′ is parallel to AP. From A′ let a perpendicular A′M be drawn to the original line of action of the force, so that AM is the orthogonal projection of AA′ on that line: AM is called the *virtual velocity* of the force P; because it is the infinitesimal distance over which the point of application of P moves in its own line of action during the first time-element of its motion. If, as in the first figure of fig. 140, AM lies along AP in the direction in which P acts, the virtual velocity is taken to be positive: and if it lies in the direction of AP produced backwards, as in the second figure, then it is taken to be negative.

Hence if the displacement of the point of application takes place along the line of action of P, the whole displacement becomes the virtual velocity: and is positive or negative according as it takes place in the direction towards which P acts, or in the opposite direction.

Hence also if the point of application of the force is displaced in a line which is perpendicular to the line of action of the force, the virtual velocity of the force is zero.

If a system of forces acting on a body consists of the forces $P_1, P_2, \dots P_n$; and if their virtual velocities are symbolized by $dp_1, dp_2, \dots dp_n$; the signs of these quantities being fixed according to the preceding conventions, then the mathematical expression of the principle of virtual velocities is

$$\Sigma . P\, dp = 0. \qquad (2)$$

396.] Examples illustrative of the principle of virtual velocities.

Ex. 1. Three forces P, Q, R act in given lines on the point A, and are in equilibrium: it is required to determine the relation between them.

Let the angles severally between the lines of action of Q and R, of R and P, of P and Q, be $a, \beta, \gamma$: let the point of application of the forces be shifted from A to A′, see fig. 141; and from A′ let perpendiculars A′$m$, A′$n$, A′$p$ be drawn to the lines of action of P, Q, R respectively; then A$m$, A$n$, A$p$ are the virtual velocities of P, Q, R respectively: and (2) becomes

$$\mathrm{P} \times \mathrm{A}m + \mathrm{Q} \times \mathrm{A}n - \mathrm{R} \times \mathrm{A}p = 0. \tag{3}$$

Let $\mathrm{AA}' = \delta s$; $\mathrm{A'AP} = \theta$; $\mathrm{QAR} = a$, $\mathrm{RAP} = \beta$, $\mathrm{PAQ} = \gamma$: so that (3) becomes,

$$\mathrm{P}\,\delta s \cos\theta + \mathrm{Q}\,\delta s \cos(\theta+\gamma) + \mathrm{R}\,\delta s \cos(\beta-\theta) = 0;$$

$$\therefore \quad \mathrm{P} + \mathrm{Q}\cos\gamma + \mathrm{R}\cos\beta + (\mathrm{R}\sin\beta - \mathrm{Q}\sin\gamma)\tan\theta = 0; \tag{4}$$

and as the line along which A is displaced is indeterminate, $\theta$ is indeterminate, and therefore from (4) we have,

$$\mathrm{P} + \mathrm{Q}\cos\gamma + \mathrm{R}\cos\beta = 0, \tag{5}$$

$$\mathrm{R}\sin\beta - \mathrm{Q}\sin\gamma = 0: \tag{6}$$

from the second of which we have

$$\frac{\mathrm{P}}{\sin a} = \frac{\mathrm{Q}}{\sin\beta} = \frac{\mathrm{R}}{\sin\gamma}; \tag{7}$$

the first term of the equality being inferred by reason of the symmetry.

Also we have

$$\mathrm{R}\cos\beta = -\mathrm{P} - \mathrm{Q}\cos\gamma, \qquad \mathrm{R}\sin\beta = \mathrm{Q}\sin\gamma;$$

whence, squaring and adding,

$$\mathrm{R}^2 = \mathrm{P}^2 + 2\mathrm{PQ}\cos\gamma + \mathrm{Q}^2: \tag{8}$$

(7) and (8) are respectively the mathematical expressions of the triangle and of the parallelogram of forces.

Ex. 2. To determine the conditions of equilibrium of the straight lever.

Let ACB be the lever, fig. 142, which turns about a horizontal axis through C: let the forces P and Q act at the ends A and B along lines of action which are inclined to ACB at angles $a$ and $\beta$ respectively: let $\mathrm{AC} = a$, $\mathrm{CB} = b$.

Let the lever be turned about the horizontal axis through an infinitesimal angle $d\theta$, so that $\text{AA}' = a\,d\theta$, $\text{BB}' = b\,d\theta$: then the projections of these quantities on the lines of action of P and Q respectively are $a\,d\theta \sin a$, $b\,d\theta \sin\beta$; and as the virtual velocity of Q is negative according to our figure, equation (2) becomes,

$$\text{P}\,a\,d\theta \sin a - \text{Q}\,b\,d\theta \sin\beta = 0,$$

$$\therefore \quad \text{P}\,a \sin a = \text{Q}\,b \sin\beta:$$

which is the ordinary equation of moments about C.

Ex. 3. To determine the conditions of equilibrium of the wheel and axle.

Let $a =$ the radius of the wheel on which P acts: $b =$ the radius of the axle on which W acts: and let the system be turned through a small angle $d\theta$, so that P (say) descends through a vertical distance $a\,d\theta$, and W ascends through a vertical space $b\,d\theta$: then equation (2) becomes,

$$-a\,d\theta\,\text{P} + b\,d\theta\,\text{W} = 0, \qquad \therefore \quad \text{P}a = \text{W}b.$$

Ex. 4. To find the conditions of equilibrium in the screw.

In this mechanical power (as it is called) I shall assume that there is no friction. Let $h$ be the vertical distance between two successive winds of the thread: let $l$ be the length of the lever, measured from the axis of the screw, at the end of which P acts: let W be the weight on the screw. Then as W descends through a vertical distance equal to $h$, the point of application of P moves round the circumference of a circle whose radius is $b$: so that $h$ and $2\pi b$ are evidently proportional to the virtual velocities of W and P; and equation (2) becomes,

$$-2\pi b\text{P} + \text{W}h = 0,$$

$$\therefore \quad \text{P} = \frac{h}{2\pi b}\text{W}. \tag{9}$$

397.] Let us now investigate the principle of virtual velocities by means of the general principles of dynamics.

When a particle, whose mass is $m$, moves with the velocity $v$, the momentum, or the *quantity of velocity*, which it has is $mv$; see Art. 210: and this is capable of being transferred to any other mass: this therefore we take to be the measure of the effect which $m$ moving with the velocity $v$ is capable of producing.

Now the effect of all mechanical forces is velocity: that is, whatever effects of other kinds they may produce, the velocity which they communicate is the only one of which, in the science of motion, we take cognizance: and this velocity may be either actual or virtual; it may be communicated, or only be capable of being communicated; and it is in either case the measure of the force's effect. In pressures such as we are about to investigate a law of, the velocity impressed is not actual; it is only virtual or potential: but the virtual $mv$ is the measure of the force's effect. To simplify our considerations, I shall suppose all the forces to be of the same kind, and to have a common source: and I will assume that they are weights. That is, the attraction of the earth acting on different quantities of matter produces, so I assume, the different acting forces: the various lines of action and the direction of such forces may be produced by a simple contrivance, such as a pulley: let us suppose the impressed momentum-increments (virtual) to be $\text{P}_1, \text{P}_2, \ldots \text{P}_n$, and acting uniformly during a second of time, to be capable of producing in that second of time an increase of momentum in the masses $m_1, m_2, \ldots m_n$ respectively equal to $m_1 g, m_2 g, \ldots m_n g$; so that we have,

$$\text{P}_1 = m_1 g, \qquad \text{P}_2 = m_2 g, \ldots\ldots \text{P}_n = m_n g; \tag{10}$$

then $\text{P}_1, \text{P}_2, \ldots \text{P}_n$ may be replaced by these several equivalents, the lines of action and the directions being arranged identically with the former by means of pulleys or some other equivalent contrivance. Of the pressures and masses in (10) I shall take $\text{P}$ and $m$ to be type-expressions, so that

$$\text{P} = mg.$$

Let a system of forces act on a particle or a rigid body, of which let $\text{P}$, or its equivalent $mg$, be the type: let a small geometrical displacement be given to the body, and such as is compatible with the arrangement of its particles, and does not cause any rupture of them. Let the point of application of the force $\text{P}$ move over a distance, the projection of which on the line of action of $\text{P}$ is $p$: and let us suppose the same result to take place with all the other forces: then the distance $p$ is the proportional measure of the velocity which $m$ receives by reason of the displacement, and therefore the measure of $m$'s effect is $mp$. Here however it must be observed that this momentum in its action on the body will be affected with a different sign according as the projection of the line through which the point of application moves is towards or opposite to the direction in which $\text{P}$ acts: that is, we consider $p$ to be positive or negative according as it falls in the line along which P acts, or in that line produced backwards. And with this convention the effects of all the forces will be proportional to $m_1 p_1, m_2 p_2, \dots m_n p_n$.

398.] Suppose all the pressures thus estimated to act simultaneously on the system of particles, and to keep it at rest. By the law of inertia, the system cannot be disturbed except by a new force. Let us suppose such a new force to be introduced, and the system thereby to receive a slight displacement; so that the points of application of the forces may be changed, but that their present lines of action may be parallel to the former ones. Then the displaced system will be in some one of the three following states. If the displacing force is removed, (1) the system may return to its former state: (2) it may remain at rest in its new state: (3) it may depart further from its original state. In the first state, the disturbing force acts and prevents the return: in the second state, the disturbing force has moved the system, but no longer acts to retain it in its new position: in the third state, it acts and hinders a farther departure from the original state. Now taking all the forces to act as in the original state of rest; in (1) the sum of these effects is greater in the disturbed than in the balanced state, because it requires another force (viz. that which prevents the return) to be in equilibrium with the original forces: then the sum of the effects of the forces is a minimum in the balanced state for this particular disturbance: and if the same result follows for every other possible disturbance, the sum of the effects of the forces is a minimum.

In (2) if, after the system has been disturbed, no force is required to keep it in its displaced state, the forces are in equilibrium in the displaced just as in the original state; and therefore as the displacement may be infinitesimal, the sum of the effects is the same in both states.

In (3) if, after the displacement, a force is needed to hinder the system from farther departure from the original state of equilibrium, the sum of the effects of the original forces is, in the disturbed state, less than it was in the balanced state of the system; and if this is true for every other possible displacement, the sum of the effects in the original balanced state is a maximum. Now although for a state of equilibrium of a system neither of these circumstances may exist for all the possible displacements, yet one or other of them must; and they all have this common characteristic; viz. the differential of the sum of the effects of the forces is zero. Having therefore in the preceding Article investigated a method of estimating the effects of forces, we have in this Article discovered a relation which these forces must satisfy, when they produce no motion; viz. the variation of the sum of their effects for a small geometrical displacement is zero.

Let the sum of the effects of all the forces be represented by $\Pi$: so that

$$\Pi = m_1 p_1 + m_2 p_2 + m_3 p_3 + \dots\dots + m_n p_n \qquad (11)$$

$$= \Sigma . mp. \qquad (12)$$

In differentiating (11) the $m$'s do not vary: therefore

$$d\Pi = m_1\, dp_1 + m_2\, dp_2 + \dots\dots + m_n\, dp_n$$

$$= 0:$$

and replacing $m_1, m_2, \dots m_n$ by their proportionals $P_1, P_2, \dots P_n$, we have

$$P_1\, dp_1 + P_2\, dp_2 + \dots\dots + P_n\, dp_n = 0,$$

$$\therefore \quad \Sigma . P\, dp = 0; \qquad (13)$$

which is the mathematical statement of the principle of Virtual Velocities.

399.] In the preceding investigations we have supposed the points of application of all the forces to be capable of geometrical displacement: there are however cases where such is not geometrically possible. Suppose a body to rest upon a surface, so that the normal reaction of the surface is one of the acting forces: if the body is removed from the surface, the reaction may be said no longer to exist. In this case, if it is required to determine the relations between the forces exclusive of the reaction, the body may be moved to some other point on the surface infinitesimally near to the former one: in which case the virtual velocity of the normal pressure will be zero, and the normal pressure will not enter into the equation (13). If on the other hand the normal pressure is required to be determined, then the normal pressure must be supposed to be like any other pressure, and to be susceptible of a change in its point of application, and it will be convenient to shift the body along a line normal to the surface.

In Art. 104, the six general equations of statical equilibrium have been deduced from the principle of virtual velocities; and

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